Unexpected Default in Information based Model

Rainer Buckdahn, Université de Bretange Occidentale, Brest, France/
Shandong University, Jinan, P.R. China

University of Loughborough, December, 5-9, 2016
Outline

Information process $\beta$ for default time $\tau$

Preliminaries and Markov Property of $(\beta, F^\beta)$

Semimartingale decomposition of $(\beta, F^\beta)$

Compensator of the default time $\tau$ w.r.t. $F^\beta$

Sketch of the proof of the main result
Information process $\beta$ for default time $\tau$

Consider the (future) default time $\tau > 0$ of, e.g., a financial company; $\tau$ is described by a r.v. over a complete probability space $(\Omega, \mathcal{F}, P)$;

Problem of modelling information concerning the default time $\tau$ := problem of describing the flow of information about $\tau$ by a process (= the so-called information process).

Information process here modelled by a $(0-0-)\text{Brownian bridge } \beta$ on $[[0, \tau]]$:

$$\beta_t := W_t - \frac{t}{\tau \lor t} W_{\tau \lor t}, \quad t \geq 0,$$

where: $W$ is a 1-dim. BM independent of $\tau$.

The filtration generated by $\beta$: $\mathbf{F}^\beta := \left( \mathcal{F}_t^\beta : \sigma\{\beta_s, s \leq t\} \lor \mathcal{N}_P \right)_{t \geq 0}$

+ provides partial information on the default time $\tau$ before it occurs:
+ The intuitive idea: If $\beta_t$ is away from 0 - no imminent default; if $\beta_t$ close to 0 - danger of default.

**Our objective**: Study of $(\beta, F^\beta)$ as information process for $\tau$.

Talk is based on joint works with
- Matteo Bedini (Banca Intesa San Paolo, Milano, Italy),
- Hans-Jürgen Engelbert (Friedrich Schiller Universität Jena, Germany):
  + *Brownian bridges on random intervals*, Teoria Verojatnost. i Primen. (2016);
  + *Unexpected default in an information based model*, arxiv (2016).

Our works are greatly inspired by:
- Information-based approach by Brody, Hughston, Macrina (2007): They model information about a random pay-off $D_T$ at some fixed (future) date $T > 0$ by...
\[ \xi_t := \beta_t^T + \alpha t D_T, \quad t \in [0, T], \]

where \( \beta^T = (\beta_t^T)_{t\in[0,T]} = (0 - 0-) \)-Brownian bridge over \([0, T]\), independent of \(D_T\), and \(\alpha > 0\).

This model considers the information on the cash-flow \(D_T\), but \(T\) is deterministic, not a default time.

- Models with reduced-form approach for credit risk (Bielecki/Rutkowski [2001], Jeanblanc/Le Cam [2009, 2010,..],..): information process \((H_t = I\{\tau \leq t\}, F^H)\).

- Our approach: brings both models together; makes the information process richer in information on \(\tau\); Objective: Study of \((\beta, F^\beta)\); characterisation of \(\tau\) is totally inaccessible, which makes it acceptable for default time modelling.
Preliminaries and Markov Property of \((\beta, \mathcal{F}^\beta)\)

\((\Omega, \mathcal{F}, P)\) - complete probability space; \(\mathcal{N}_P\) - the collection of the \(P\)-null sets;
\(W\) - 1-dim. BM over \((\Omega, \mathcal{F}, P)\);
For \(r \geq 0\): Brownian bridge over \([0, r]\): \(\beta^r_t := W_t - \frac{t}{r\sqrt{t}} W_{r\sqrt{t}}, \ t \geq 0\);
\(\beta^r\) is a Gaussian process:
Density of \(\beta^r_t\) \((0 < t < r)\): \(\varphi_t(r, x) := p\left(\frac{t(r-t)}{r}, x\right)\);
\(\varphi_t(r, x) = 0\ \ t \geq r \geq 0\), where \(p(t, .)\) is the density of \(\mathcal{N}(0, t)\).

Recall: \(B^r_t = \beta^r_t + \int_0^t \frac{\beta^r_s}{r-s} ds, \ t \geq 0\), is an \(\mathcal{F}^\beta\) -BM stopped at \(r\).

Letting \(\tau\) be a positive r.v. over \((\Omega, \mathcal{F}, P)\), we define the process \(\beta = (\beta_t)_{t \geq 0}\):
\(\beta_t(\omega) := \beta^{\tau(\omega)}_t(\omega), \ \omega \in \Omega, \ t \geq 0\).
\(\mathcal{F}^\beta = \left(\mathcal{F}_t^\beta := \sigma\{\beta_s, s \leq t\}\right)_{t \geq 0}, \ \mathcal{F}^\beta = \mathcal{F}^\beta \vee \mathcal{N}_P, \ \mathcal{F}_+^\beta = \left(\mathcal{F}^\beta_t := \cap_{s:s > t}\mathcal{F}^\beta_s\right)_{t \geq 0}\).
**Assumption:** $\tau$ and $W$ are independent.

**First results**

**Lemma.** For all $t > 0$, $\{\beta_t = 0\} = \{\tau \leq t\}$, $P$-a.s., i.e., $\tau$ is an $F^\beta$-s.t.

**Proof.** Clear: $\{\tau \leq t\} \subset \{\beta_t = 0\}$; On the other hand:

$$P\{\beta_t = 0, \tau > t\} = \int_{(t, +\infty)} P\{\beta_t = 0 | \tau = r\} P_\tau(dr)$$

$$= \int_{(t, +\infty)} P\{\beta_t^r = 0\} P_\tau(dr) = 0,$$

Consequently, $\{\beta_t = 0\} \setminus \{\tau \leq t\} \in \mathcal{N}_P$.

**Corollary.** $\tau$ is an $F^\beta$-stopping time.

**Proposition.** $(\beta, \hat{F}^\beta)$ is a Markov process, i.e., for all $f \in bB(R)$, $t, h \geq 0$,

$$E[f(\beta_{t+h}) | \hat{F}_t^\beta] = E[f(\beta_{t+h}) | \beta_t], \text{ P.a.s.}$$
Proof Let $f \in b\mathcal{B}(R), t, h \geq 0$. We have to show:

$E[f(\beta_{t+h})|\hat{\mathcal{F}}^\beta_t] = E[f(\beta_{t+h})|\beta_t], \text{ P.a.s.}$

- On $\{\tau \leq t\}, \text{ P-a.s.}$

$E[f(\beta_{t+h})|\hat{\mathcal{F}}^\beta_t]I\{\tau \leq t\} = E[f(0)I\{\tau \leq t\}|\hat{\mathcal{F}}^\beta_t] = f(0)I\{\tau \leq t\} = f(0)I\{\beta_t = 0\} \in \sigma\{\beta_t\}.$

- Still to show:

$E[f(\beta_{t+h})|\hat{\mathcal{F}}^\beta_t]I\{\tau > t\} = E[f(\beta_{t+h})|\beta_t]I\{\tau > t\}, \text{ P.a.s.,}$

or equivalently: for all $\xi \in b\hat{\mathcal{F}}^\beta_t$,

$E[f(\beta_{t+h})\xi I\{\tau > t\} = E[E[f(\beta_{t+h})|\beta_t]\xi I\{\tau > t\}].$

But, as

$\hat{\mathcal{F}}^\beta_t = \sigma\left\{\beta_{t_n}, \xi_k := \frac{\beta_{t_k}}{t_k} - \frac{\beta_{t_{k-1}}}{t_{k-1}}, 1 \leq k \leq n, 0 < t_1 < ... < t_n = t, n \geq 1\right\},$

it is sufficient to prove that, for all $g \in b\mathcal{B}(R), h \in b\mathcal{B}(n), \text{ P.a.s.}$
\[ E \left[ f(\beta_{t+h})g(\beta_t)h(\xi_1, \ldots, \xi_n)I\{\tau > t\} \right] = E \left[ E[f(\beta_{t+h})|\beta_t]g(\beta_t)h(\xi_1, \ldots, \xi_n)I\{\tau > t\} \right]. \]

But, for \( \eta_k := \frac{W_{t_k}}{t_k} - \frac{W_{t_{k-1}}}{t_{k-1}} \), and recalling that \( \xi_k := \frac{\beta_{t_k}}{t_k} - \frac{\beta_{t_{k-1}}}{t_{k-1}} \),
on \{\tau > t\}: \xi_k = \eta_k, \ 1 \leq k \leq n.

Consequently,

\[ E \left[ f(\beta_{t+h})g(\beta_t)h(\xi_1, \ldots, \xi_n)I\{\tau > t\} \right] = E \left[ f(\beta_{t+h})g(\beta_t)h(\eta_1, \ldots, \eta_n)I\{\tau > t\} \right] \]

\[ = \int_{(t, +\infty)} E \left[ f(\beta_{t+h}^r)g(\beta_t^r)h(\eta_1, \ldots, \eta_n)|\tau = r\right] P_\tau(dr) \]

\[ = \int_{(t, +\infty)} E \left[ f(\beta_{t+h}^r)g(\beta_t^r)h(\eta_1, \ldots, \eta_n)\right] P_\tau(dr). \]

But \((\beta_{t+h}^r, \beta_t^r, \eta_1, \ldots, \eta_n)\) is Gaussian, and \((\eta_1, \ldots, \eta_n), (\beta_t^r, \beta_{t+h}^r)\) are non correlated. Thus, \((\eta_1, \ldots, \eta_n)\) is independent of \((\beta_t^r, \beta_{t+h}^r)\) and
\[
E \left[ f(\beta_{t+h})g(\beta_t)h(\xi_1, \ldots, \xi_n)I\{\tau > t}\right] \\
= \int_{(t, +\infty)} E \left[ f(\beta_{t+h})g(\beta_t)h(\eta_1, \ldots, \eta_n)\right] P_\tau(dr) \\
= \int_{(t, +\infty)} E \left[ f(\beta_{t+h})g(\beta_t)\right] P_\tau(dr) \cdot E \left[ h(\eta_1, \ldots, \eta_n)\right] \\
= E \left[ f(\beta_{t+h})g(\beta_t)I\{\tau > t}\right] E \left[ h(\eta_1, \ldots, \eta_n)\right] \\
= E \left[ f(\beta_{t+h})g(\beta_t)I\{\beta_t \neq 0\}\right] E \left[ h(\eta_1, \ldots, \eta_n)\right] \\
= E \left[ E \left[ f(\beta_{t+h})|\beta_t\right] g(\beta_t)I\{\beta_t \neq 0\}\right] E \left[ h(\eta_1, \ldots, \eta_n)\right] \\
= E \left[ E \left[ f(\beta_{t+h})|\beta_t\right] g(\beta_t)I\{\tau > t\}\right] E \left[ h(\eta_1, \ldots, \eta_n)\right].
\]

As \( E \left[ f(\beta_{t+h})|\beta_t\right] g(\beta_t) \) is a bounded Borel function of \( \beta_t \), the above result also gives:
\[
E \left[ E \left[ f(\beta_{t+h})|\beta_t\right] g(\beta_t)I\{\tau > t\}\right] E \left[ h(\eta_1, \ldots, \eta_n)\right] \\
E \left[ E \left[ f(\beta_{t+h})|\beta_t\right] g(\beta_t)h(\xi_1, \ldots, \xi_n)I\{\tau > t}\right].
\]
Bayes estimates for the default time $\tau$

We have seen: $(\beta, \hat{F}^\beta)$ and, thus, $(\beta, F^\beta)$ are Markov processes. An immediate consequence of this and the Bayes formula:

Bayes prior probability for $\tau = \text{conditional law of the unknown } \tau \text{ based on the observation of the information process } \beta \text{ until time } t$:

**Proposition.**

\[
P\{\tau \leq u | \mathcal{F}_t^\beta\} = I\{\tau \leq t \land u\} + P\{t < \tau \leq u | \beta_t\}I\{t < \tau\}
\]

\[
= I\{\tau \leq t \land u\} + \int_{(t,u]} \phi_t(r, \beta_t) P_\tau(dr) I\{t < \tau\},
\]

with $\phi_t(r, x) := \frac{\varphi_t(r, x)}{\int_{(t, +\infty)} \varphi_t(v, x) P_\tau(dv)}$, $(r, t) \in R_+^2$, $x \in R$. 

Rainer Buckdahn, Université de Bretagne Occidentale, Brest, France/ Shandong University, Jinan, P.R. China
Proof. As \( \{\tau \leq t \land u\} \in \mathcal{F}_t^\beta \) and
\[
I\{t < \tau \leq u\} = I\{\beta_t \neq 0, \beta_u = 0\} \in \sigma\{\beta_v, v \geq t\}, \ \text{P-a.s.},
\]
\[
P\{\tau \leq u | \mathcal{F}_t^\beta\} = E\left[I\{\tau \leq t \land u\} + I\{t < \tau \leq u\} | \mathcal{F}_t^\beta\right]
\]
\[
= I\{\tau \leq t \land u\} + P\{t < \tau \leq u | \beta_t\}I\{t < \tau\}. 
\]

Computation of \( P\{t < \tau \leq u | \beta_t\}I\{t < \tau\} \) with help of Bayes formula.

Recall (e.g., Shiryaev, Probability, 2nd edition): Let \( \tau : (\Omega, \mathcal{F}, P) \to (E_1, \mathcal{E}_1) \), \( X : (\Omega, \mathcal{F}, P) \to (E_2, \mathcal{E}_2) \) be r.v., \( P_r \) a regular conditional distribution of \( X \) knowing \( \tau = r \), i.e.,
\[
P_r(B) = P\{X \in B | \tau = r\}, \ P_\tau(dr)-\text{a.s.}, \text{ for all } B \in \mathcal{E}_2,
\]
with a priori distribution of \( \tau \): \( P_\tau = P \circ [\tau]^{-1} \).

Let \( G_C(B) := \int_C P_r(B)P_\tau(dr)(= P\{X \in B, \tau \in C\}), \ B \in \mathcal{E}_2, C \in \mathcal{E}_1. \)
Bayes formula: A posteriori probability: \( P\{\tau \in C|X = x\} = \frac{G_C(dx)}{P_X(dx)} , P_X(dx)\)-a.s.

If, moreover, there exists a \( \sigma \)-finite measure \( \mu \) on \((E_2, \mathcal{E}_2)\) with \( P_r \ll \mu \), for all \( r \in E_1 \), and a measurable function \( p : (E_1 \times E_2, \mathcal{E}_1 \otimes \mathcal{E}_2) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R})) \) with
\[
p(r, x) = \frac{P_r(dx)}{\mu(dx)} , \mu(dx)\)-a.e. on \( E_2 \), for all \( r \in E_1 \),
\]
then:
\[
+ \ G_C \ll \mu , \ \frac{G_C(dx)}{\mu(dx)} = \int_C p(r, x)P_\tau(dr) , \mu(dx)\)-a.e.;
\]
\[
+ \ P_X \ll \mu , \ \frac{P_X(dx)}{\mu(dx)} = \int_{E_1} p(r, x)P_\tau(dr) , \mu(dx)\)-a.s., and thus:
\]
\[
+ \ P\{\tau \in C|X = x\} = \frac{\int_C p(r, x)P_\tau(dr)}{\int_{E_1} p(r, x)P_\tau(dr)} , P_X(dx)\)-a.s., \( C \in \mathcal{E}_1 \).
For our proposition we have to choose:

+ \((E_1, \mathcal{E}_1) = (R_+, \mathcal{B}(R_+))\), \(\tau\) - the default time;
+ \((E_2, \mathcal{E}_2) = (R, \mathcal{B}(R))\), \(X = \beta_t\),

+ \(\mu(dx) = \delta_0(dx) + \lambda(dx)\), with \(\lambda\) - Borel measure on \(R\);

Note: With this choice,

\[
P_r\{\beta_t \in B\} = P\{\beta^\tau_t \in B\} = \int_B \varphi_t(r, x)\mu(dx), \ B \in \mathcal{B}(R),
\]

where: \(\varphi_t(r, .)\) - density of \(\beta^\tau_t\), for \(0 < t < r\) (with \(\varphi_t(r, 0) := 0\), and \(\varphi_t(r, \cdot) = I_{\{0\}}\), otherwise.

Substituting this in the above Bayes formula proves the proposition.

**Extension of this proposition:** Then we can show:

**Proposition.** For all \(g \in b\mathcal{B}(R_+ \times R)\), \(0 < t < u\), \(P\)-a.s.,
\[ E[g(\tau, \beta_u)|\mathcal{F}_t^\beta] \]

\[ = g(\tau, 0)I\{\tau \leq t\} + \int_{(t,u]} g(r, 0)\phi_t(r, \beta_t)P_\tau(dr)I\{t < \tau\} \]

\[ + \int_{(u, +\infty)} \int_R g(r, y)\varphi_{u-t}\left(r - t, y - \frac{r - u}{r - t}\beta_t\right)dy\phi_t(r, \beta_t)P_\tau(dr)I\{t < \tau\}. \]

**Proof.**

\[ + \text{ As } \tau \text{ is an } \mathbf{F}^\beta\text{-stopping time and } \beta_u = 0 \text{ on } \{\tau \leq t < u\}: \]

\[ E[g(\tau, \beta_u)|\mathcal{F}_t^\beta]I\{\tau \leq t\} = g(\tau, 0)I\{\tau \leq t\}, \text{ } P\text{-a.s.} \]

\[ + \text{ As } g(\tau, 0)I\{t < \tau \leq u\} \in \sigma\{\beta_v, v \geq t\} \vee \mathcal{N}_P \text{ and } (\beta, \mathbf{F}^\beta) \text{ is a Markov process,} \]

\[ E[g(\tau, \beta_u)I\{\tau \leq u\}|\mathcal{F}_t^\beta]I\{\tau > t\} = E[g(\tau, 0)I\{t < \tau \leq u\}|\mathcal{F}_t^\beta]I\{\tau > t\} \]

\[ = E[g(\tau, 0)I\{t < \tau \leq u\}|\beta_t]I\{\tau > t\} \]

\[ = \int_{(t,u]} g(r, 0)\phi_t(r, \beta_t)P_\tau(dr)I\{t < \tau\} \quad \text{Bayes formula (proposition above);} \]
As \( g(\tau, \beta_u)I\{t < u < \tau\} \in \sigma\{\beta_v, v \geq t\} \vee \mathcal{N}_P \) and 

\((\beta, F^\beta)\) is a Markov process

\[
E[g(\tau, \beta_u)I\{\tau > u\}|F_t^\beta]I\{\tau > t\} = E[g(\tau, \beta_u)I\{\tau > u\}|\beta_t]I\{\tau > t\}
\]

\[
= E\left[ E\left[ g(r, \beta^r_u)I\{r > u\}|\beta^r_t|r=\tau \right]|\beta_t\right]I\{\tau > t\}
\]

where the latter relation follows from the fact that

\[
E \left[ (g(\tau, \beta_u)I\{\tau > t\}) h(\beta_t) \right] = \int_{(t, +\infty)} E\left[ g(r, \beta^r_u)h(\beta^r_t) \right] P_\tau(dr)
\]

\[
= \int_{(t, +\infty)} E\left[ E\left[ g(r, \beta^r_u)|\beta^r_t\right]|\beta_t\right] h(\beta_t)I\{\tau > t\}
\]

for all \( h \in bB(R) \), i.e.,

\[
E \left[ g(\tau, \beta_u)|\beta_t\right] I\{\tau > t\} = E \left[ E\left[ g(r, \beta^r_u)|\beta^r_t\right]|r=\tau \right]|\beta_t\right] I\{\tau > t\}.
\]
Finally, from some computation with the Bayes formula we obtain:

\[
E[g(\tau, \beta_u)I\{\tau > u\}|\mathcal{F}_t^\beta]I\{\tau > t\} = E[g(\tau, \beta_u)I\{\tau > u\}|\beta_t]I\{\tau > t\}
\]

\[
= \int_{(u, +\infty)} \int_{\mathbb{R}} g(r, y) \varphi_{u-t} \left( r - t, y - \frac{r - u}{r - t} \beta_t \right) dy \phi_t(r, \beta_t) P_\tau(dr) I\{t < \tau\}.
\]

**Markov property**

We have already seen the Markov property of \((\beta, \hat{\mathcal{F}}^\beta)\) and, thus, that of \((\beta, \mathcal{F}^\beta = \hat{\mathcal{F}}^\beta \lor \mathcal{N}_P)\). With a rather technical proof we obtain also:

**Theorem.** \((\beta, \mathcal{F}^\beta_+)\) is a Markov process.

**Corollary.** \(\mathcal{F}^\beta_+ = \mathcal{F}^\beta\), i.e., the filtration \(\mathcal{F}^\beta\) satisfies the usual conditions of completeness and right-continuity.

**Proof** (of the Corollary): Classical result, see e.g., Blumenthal, Getoor (1968).
Sketch of the proof (of the Theorem):

To show: For all \( g \in C_b(R), u > t \geq 0 \),
\[
E \left[ g(\beta_u) | \mathcal{F}_{t+}^\beta \right] = E \left[ g(\beta_u) | \beta_t \right], \text{ P-a.s.}
\]

Case 1: \( t > 0 \)

Let \( 0 < t < \cdots < t_{n+1} < t_n < \cdots < t_0 = u, \ t_n \downarrow t (n \nearrow +\infty) \). Then, P-a.s.,
\[
E \left[ g(\beta_u) | \mathcal{F}_{t+}^\beta \right] = \lim_{n \to +\infty} E \left[ g(\beta_u) | \mathcal{F}_{t_n}^\beta \right].
\]

On the other hand, from the Proposition concerning the Bayes formula, as \( t < t_n < u, \)
\[
E \left[ g(\beta_u) | \mathcal{F}_{t_n}^\beta \right] = E \left[ g(\beta_u) | \beta_{t_n} \right]
\]
\[
= g(0) I_{\{\tau \leq t_n\}} + g(0) \int_{(t_n, u]} \phi_{t_n}(r, \beta_{t_n}) P_\tau (dr) I_{\{t_n < \tau\}}
\]
\[
+ \int_{(u, +\infty)} G_{t_n, u}(r, \beta_{t_n}) \phi_{t_n}(r, \beta_{t_n}) P_\tau (dr) I_{\{t_n < \tau\}},
\]
where
\[ G_{t,u}(r,x) : = \int_{R} g(y) p \left( \frac{r-u}{r-t}(u-t), y - \frac{r-u}{r-t}x \right) dy \]

\( = E \left[ g(r, \beta^r_u) \mid \beta^r_t = x \right] \).

Hence, we shall prove that, \( P \)-a.s.,
\[ \left( \begin{array}{c} \left( E \left[ g(\beta_u) \mid \mathcal{F}_{t+}^\beta \right] = \lim_{n \to +\infty} E \left[ g(\beta_u) \mid \mathcal{F}_{t_n}^\beta \right] = \right) \\
\lim_{n \to +\infty} E \left[ g(\beta_u) \mid \beta_{t_n} \right] = E \left[ g(\beta_u) \mid \beta_t \right].\right. \]

From above we see that this latter relation holds true if the following two identities are satisfied, \( P \)-a.s. on \( \{ t < \tau \} \):
\[ \lim_{n \to \infty} \int_{(t_n,u]} \phi_{t_n}(r, \beta_{t_n}) P_\tau(dr) = \int_{(t,u]} \phi_t(r, \beta_t) P_\tau(dr) \]

and
\[
\lim_{n \to \infty} \int_{(u, +\infty)} G_{t_n, u}(r, \beta_{t_n}) \phi_{t_n}(r, \beta_{t_n}) P_{\tau}(dr) \]

\[
= \int_{(u, +\infty)} G_{t, u}(r, \beta_{t}) \phi_{t}(r, \beta_{t}) P_{\tau}(dr)
\]

Proof of these both convergences: rather technical; one shows that Lebesgue’s bounded convergence theorem can be applied \(\omega\)-wise. For the proof it is essential that \(t > 0\).

Case 2: \(t = 0\).

Case 2.1: There exists \(\varepsilon > 0\) such \(P\{\tau > \varepsilon\} = 1\).

Let \(0 < \ldots < t_{n+1} < t_n < \ldots < t_1 < \varepsilon : t_n \searrow 0\), as \(n \to \infty\). We have to show

\[
\left( E \left[ g(\beta_u) | \mathcal{F}_{0+}^{\beta} \right] = \lim_{n \to +\infty} E \left[ g(\beta_u) | \mathcal{F}_{t_n}^{\beta} \right] = \right)
\]

\[
\lim_{n \to +\infty} E \left[ g(\beta_u) | \beta_{t_n} \right] = E \left[ g(\beta_u) | \beta_0 \right] = E \left[ g(\beta_u) \right], \text{ i.e.,}
\]
\[
E[g(\beta_u) | \beta_{t_n}] = g(0) I_{\{\tau \leq t_n\}} + g(0) \int_{(t_n,u]} \phi_{t_n}(r, \beta_{t_n}) P_\tau(dr) I_{\{t_n < \tau\}} \\
+ \int_{(u, +\infty)} G_{t_n,u}(r, \beta_{t_n}) \phi_{t_n}(r, \beta_{t_n}) P_\tau(dr) I_{\{t_n < \tau\}}.
\]

\[
\rightarrow E[g(\beta_u)] = g(0) P\{\tau \leq u\} + \int_{(u, +\infty)} \int_{\mathbb{R}} g(y) p\left(\frac{r - u}{r}u, y, 0\right) dy P_\tau(dr).
\]

+ For the first term: \( g(0) I_{\{\tau \leq t_n\}} \rightarrow 0, \text{ as } t_n \searrow 0; \)

+ For the second term: Note that

\[
\phi_{t_n}(r, \beta_{t_n}) = \frac{\varphi_{t_n}(r, \beta_{t_n})}{\int_{(t_n, +\infty)} \varphi_{t_n}(v, \beta_{t_n}) P_\tau(dv)} I_{\{t_n < r\}}
\]

\[
= \frac{1}{\sqrt{2\pi t_n}} \sqrt{\frac{r}{r-t_n}} \exp\left\{-\frac{\beta_{t_n}^2 r}{2t_n(r-t_n)}\right\} I\{t_n < r\}.
\]

Rainer Buckdahn, Université de Bretagne Occidentale, Brest, France/ Shandong University, Jinan, P.R. China
With some subtle estimates the following can be shown:

**Auxiliary result:** Suppose that $P(\tau > \varepsilon) = 1$. Then the function $r \mapsto \phi_{t_n}(r, \beta_{t_n})$ is $P_\tau$-a.s. uniformly bounded by some constant $K = K(\varepsilon, \omega) < +\infty$ and, for all $r > 0$, \(\lim_{n \to \infty} \phi_{t_n}(r, \beta_{t_n}) = 1, \ P\text{-a.s.}\)

Consequently:

\[
g(0) \int_{(t_n,u]} \phi_{t_n}(r, \beta_{t_n}) P_\tau (dr) I_{\{t_n < \tau\}} \to g(0) P\{\tau \leq u\}, \text{ as } n \nearrow +\infty.
\]

For the third term: One uses the above auxiliary result for $\phi_{t_n}(r, \beta_{t_n})$ and shows that $s \mapsto G_{s,u}(r, \beta_s(\omega))$ is $\omega$-wise bounded and continuous. This gives the wished limit.

**Case 2.2: General case $\tau > 0$:** Let $\varepsilon > 0$:

+ $\varepsilon\beta = (\varepsilon\beta_t = (\beta^r_t)_{r=\tau \lor \varepsilon}, \ t \geq 0)$;
+ $\hat{F}^\varepsilon = (\hat{F}^\varepsilon_t)_{t \geq 0}$, where $\hat{F}^\varepsilon_t := \sigma(\varepsilon\beta_s, \ 0 \leq s \leq t)$.
Unexpected Default in Information based Model

Obviously: For proving that \((\beta, F_{\beta})\) is Markov at \(t = 0\) it is sufficient to show that \(F_{\beta}^{\beta} = \hat{F}_{0+}^{0} \lor \mathcal{N}_P\) is \(P\)-trivial. For this we consider a set \(A \in \hat{F}_{0+}^{0}\) and we show that if \(P(A) > 0\), then \(P(A) = 1\).

- If \(P(A) > 0\), then \(\exists \varepsilon > 0\) s.t. \(P(A \cap \{\tau > \varepsilon\}) > 0\).
- As \(A \in \hat{F}_{0+}^{0} \subset \hat{F}_u^{0}, u > 0\): \(A \cap \{\tau > \varepsilon\} \in \hat{F}_u^{0}|_{\{\tau > \varepsilon\}} \lor \mathcal{N}_P\),
  
  where \(\hat{F}_u^{0}|_{\{\tau > \varepsilon\}} := \left\{ B \cap \{\tau > \varepsilon\}; \ B \in \hat{F}_u^{0} \right\}\).
- On the other hand: on \(\{\tau > \varepsilon\}\), \(\beta_t = \varepsilon \beta_t, t \geq 0\), i.e.,
  \[A \cap \{\tau > \varepsilon\} \in \hat{F}_u^{\varepsilon}|_{\{\tau > \varepsilon\}} \lor \mathcal{N}_P = \hat{F}_u^{\varepsilon}|_{\{\tau > \varepsilon\}} \lor \mathcal{N}_P.\]
  
  \[\rightarrow \exists A_n \in \hat{F}_{1/n}^{\varepsilon}\) s.t. \(A \cap \{\tau > \varepsilon\} = A_n \cap \{\tau > \varepsilon\}, P\)-a.s., \(n \geq 1\).
- Then \(A_0 := \limsup_{n \to \infty} A_{1/n} \in \hat{F}_{0+}^{\varepsilon}\), and \(A \cap \{\tau > \varepsilon\} = A_0 \cap \{\tau > \varepsilon\}, P\)-a.s.

However, from Case 2.1: \(\hat{F}_{0+}^{\varepsilon}\) is \(P\)-trivial. Thus, \(P(A_0) \in \{0, 1\}\). But,
\[ P(A_0) \geq P(A_0 \cap \{\tau > \varepsilon\}) = P(A \cap \{\tau > \varepsilon\}) > 0. \]

- Hence, \( P(A_0) = 1 \), and \( A \cap \{\tau > \varepsilon\} = A_0 \cap \{\tau > \varepsilon\} = \{\tau > \varepsilon\}, \) \( P\)-a.s., i.e.,

\[ A \supset \{\tau > \varepsilon\} \nearrow \{\tau > 0\} = \Omega, \] \( P\)-a.s.

Consequently, \( P(A) = 1 \), i.e., \( \hat{\mathcal{F}}_{0+}^0 \) is trivial.

This ends the proof.
Semimartingale decomposition of \((\beta, F^\beta)\)

After having studied \((\beta, F^\beta)\) as a Markov process, we characterise it now as a semimartingale.

Theorem. The process

\[
B_t := \beta_t + \int_0^t E \left[ \frac{\beta_s}{\tau - s} I\{s < \tau\} \right] ds
\]

\[
= \beta_t + \int_0^t \beta_s \int_{(s, +\infty)} \frac{1}{r - s} \phi_s(r, \beta_s) P_\tau(dr) I\{s < \tau\} ds, \text{ P-a.s.}
\]

is an \(F^\beta\)-BM stopped at \(\tau\). Thus, \((\beta, F^\beta)\) is a semimartingale with the above semimartingale decomposition.

The key for the proof is a slight extension of a well known result from filtering theory. For this first a recall:
Recall (Optional projection):

**Proposition.** Let $Z \geq 0$ be a measurable process and $\mathcal{G}$ be a filtration satisfying the usual conditions. Then there exists a unique (up to indistinguishability) $\mathcal{G}$-optional process $^oZ$ s.t.

$$E[Z_T I\{T < +\infty\}|\mathcal{G}_T] = ^oZ_T I\{T < +\infty\}, \text{ P-a.s., for all } \mathcal{G}\text{-s.t. } T.$$ 

+ $^oZ := \mathcal{G}$-optional projection of $Z$;
+ For general $Z$ measurable process:

$$^oZ := \begin{cases} ^o(Z^+) - ^o(Z^-), & ^o(Z^+) \wedge ^o(Z^-) < +\infty, \\ +\infty, & \text{otherwise.} \end{cases}$$

We can now formulate the Innovation Lemma:
**Innovation Lemma** (e.g. Roger, LC.G., Williams, D., Vol.2, 2000): Let be

+ $\mathcal{G}$ - a filtration satisfying the usual conditions,
+ $T$ a $\mathcal{G}$-stopping time,
+ $B$ a $\mathcal{G}$-BM stopped at $T$ (i.e., $\mathcal{G}$-martingale with $<B>_t = t \land T$, $t \geq 0$);
+ $Z = (Z_t)_{t \geq 0}$ an $\mathcal{G}$-optional process s.t. $E \left[ \int_0^t |Z_s| ds \right] < +\infty$;
+ $X_t := B_t + \int_0^t Z_s ds$, $t \geq 0$;
+ $^oZ = (^oZ_t)_{t \geq 0}$ the optional projection of $Z$ w.r.t. $\mathcal{F}^X = (\mathcal{F}_t^X)_{t \geq 0}$.

Then

$$b_t = X_t - \int_0^t ^oZ_s ds, \ t \geq 0,$$

is an $\mathcal{F}^X$-BM stopped at $T$. 

Rainer Buckdahn, Université de Bretagne Occidentale, Brest, France/ Shandong University, Jinan, P.R. China

27/39
Application of the Innovation Lemma to our situation:

\[ G_t := \mathcal{F}_t^\beta \vee \sigma\{\tau\} \quad (t \geq 0); \]

\[ Z_t := \frac{\beta_t}{\tau - t} I\{t < \tau\}, \quad t \geq 0, \text{ is } G\text{-optional (since càd and } G\text{-adapted);} \]

it can be shown: \( E \left[ \int_0^t |Z_s|ds \right] < +\infty, \) and

\[ B_t := \beta_t + \int_0^t Z_s ds, \quad t \geq 0, \text{ is a } G\text{-BM stopped at } \tau. \]

+ Computation of the \( F^\beta \)-optional projection \( oZ \) of \( Z \):

Obviously, \( E \left[ |Z_t| \right] < +\infty, \ kcal\quad dt\text{-a.e.}, \) and for all such \( t \geq 0, \) using the Bayes formula

\[
oZ_t = E \left[ Z_t | \mathcal{F}_t^\beta \right] = E \left[ \frac{\beta_t}{\tau - t} I\{\tau > t\} | \beta_t \right] \\
= \beta_t \int_{(t, +\infty)} \frac{1}{r - t} \phi_t(r, \beta_t) P_\tau(dr) I\{\tau > t\}
\]

The right-hand side is well defined for all \( t \geq 0, \) and it can be shown that the right-hand side is really the \( F^\beta \)-optional projection \( oZ \) of \( Z. \)
Compensator of the default time $\tau$ w.r.t. $F^\beta$

The default process in our model: $H_t = I\{\tau \leq t\}, \ t \geq 0$
(equals to 0 before default, and to 1 after default).

$(H = (H_t)_{t \geq 0}, F^\beta)$ is an adapted increasing càdlàg process; we’ll compute its compensator in order to categorise the stopping time $\tau$.

Recall: The compensator $K$ of $(H, F^\beta)$ is the dual predictable projection of $H$, i.e., the unique $F^\beta$-predictable increasing càdlàg process with $K_0^- = 0$ such that $(H - K, F^\beta)$ is a martingale.

Assumption: $P_\tau$ admits a continuous density function $f$ w.r.t. the Lebesgue on $R_+$.

Our main result is as follows
Theorem. The compensator $K$ of the default process $(H, F^\beta)$ is given by

$$K_t = \int_0^{t \wedge \tau} \frac{f(s)}{\int_s^{+\infty} \varphi_s(v,0)f(v)dv} dL^\beta(s,0), \ t \geq 0. $$

Here $L^\beta(t, x)$ denotes the local time of $\beta$ up to time $t$ at level $x$.

Corollary. The default time $\tau$ is in the filtration $F^\beta$ totally inaccessible (i.e., doesn’t coincide with positive probability with an $F^\beta$-predictable stopping time).

Remark. Knowing whether a default time is predictable, accessible or totally inaccessible of great importance in mathematical credit risk models. 

*Predictable default times* are typical of structural credit risk models: the default can be foreseen; 

* Totally inaccessible default times* which occur by surprise are considered by reduced-form models.
Remarkable here in our model: Under the common assumption that $\tau$ has a continuous density, in spite of the rather rich information process $\beta$, the default time $\tau$ remains totally inaccessible. Recall, in the standard approach the default is modelled by $(H, F^H)$, i.e., people just know if the default has occurred or not.

Remaining part of the talk:
Sketch of the proof of the main result

We begin with some recalls:

1) The (right) local time of the continuous semimartingale \((\beta, F^\beta)\), defined through Tanaka’s formula:

\[
L^\beta(t, x) = |\beta_t - x| - |\beta_0 - x| - \int_0^t \text{sign}(\beta_s - x) d\beta_s, \ t \geq 0,
\]

where \text{sign}(x) = -1, if \(x \leq 0\) and = 1, if \(x > 0\).

Well known: There exists a modification of \(L^\beta(t, x), \ t \geq 0, x \in \mathbb{R}\), s.t. \((t, x) \rightarrow L^\beta(t, x)\) continuous in \(t\) and càdlàg in \(x\).

Lemma: Existence of a modification s.t. \((t, x) \rightarrow L^\beta(t, x)\) jointly continuous.

In particular: \((L^\beta(t, 0))_{t \geq 0}\) is a continuous increasing process. Hence, the compensator \(K\) given in the Theorem is continuous, and \(\tau\) is totally inaccessible stopping time.
2) Sketch of the computation of the compensator $K$:

We follow the so-called Laplacian approach by P.-A. Meyer (1966):

\[ L_t = E[C\infty|\mathcal{F}_t], \quad t \geq 0; \]

Potential generated by $C$: \( X_t := L_t - C_t, \quad t \geq 0 \). Suppose that $X \in (D)$.

**Notation:** For $h > 0$:

- \( p_hX = (p_hX_t, t \geq 0) \) is the càd modification of the supermartingale \( p_hX_t = E[X_{t+h}|\mathcal{F}_t], \quad t, h \geq 0; \)

- \( A^h_t := \frac{1}{h} \int_0^t (X_s - p_hX_s) ds, \quad s \geq 0 \) (integrable increasing process).

**Theorem** (P.-A. Meyer, 1966): There exists a unique integrable $\mathcal{F}$-predictable increasing process $A$ generating the potential $X$. For every s.t. $\eta$ it holds

\[ A^h_\eta \rightarrow A_\eta \text{ in } \sigma(L^1, L^\infty), \quad \text{as } h \downarrow 0. \]
Here, for us: $\mathbf{F} = \mathbf{F}_\beta^\beta$, $C_t = H_t = I\{\tau \leq t\}, \ t \geq 0$; $C_\infty = H_\infty = 1$;
potential generated by $H$: $X_t = 1 - H_t = I\{\tau > t\}, \ t \geq 1$;
integrable increasing process: Thanks to our Bayes formula for the prior probability of default, for $0 < t_0 < t < +\infty$,

$$K_t^h - K_{t_0}^h = \int_{t_0}^t \left( I\{s < \tau\} - E[I\{s + h < \tau\}|\mathcal{F}_s^\beta] \right) ds$$

$$= \int_{t_0}^t \frac{1}{h} P\{s < \tau \leq s + h|\mathcal{F}_s\} ds$$

$$= \int_{t_0}^{t \wedge \tau} \frac{1}{h} \left( \int_s^{s + h} \varphi_s(r, \beta_s)f(r)dr \right) ds$$

$$= \int_{t_0 \wedge \tau}^{t \wedge \tau} \frac{1}{h} \left( \int_s^{+\infty} \varphi_s(v, \beta_s)f(v)dv \right) ds$$

$$=: I_{t_0,t}^h.$$ We have to study $I_{t_0,t}^h$, as $h \downarrow 0$.

Put: $g(s, x) := \left( \int_s^{+\infty} \varphi_s(v, \beta_s)f(v)dv \right)^{-1}$, $s > 0, x \in \mathbb{R}$. 

---

Rainer Buckdahn, Université de Bretagne Occidentale, Brest, France/ Shandong University, Jinan, P.R. China | 34/39
To simplify $I_{t_0,t}^h = \int_{t_0 \wedge \tau}^{t \wedge \tau} \frac{1}{h} \left( \int_s^{s+h} \varphi_s(r, \beta_s) f(r) dr \cdot g(s, \beta_s) \right) ds$:

**Lemma 1:**

$$\int_{t_0 \wedge \tau}^{t \wedge \tau} \frac{1}{h} \left( \int_s^{s+h} \varphi_s(r, \beta_s) (f(r) - f(s)) dr \cdot g(s, \beta_s) \right) ds \xrightarrow{h \downarrow 0} 0, \text{ P-a.s.}$$

Note:

$$\int_{t_0 \wedge \tau}^{t \wedge \tau} \frac{1}{h} \left( \int_s^{s+h} \varphi_s(r, \beta_s) dr \cdot g(s, \beta_s) f(s) \right) ds$$

$$= \int_{t_0 \wedge \tau}^{t \wedge \tau} \frac{1}{h} \left( \int_0^h p \left( \frac{su}{s+u}, \beta_s \right) \cdot g(s, \beta_s) f(s) \right) ds$$

**Lemma 2:**

$$\int_{t_0 \wedge \tau}^{t \wedge \tau} \frac{1}{h} \int_0^h \left| p \left( \frac{su}{s+u}, \beta_s \right) - p(u, \beta_s) \right| du \cdot g(s, \beta_s) f(s) ds \xrightarrow{h \downarrow 0} 0, \text{ P-a.s.}$$
Hence, with the notation \( q(h, x) := \frac{1}{h} \int_0^h p(u, x) du, \ 0 < h \leq 1, \ x \in R, \)

\[
\lim_{h \downarrow 0} (K_t^h - K_{t_0}^h) = \lim_{h \downarrow 0} I_{t_0, t}^h = \lim_{h \downarrow 0} \int_{t_0}^{t \wedge \tau} q(h, \beta_s) \cdot g(s, \beta_s)f(s)ds,
\]

and from the occupation time formula

\[
\int_{t_0}^{t \wedge \tau} q(h, \beta_s) \cdot g(s, \beta_s)f(s)ds
\]

\[
= \int_{R} \left( \int_{t_0}^{t} g(s, x)f(s)dL^\beta(s, x) \right) q(h, x)dx, \ P\text{-a.s.}
\]

Now using that:

+ \( q(h, x)dx \rightarrow \delta_0(dx) \) weak convergence of probability measures;
+ It can be shown: There is a modification of \( L^\beta \) such \( x \rightarrow \int_{t_0}^{t} g(s, x)f(s)dL^\beta(s, x) \) is continuous and bounded; the bound may depend on \((t, \omega)\),

we obtain:
\[ \lim_{h \downarrow 0} \left( K^h_t - K^h_{t_0} \right) = \lim_{h \downarrow 0} \int_R \left( \int_{t_0}^t g(s, x) f(s) dL^\beta(s, x) \right) q(h, x) dx \]

\[ = \int_{t_0}^t g(s, 0) f(s) dL^\beta(s, 0) \]

\[ = K_t - K_{t_0}, \ P\text{-a.s.}, \]

where the increasing process \( K \) is defined in the Theorem.

It remains to identify \( K \) with the compensator \( pH \) of the default process \( H \). By P.-A.Meyer (1966) we know that, for all \( F^\beta \)-stopping times \( \eta \):

\[ K^h_{\tau} \xrightarrow{\sigma(L^1, L^\infty)} h \downarrow 0 \]

\[ pH_{\tau}. \]

For this we note that, for \( t > t_0 > 0, h_n \downarrow 0 \) as \( n \uparrow +\infty \):

+ From the Compactness criterion of Dunford-Pettis: As \( (K^h_{t_n} - K^h_{t_0})_{n \geq 1} \) is relatively compact in the weak topology \( \sigma(L^1, L^\infty) \), it is uniformly integrable.
+ Thus, the $P$-a.s. convergence of $(K_t^{h_n} - K_{t_0})_{n \geq 1}$ implies its $L^1$-convergence and, hence, also its convergence in $\sigma(L^1, L^\infty)$ to $K_t - K_{t_0}$.

+ The uniqueness of the limit in the weak topology $\sigma(L^1, L^\infty)$ implies that

$$K_t - K_{t_0} = p H_t - p H_{t_0}, \text{ for all } t > t_0 > 0.$$ 

As $K_{0+} = 0 = p H_{0+}$, the result follows.
Thank you very much for your attention.