Week 5: More Calculus, Integration

Solutions

1. In each case, find any values of x for which $\frac{dy}{dx} = 0$

   $y = x^2 + 6x$

   Solution: $x = -3$

   $y = 4x^2 + 2x + 1$

   Solution: $x = -\frac{1}{4}$

   $y = x^3 - 12x$

   Solution: $x = \pm 2$

   $y = 4 + 9x^2 - x^3$

   Solution: $x = 0, 6$

2. Find the coordinates of any stationary points on each curve.

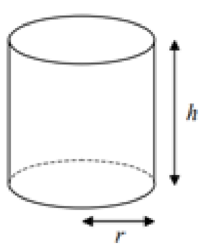
   $y = x^2 + 2x$

   Solution: Has a stationary point at $(-1, -1)$

   $y = 5x^2 - 4x + 1$

   Solution: Has a stationary point at $(0.4, 0.2)$

3. The diagram shows a closed plastic cylinder used for making compost.

   The radius of the base and the height of the cylinder are r cm and h cm respectively and the surface area of the cylinder is 30 000 cm^2

   (a) Show that the volume of the cylinder, V cm^3 , is given by $V = 15000 - \pi r^3$.

   Solution:

   $SA = 2\pi r^2 + 2\pi rh = 30000$

   $\therefore \pi rh = 15000 - \pi r^2$

   $h = \frac{15000}{\pi r} - r$

   $V = \pi r^2 h$

   $= \pi r^2 (\frac{15000}{\pi r} - r)$

   $= 15000r - \pi r^3$

(b) Find the maximum volume of the cylinder and show that your value is a maximum.
Solution:
$\frac{dV}{dr} = 15000 - 3\pi r^2$
$SP : 15000 - 3\pi r^2 = 0$
$r^2 = \frac{5000}{\pi}$
$r = \sqrt{\frac{5000}{\pi}}$ [=39.9 (3sf)]
max volume = 399 000 cm^3 (3sf)
$\frac{d^2V}{dr^2} = -6\pi r$
when $r = \sqrt{\frac{5000}{\pi}}$, $\frac{d^2V}{dr^2} = -752$
$\frac{d^2V}{dr^2} < 0 \therefore$ maximum

4. Integrate with respect to y: $y^{\frac{1}{2}}$
Solution: $\frac{2}{3}y^{\frac{3}{2}} + c$

5. Find $\int y \, dx$ when

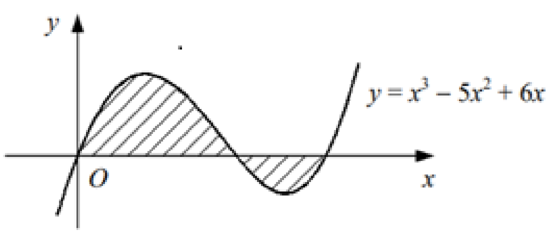
(a) $y = 3x^2 - x + 6$
Solution: $x^3 - \frac{1}{2}x^2 + 6x + c$

(b) $y = x^6 - x^3 + 2x - 5$
Solution: $\frac{1}{7}x^7 - \frac{1}{4}x^4 + x^2 - 5x + c$

(c) $\sin 2x + 3\cos 3x$
Solution: $\sin(3x) - \frac{\cos 2x}{2} + c$

(d) $y = -e^{2x} + \frac{4}{x}$
Solution: $4\ln(|x|) - \frac{e^{2x}}{2} + c$

6. The diagram shows the curve with the equation $y = x^3 - 5x^2 + 6x$.



(a) Find the coordinates of the points where the curve crosses the x-axis.
Solution: $(0, 0), (2, 0)$ and $(3, 0)$

(b) Show that the total area of the shaded regions enclosed by the curve and the x-axis is $3\frac{1}{12}$
Solution:
$\int_0^2 (x^3 - 5x^2 + 6x), dx$
$= [\frac{1}{4}x^4 - \frac{5}{3}x^3 + 3x^2]_0^2$
$= (4 - \frac{40}{3} + 12) - 0 = \frac{8}{3}$
$\int_2^3 (x^3 - 5x^2 + 6x), dx$
$= [\frac{1}{4}x^4 - \frac{5}{3}x^3 + 3x^2]_2^3$
$= (\frac{81}{4} - 45 + 27) - \frac{8}{3} = -\frac{5}{12}$
total area $= \frac{8}{3} + 512 = 3\frac{1}{12}$

7. Evaluate:

(a) $\int_2^3 \frac{1}{x^2} dx$;

(b) $\int_0^{\frac{\pi}{3}} \cos 2x dx$;

(c) $\int_1^3 e^{2t} dt$.

Solution:

(a) $\int_2^3 \frac{1}{x^2} dx = \frac{1}{6}$,

(b) $\int_0^{\frac{\pi}{3}} \cos 2x dx = \frac{\sqrt{3}}{4} = 0.4330$,

(c) $\int_1^3 e^{2t} dt = \frac{1}{2}(e^6 - e^2) = 198.02$.