Contents 28

Differential Vector Calculus

Learning outcomes

In this Workbook you will learn about scalar and vector fields and how physical quantities can be represented by such fields. You will be able to 'differentiate' such fields i.e. to find how rapidly the scalar or vector field varies with position. Depending on whether the original function and the intended derivative are scalars or vectors, there are three such derivatives known as the 'gradient', the 'divergence' and the 'curl'. You will be able to evaluate these derivatives for given fields. In addition, you will be able to work out the derivatives while using polar coordinate systems.

Background to Vector Calculus

28.1

Introduction

Vector Calculus is the study of the various derivatives and integrals of a scalar or vector function of the variables defining position (x,y,z) and possibly also time (t). This Section considers functions of several variables and introduces scalar and vector fields.

Prerequisites

Before starting this Section you should ...

- be familiar with the concept of a function of two variables
- be familiar with the concept of partial differentiation
- be familiar with the concept of vectors

Learning Outcomes

On completion you should be able to ...

- state the properties of scalar and vector fields
- work with a vector function of a variable

1. Functions of several variables and partial derivatives

These functions were first studied in HELM 18. As a reminder:

- a function of the two independent variables x and y may be written as $f(x,y)$
- the first and second order partial derivatives are $\dfrac{\partial f}{\partial x}$, $\dfrac{\partial f}{\partial y}$, $\dfrac{\partial^2 f}{\partial x^2}$, $\dfrac{\partial^2 f}{\partial y^2}$ and $\dfrac{\partial^2 f}{\partial x \partial y}$.

Consider, for example, the function $f(x,y) = x^2 + 5xy + 3y^4 + 1$. The first and second partial derivatives are

$$\begin{aligned}
\frac{\partial f}{\partial x} &= 2x + 5y \qquad \text{(differentiating with respect to } x \text{ keeping } y \text{ constant)}\\
\frac{\partial f}{\partial y} &= 5x + 12y^3 \qquad \text{(differentiating with respect to } y \text{ keeping } x \text{ constant)}\\
\frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(2x+5y) = 2\\
\frac{\partial^2 f}{\partial y^2} &= \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(5x+12y^3) = 36y^2\\
\frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(2x+5y) = 5
\end{aligned}$$

The number of independent variables is not restricted to two. For example, if u is a function of the three variables x, y and z, say $u = x^2 + y^2 + z^2$ then:

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = 2y, \quad \frac{\partial u}{\partial z} = 2z, \quad \frac{\partial^2 u}{\partial x^2} = 2, \quad \frac{\partial^2 u}{\partial y^2} = 2, \quad \frac{\partial^2 u}{\partial z^2} = 2$$

Similarly, if u is a function of the four variables x, y, z and t say $u = xy^2z^3\mathrm{e}^t$ then

$$\frac{\partial u}{\partial x} = y^2z^3\mathrm{e}^t, \quad \frac{\partial u}{\partial t} = xy^2z^3\mathrm{e}^t, \quad \frac{\partial^2 u}{\partial z^2} = 6xy^2z\mathrm{e}^t, \text{ etc.}$$

2. Vector functions of a variable

Vectors were first studied in HELM 9. A vector is a quantity that has magnitude and direction and combines together with other vectors according to the triangle law. Examples are (i) a velocity of 60 mph West and (ii) a force of 98.1 newtons vertically downwards.
It is often convenient to express vectors in terms of $\underline{i}$, $\underline{j}$ and $\underline{k}$, which are unit vectors in the x, y and z directions respectively. Examples are $\underline{a} = 3\underline{i} + 4\underline{j}$ and $\underline{b} = 2\underline{i} - 2\underline{j} + \underline{k}$
The magnitudes of these vectors are $|\underline{a}| = \sqrt{3^2+4^2} = 5$ and $|\underline{b}| = \sqrt{2^2 + (-2)^2 + 1^2} = 3$ respectively. In this case $\underline{a}$ and $\underline{b}$ are constant vectors, but a vector could be a function of an independent variable such as t (which may represent time in certain applications).

Example 1

A particle is at the point A(3,0). At time $t = 0$ it starts moving at a constant speed of 2 m s^{-1} in a direction parallel to the positive y-axis. Find expressions for the position vector, $\underline{r}$, of the particle at time t, together with its velocity $\underline{v} = \frac{d\underline{r}}{dt}$ and acceleration $\underline{a} = \frac{d^2\underline{r}}{dt^2}$.

Solution

In the first second of its motion the particle moves 2 metres to B and it moves a further 2 metres in each subsequent second, to C, D, ... Because it moves parallel to the y-axis its velocity is $\underline{v} = 2\underline{j}$. As its velocity is constant its acceleration is $\underline{a} = \underline{0}$.
The position of the particle at $t = 0, 1, 2, 3$ is given in the table.

Time t	0	1	2	3
Position $\underline{r}$	$3\underline{i}$	$3\underline{i} + 2\underline{j}$	$3\underline{i} + 4\underline{j}$	$3\underline{i} + 6\underline{j}$

In general, after t seconds, the position vector of the particle is $\underline{r} = 3\underline{i} + 2t\underline{j}$

Example 2

The position vector of a particle at time t is given by $\underline{r} = 2t\underline{i} + t^2\underline{j}$. Find its equation in Cartesian form and sketch the path followed by the particle.

Tabulating $\underline{r} = x\underline{i} + y\underline{j}$ at different times t:

Time t	0	1	2	3	4
x	0	2	4	6	8
y	0	1	4	9	16
$\underline{r}$	$\underline{0}$	$2\underline{i} + \underline{j}$	$4\underline{i} + 4\underline{j}$	$6\underline{i} + 9\underline{j}$	$8\underline{i} + 16\underline{j}$

Solution

To find the Cartesian equation of the curve we eliminate t between $x = 2t$ and $y = t^2$. Re-arrange $x = 2t$ as $t = \frac{1}{2}x$. Then $y = t^2 = \left(\frac{1}{2}x\right)^2 = \frac{1}{4}x^2$, which is a parabola. This is the path followed by the particle. See Figure 1.

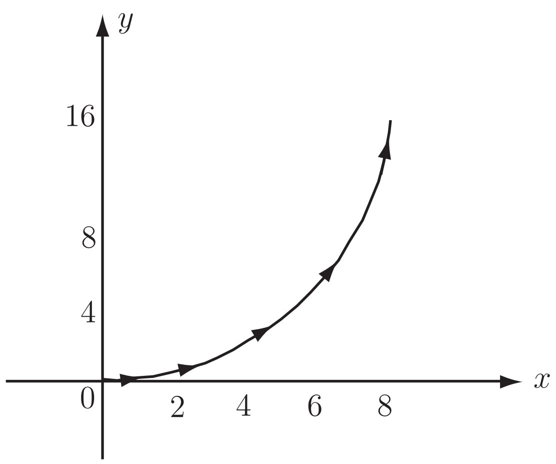


Figure 1: Path followed by a particle

In general, a three-dimensional vector function of one variable t is of the form

$$\underline{u} = x(t)\underline{i} + y(t)\underline{j} + z(t)\underline{k}.$$

Such functions may be differentiated one or more times and the rules of differentiation are derived from those for ordinary scalar functions. In particular, if $\underline{u}$ and $\underline{v}$ are vector functions of t and if c is a constant, then:

Rule 1. $\dfrac{d}{dt}(\underline{u}+\underline{v}) = \dfrac{d\underline{u}}{dt} + \dfrac{d\underline{v}}{dt}$

Rule 2. $\dfrac{d}{dt}(c\underline{u}) = c\dfrac{d\underline{u}}{dt}$

Rule 3. $\dfrac{d}{dt}(\underline{u}\cdot\underline{v}) = \underline{u}\cdot\dfrac{d\underline{v}}{dt} + \dfrac{d\underline{u}}{dt}\cdot\underline{v}$

Rule 4. $\dfrac{d}{dt}(\underline{u}\times\underline{v}) = \underline{u}\times\dfrac{d\underline{v}}{dt} + \dfrac{d\underline{u}}{dt}\times\underline{v}$

Also, if a particle moves so that its position vector at time t is $\underline{r}(t) = x(t)\underline{i} + y(t)\underline{j} + z(t)\underline{k}$ then the velocity of the particle is

$$\underline{v} = \frac{d\underline{r}}{dt} = \dot{\underline{r}} = \frac{dx(t)}{dt}\underline{i} + \frac{dy(t)}{dt}\underline{j} + \frac{dz(t)}{dt}\underline{k} = \dot{x}\underline{i} + \dot{y}\underline{j} + \dot{z}\underline{k}$$

and its acceleration is

$$\underline{a} = \frac{d\underline{v}}{dt} = \frac{d^2\underline{r}}{dt^2} = \ddot{\underline{r}} = \frac{d^2x(t)}{dt^2}\underline{i} + \frac{d^2y(t)}{dt^2}\underline{j} + \frac{d^2z(t)}{dt^2}\underline{k} = \ddot{x}\underline{i} + \ddot{y}\underline{j} + \ddot{z}\underline{k}$$

Example 3

Find the derivative (with respect to t) of the position vector $\underline{r} = t^2\underline{i} + 3t\underline{j} + 4\underline{k}$. Also find a unit vector tangential to the curve traced out by the position vector at the point where $t = 2$.

Solution

Differentiating $\underline{r}$ with respect to t,

$$\dot{\underline{r}} = \frac{d\underline{r}}{dt} = 2t\underline{i} + 3\underline{j}$$

so

$$\dot{\underline{r}}(2) = 4\underline{i} + 3\underline{j}$$

A unit vector in this direction, which is tangential to the curve, is

$$\frac{\dot{\underline{r}}(2)}{|\dot{\underline{r}}(2)|} = \frac{4\underline{i} + 3\underline{j}}{\sqrt{4^2+3^2}} = \frac{4}{5}\underline{i} + \frac{3}{5}\underline{j}$$

Example 4

For the position vectors (i) $\underline{r} = 3\underline{i} + 2t\underline{j}$ and (ii) $\underline{r} = 2t\underline{i} + t^2\underline{j}$ use the general expressions for velocity and acceleration to confirm the values of $\underline{v}$ and $\underline{a}$ found earlier in Examples 1 and 2.

Solution

(i) $\underline{r} = 3\underline{i} + 2t\underline{j}$. Then

$$\underline{v} = \frac{d\underline{r}}{dt} = \dot{\underline{r}} = \frac{d}{dt}(3\underline{i} + 2t\underline{j}) = \frac{d(3)}{dt}\underline{i} + \frac{d(2t)}{dt}\underline{j} = 0\underline{i} + 2\underline{j} = 2\underline{j}$$

and

$$\underline{a} = \frac{d\underline{v}}{dt} = \ddot{\underline{r}} = \frac{d}{dt}(2\underline{j}) = \frac{d(2)}{dt}\underline{j} = 0\underline{j} = \underline{0}$$

which agree with those found earlier.

(ii) $\underline{r} = 2t\underline{i} + t^2\underline{j}$. Then

$$\underline{v} = \frac{d\underline{r}}{dt} = \dot{\underline{r}} = \frac{d}{dt}(2t\underline{i} + t^2\underline{j}) = \frac{d(2t)}{dt}\underline{i} + \frac{d(t^2)}{dt}\underline{j} = 2\underline{i} + 2t\underline{j}$$

and

$$\underline{a} = \frac{d\underline{v}}{dt} = \ddot{\underline{r}} = \frac{d}{dt}(2\underline{i} + 2t\underline{j}) = \frac{d(2)}{dt}\underline{i} + \frac{d(2t)}{dt}\underline{j} = 0\underline{i} + 2\underline{j} = 2\underline{j}$$

which agree with those found earlier.

Example 5

A particle of mass $m = 1$ kg has position vector $\underline{r}$. The torque (moment of force) $\underline{H}$ relative to the origin acting on the particle as a result of a force $\underline{F}$ is defined as $\underline{H} = \underline{r} \times \underline{F}$, where, by Newton's second law, $\underline{F} = m\ddot{\underline{r}}$. The angular momentum (moment of momentum) $\underline{L}$ of the particle is defined as $\underline{L} = \underline{r} \times m\dot{\underline{r}}$. Find $\underline{L}$ and $\underline{H}$ for the particle where (i) $\underline{r} = 3\underline{i} + 2t\underline{j}$ and (ii) $\underline{r} = 2t\underline{i} + t^2\underline{j}$, and show that in each case the torque law $\underline{H} = \dot{\underline{L}}$ is satisfied.

Solution

(i) Here $\underline{r} = 3\underline{i} + 2t\underline{j}$ so $\dot{\underline{r}} = 2\underline{j}$ and $\underline{a} = \underline{0}$. Then

$$\underline{L} = \underline{r} \times m\dot{\underline{r}} = (3\underline{i} + 2t\underline{j}) \times 2\underline{j} = 6\underline{k} \quad \text{so} \quad \dot{\underline{L}} = \frac{d}{dt}(6)\underline{k} = \underline{0}$$

and

$$\underline{H} = \underline{r} \times \underline{F} = \underline{r} \times m\ddot{\underline{r}} = (3\underline{i} + 2t\underline{j}) \times \underline{0} = \underline{0} \quad \text{giving} \quad \underline{H} = \dot{\underline{L}} \quad \text{as required.}$$

Solution (contd.)

(ii) Here $\underline{r} = 2t\underline{i} + t^2\underline{j}$ so $\dot{\underline{r}} = 2\underline{i} + 2t\underline{j}$ and $\underline{a} = 2\underline{j}$. Then

$$\underline{L} = \underline{r} \times m\dot{\underline{r}} = (2t\underline{i} + t^2\underline{j}) \times (2\underline{i} + 2t\underline{j}) = (4t^2 - 2t^2)\underline{k} = 2t^2\underline{k} \quad \text{so} \quad \dot{\underline{L}} = 4t\underline{k}$$

and

$$\underline{H} = \underline{r} \times \underline{F} = \underline{r} \times m\ddot{\underline{r}} = (2t\underline{i} + t^2\underline{j}) \times 2\underline{j} = 4t\underline{k} \quad \text{giving} \quad \underline{H} = \dot{\underline{L}} \text{ as required.}$$

Task

A particle moves so that its position vector is $\underline{r} = 12t\underline{i} + (19t - 5t^2)\underline{j}$.

(a) Find $\dfrac{d\underline{r}}{dt}$ and $\dfrac{d^2\underline{r}}{dt^2}$.

(b) When is the $\underline{j}$-component of $\dfrac{d\underline{r}}{dt}$ equal to zero?

(c) Find a unit vector normal to its trajectory when $t = 1$.

Your solution

Answer

(a) $\dfrac{d\underline{r}}{dt} = 12\underline{i} + (19 - 10t)\underline{j}, \qquad \dfrac{d^2\underline{r}}{dt^2} = -10\underline{j}.$

(b) The $\underline{j}$-component of $\dfrac{d\underline{r}}{dt}$, (also written $\dot{\underline{r}}$) is zero when $t = 1.9$.

(c) When $t = 1$ $\dot{\underline{r}} = 12\underline{i} + 9\underline{j}$. A vector perpendicular to this is $\dot{\underline{r}} = 9\underline{i} - 12\underline{j}$. Its magnitude is $\sqrt{81 + 144} = 15$. So a unit vector in this direction is $\frac{9}{15}\underline{i} - \frac{12}{15}\underline{j} = \frac{3}{5}\underline{i} - \frac{4}{5}\underline{j}$. The unit vector $-\dfrac{3}{5}\underline{i} + \dfrac{4}{5}\underline{j}$ is also a solution.

Task

A particle moving at a constant speed around a circle moves so that

$\underline{r} = \cos(\pi t)\underline{i} + \sin(\pi t)\underline{j}$

(a) Find $\dfrac{d\underline{r}}{dt}$ and $\dfrac{d^2\underline{r}}{dt^2}$.

(b) Find $\underline{r} \cdot \dfrac{d\underline{r}}{dt}$ and $\underline{r} \times \dfrac{d^2\underline{r}}{dt^2}$.

Your solution

Answer

(a) $\dfrac{d\underline{r}}{dt} = -\pi \sin \pi t \underline{i} + \pi \cos \pi t \underline{j}, \quad \dfrac{d^2\underline{r}}{dt^2} = -\pi^2 \cos \pi t \underline{i} - \pi^2 \sin \pi t \underline{j} = -\pi^2 \underline{r},$

(b) $\underline{r}.\dfrac{d\underline{r}}{dt} = -\pi \cos \pi t \sin \pi t + \pi \cos \pi t \sin \pi t = 0 \;\Rightarrow\; \dfrac{d\underline{r}}{dt}$ is perpendicular to $\underline{r}$

$$\underline{r} \times \frac{d^2\underline{r}}{dt^2} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \cos \pi t & \sin \pi t & 0 \\ -\pi^2 \cos \pi t & -\pi^2 \sin \pi t & 0 \end{vmatrix} = \underline{0} \;\Rightarrow\; \frac{d^2\underline{r}}{dt^2} \text{ is parallel to } \underline{r}.$$

Task

If $\underline{r} = \sin(2t)\underline{i} + \cos(2t)\underline{j} + t^2\underline{k}$ and $(1+t^2)\,|\ddot{\underline{r}}|^2 = c\,|\dot{\underline{r}}|^2$, find the value of c.

Your solution

Answer

$\dot{\underline{r}} = 2\cos(2t)\underline{i} - 2\sin(2t)\underline{j} + 2t\underline{k}, \quad \ddot{\underline{r}} = -4\sin(2t)\underline{i} - 4\cos(2t)\underline{j} + 2\underline{k}$

$|\ddot{\underline{r}}|^2 = 16\sin^2(2t) + 16\cos^2(2t) + 4 = 20 \quad |\dot{\underline{r}}|^2 = 4\cos^2(2t) + 4\sin^2(2t) + 4t^2 = 4(1+t^2)$

$\therefore \quad 20(1+t^2) = 4c(1+t^2)$ so that $c = 5$.

3. Scalar fields

A **scalar field** is a distribution of scalar values over a region of space (which may be 1D, 2D or 3D) so that a scalar value is associated with each point of space. Examples of scalar fields follow.

1.

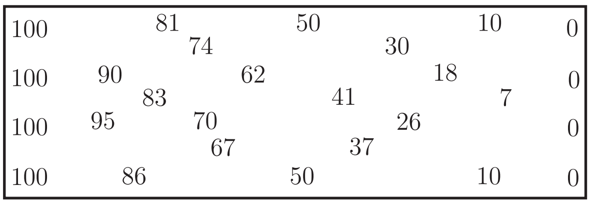


Figure 2: Temperature in a plate, one side held at 100°C the other at 0°C

2.

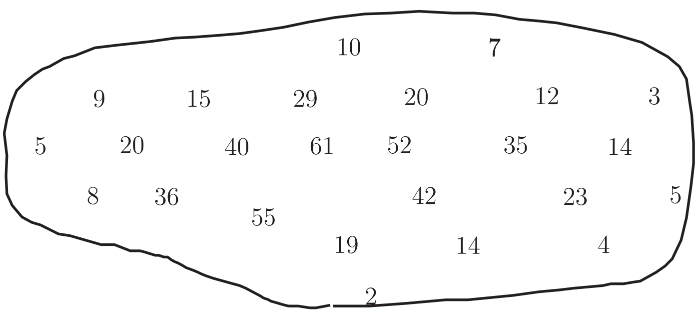


Figure 3: Height of land above sea level

3. The mean annual rainfall at different locations in Britain.

4. The light intensity near a 100 watt light bulb.

To define a **scalar field** we need to:

- Describe the region of space where it is found (this is the **domain**)
- Give a rule to show how the value of the scalar is related to every point in the domain.

Consider the scalar field defined by $\phi(x, y) = x + y$ over the rectangle $0 \le x \le 4$, $0 \le y \le 2$. We can calculate, and plot, values of ϕ at different (x, y) points. For example $\phi(0, 2) = 0 + 2 = 2$, $\phi(4, 1) = 4 + 1 = 5$ and so on.

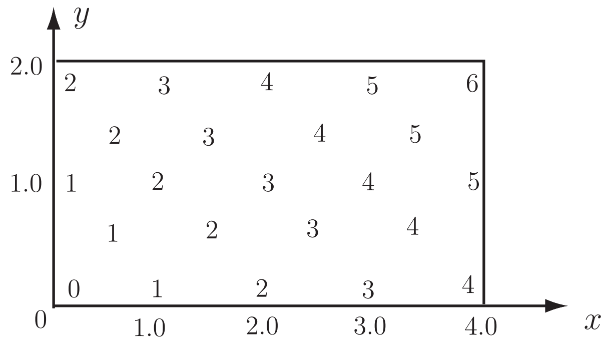


Figure 4: The scalar field $\phi(x, y) = x + y$

Contours

A contour on a map is a curve joining points that are the same height above sea level. These contours give far more information about the shape of the land than selected spot heights.

For example, the contours near the top of a hill might look like those shown in Figure 5 where the numbers are the values of the heights above sea level.

In general for a scalar field $\phi(x, y, z)$, contour curves are the family of curves given by $\phi = c$, for different values of the constant c.

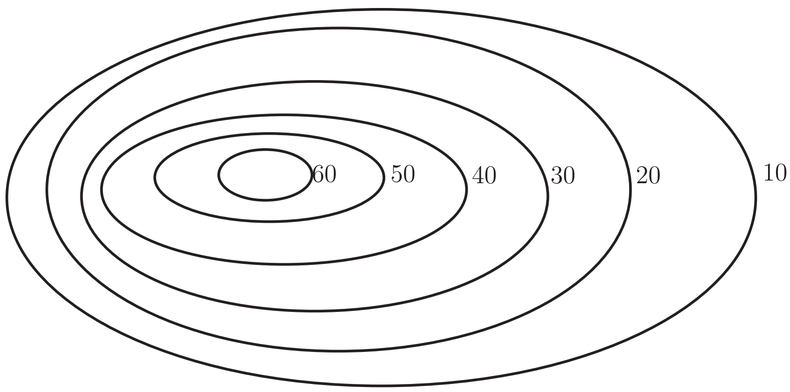


Figure 5: Contour lines

Example 6

Describe contour curves for the following scalar fields and sketch typical contours for (a) and (b).

(a) $\phi(x,y) = x + y$

(b) $\phi(x,y) = 9 - x^2 - y^2$

(c) $\phi(x,y) = \dfrac{1}{x^2 + y^2 + z^2}$

Solution

(a) The contour curves for $\phi(x,y) = x + y$ are $x + y = c$ or $y = -x + c$. These are straight lines of gradient -1. See Figure 6(a).

(b) For $\phi(x,y) = 9 - x^2 - y^2$, the contour curves are $9 - x^2 - y^2 = c$, or $x^2 + y^2 = 9 - c$. See Figure 6(b). These are circles, centered at the origin, radius $\sqrt{9-c}$.

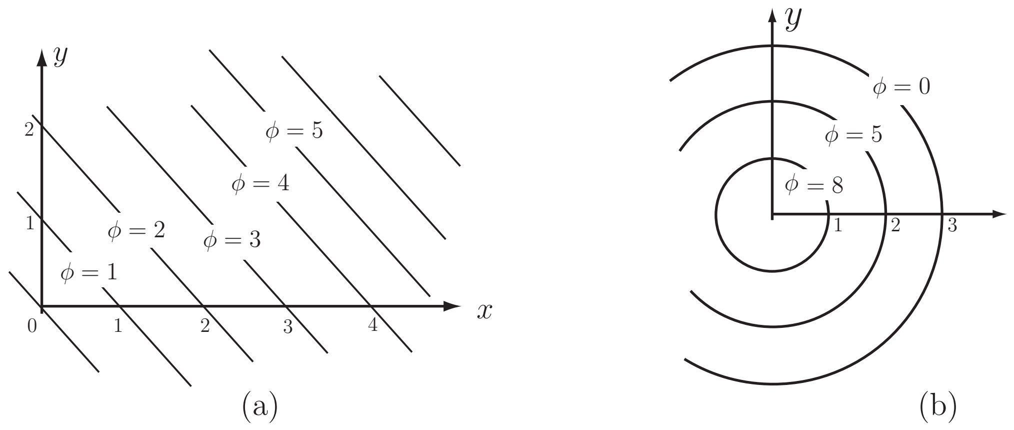


Figure 6: Contours for (a) $x + y$ (b) $9 - x^2 - y^2$

(c) For the three-dimensional scalar field $\phi(x,y,z) = \dfrac{1}{x^2 + y^2 + z^2}$ the contour surfaces are $\dfrac{1}{x^2 + y^2 + z^2} = c$ or $x^2 + y^2 + z^2 = \dfrac{1}{c}$. These are spheres, centered at the origin and of radius $\dfrac{1}{\sqrt{c}}$.

Describe the contours for the following scalar fields

(a) $\phi = y - x$ (b) $\phi = x^2 + y^2$ (c) $\phi = y - x^2$

Your solution

Answer

(a) Straight lines of gradient 1, (b) Circles; centred at origin, (c) Parabolas $y = x^2 + c$.

Key Point 1

A scalar field F (in three-dimensional space) returns a real value for the function F for every point (x, y, z) in the domain of the field.

4. Vector fields

A vector field is a distribution of vectors over a region of space such that a vector is associated with each point of the region. Examples are:

1. The velocity of water flowing in a river (Figure 7).

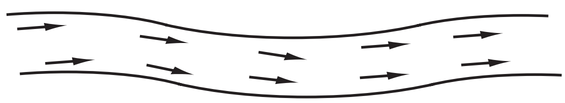


Figure 7: Velocity of water in a river

2. The gravitational pull of the Earth (Figure 8). At every point there is a gravitational pull towards the centre of the Earth.

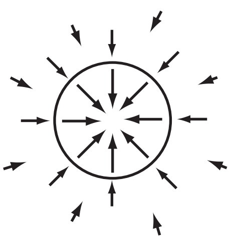


Figure 8: Gravitational pull of the Earth

Note: the length of the vector is used to indicate its magnitude (i.e. greater near the centre of the Earth.)

3. The flow of heat in a metal plate insulated on its sides (Figure 9). Heat flows from the hot portion on the left to the cool portion on the right.

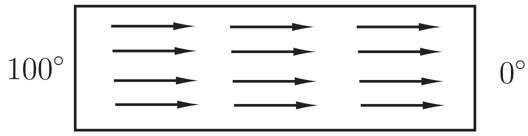


Figure 9: Flow of heat in a metal plate

To define a **vector field** we need to :

- Describe the region of space where the vectors are found (the domain)
- Give a rule for associating a vector with each point of the domain.

Note that in the case of the heat flowing in a plate, the temperature can be described by a scalar field while the flow of heat is described by a vector field.
Consider the flow of water in different situations.

(a) In a pond where the water is motionless everywhere, the velocity at all points is zero. That is, $\underline{v}(x, y, z) = \underline{0}$, or for brevity, $\underline{v} = \underline{0}$.

(b) Consider a straight river with steady flow downstream (see Figure 10). The surface velocity $\underline{v}$ can be seen by watching the motion of a light floating object, such as a leaf. The leaf will float downstream parallel to the bank so $\underline{v}$ will be a multiple of $\underline{j}$. However, the speed is usually smallest near the bank and fastest in the middle of the river. In this simple model, the velocity $\underline{v}$ is assumed to be independent of the depth z. That is, $\underline{v}$ varies, in the $\underline{i}$, or x, direction so that $\underline{v}$ will be of the form $\underline{v} = f(x)\underline{j}$.

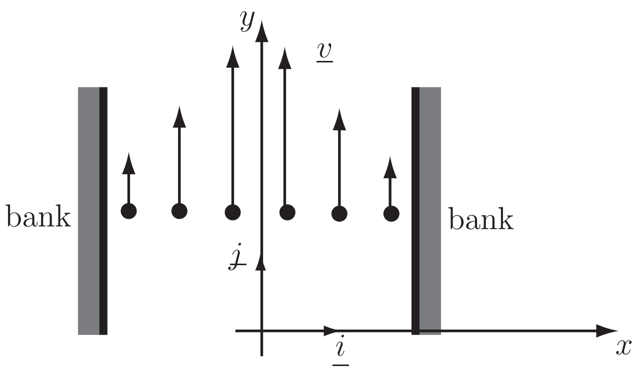


Figure 10: Flow in a straight river

(c) In a more realistic model $\underline{v}$ would vary as we move downstream and would be different at different depths due to, for example, rocks or bends. The velocity at any point could also depend on when the observation was made (for example the speed would be higher shortly after heavy rain) and so in general the velocity would be a function of the four variables x, y, z and t, and be of the form $\underline{v} = f_1(x, y, z, t)\underline{i} + f_2(x, y, z, t)\underline{j} + f_3(x, y, z, t)\underline{k}$, for suitable functions f_1, f_2 and f_3.

Example 7

Sketch sample vectors at the points $(3, 2)$, $(-2, 2)$, $(-3, -1)$, $(1, -4)$ for the two-dimensional vector field defined by $\underline{v} = x\underline{i} + 2\underline{j}$.

Solution

At $(3, 2)$, $\underline{v} = 3\underline{i} + 2\underline{j}$
At $(-2, 2)$, $\underline{v} = -2\underline{i} + 2\underline{j}$
At $(-3, -1)$, $\underline{v} = -3\underline{i} + 2\underline{j}$
At $(1, -4)$, $\underline{v} = \underline{i} + 2\underline{j}$
Plotting these vectors $\underline{v}$ gives the arrows in Figure 11.

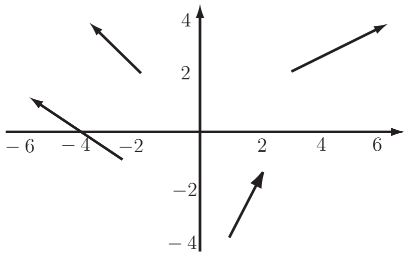


Figure 11: Sample vectors for the vector field $\underline{v} = x\underline{i} + 2\underline{j}$

It is possible to construct curves which start from and are in the same direction as any one vector and are guided by the direction of successive vectors. Starting at different points gives a set of non-intersecting lines called, depending on the context, vector field lines, lines of flow, streamlines or lines of force.

For example, consider the vector field $\underline{F} = -y\underline{i} + x\underline{j}$; $\underline{F}$ can be calculated at various points in the xy plane. Some of the individual vectors can be seen in Figure 12(a) while Figure 12(b) shows them converted seamlessly to field lines. For this function $\underline{F}$ the field lines are circles centered at the origin.

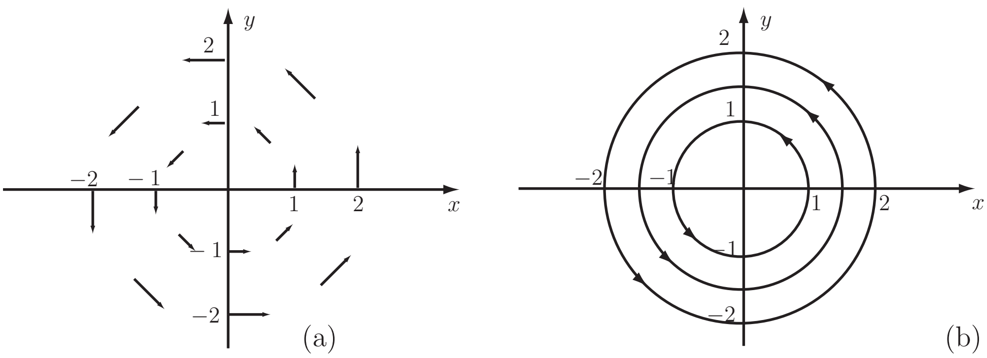


Figure 12: (a) Vectors at various points (b) Converted to field lines

Example 8

The Earth is affected by the gravitational force field of the Sun. This vector field is such that each vector $\underline{F}$ is directed towards the Sun and has magnitude proportional to $\dfrac{1}{r^2}$, where $r = \sqrt{x^2 + y^2 + z^2}$ is the distance from the Sun to the Earth. Derive an equation for $\underline{F}$ and sketch some field lines.

Solution

The field has magnitude proportional to $r^{-2} = (x^2 + y^2 + z^2)^{-1}$ and points directly towards the Sun (the origin) i.e. parallel to a unit vector pointing towards the origin. At the point given by $\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$, a unit vector pointing towards the origin is $\dfrac{-x\underline{i} - y\underline{j} - z\underline{k}}{|-x\underline{i} - y\underline{j} - z\underline{k}|} = \dfrac{-x\underline{i} - y\underline{j} - z\underline{k}}{\sqrt{x^2 + y^2 + z^2}}$.
Multiplying the unit vector by the required magnitude $r^{-2} = (x^2 + y^2 + z^2)^{-1}$ (and by a constant of proportionality c) gives $\underline{F} = c\dfrac{-x\underline{i} - y\underline{j} - z\underline{k}}{(x^2 + y^2 + z^2)^{3/2}}$. Figure 13 shows some field lines for $\underline{F}$.

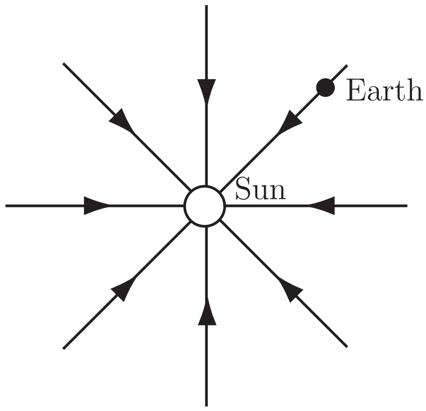


Figure 13: Gravitational field of the Sun

Key Point 2

A vector field $\underline{F}(x, y, z)$ (in three-dimensional coordinates) returns a vector $\underline{F}_0 = \underline{F}(x_0, y_0, z_0)$ for every point (x_0, y_0, z_0) in the domain of the field.

Exercises

1. Which of the following are scalar fields and which are vector fields?

   (a) $F = x^2 - yz$

   (b) $G = \dfrac{2x - z}{\sqrt{x^2 + y^2 + z^2 + 1}}$

   (c) $\underline{f} = x\underline{i} + y\underline{j} + z\underline{k}$

   (d) $H = \dfrac{y-1}{z^2+1}x + \dfrac{z-1}{x^2+1}y + \dfrac{x-1}{y^2+1}z$

   (e) $\underline{g} = (y+z)\underline{i}$

2. Draw vector diagrams for the vector fields

   (a) $\underline{f} = \underline{i} + 2\underline{j}$

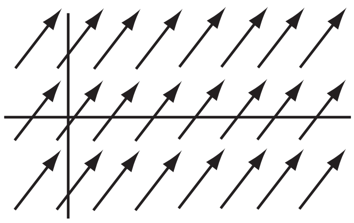
   (b) $\underline{g} = \underline{i} + y^2\underline{j}$

Answers

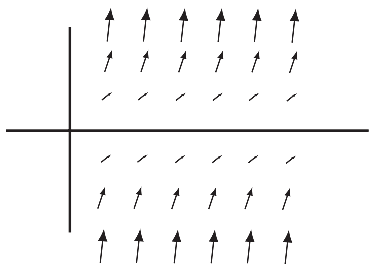
1. (a), (b) and (d) are scalar fields as the quantities defined are scalars.

   (c) and (e) are vector fields as the quantities defined are vectors.

2.



(a) The vectors point in the same direction everywhere



(b) As $|y|$ increases, the y-component increases

Differential Vector Calculus

28.2

Introduction

A vector field or a scalar field can be differentiated with respect to position in three ways to produce another vector field or scalar field. This Section studies the three derivatives, that is: (i) the gradient of a scalar field (ii) the divergence of a vector field and (iii) the curl of a vector field.

Prerequisites

Before starting this Section you should ...

- be familiar with the concept of a function of two variables
- be familiar with the concept of partial differentiation
- be familiar with scalar and vector fields

Learning Outcomes

On completion you should be able to ...

- find the divergence, gradient or curl of a vector or scalar field

1. The gradient of a scalar field

Consider the height ϕ above sea level at various points on a hill. Some contours for such a hill are shown in Figure 14.

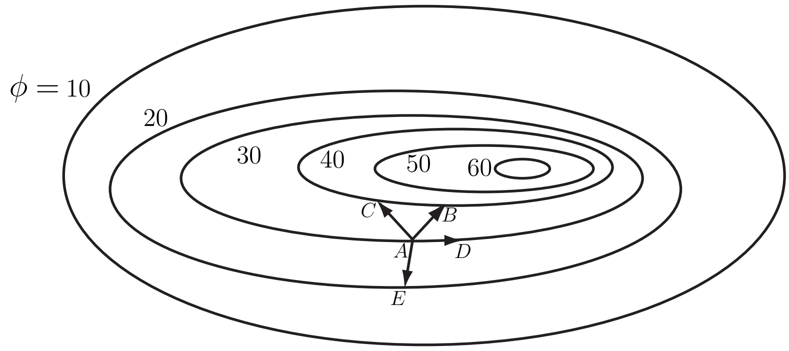


Figure 14: "Contour map" of a hill

We are interested in how ϕ changes from one point to another. Starting from A and making a displacement $\underline{d}$ the change in height (ϕ) depends on the direction of the displacement. The magnitude of each $\underline{d}$ is the same.

Displacement	Change in ϕ
AB	$40 - 30 = 10$
AC	$40 - 30 = 10$
AD	$30 - 30 = 0$
AE	$20 - 30 = -10$

The change in ϕ clearly depends on the direction of the displacement. For the paths shown ϕ increases most rapidly along AB, does not increase at all along AD (as A and D are both on the same contour and so are both at the same height) and decreases along AE.

The direction in which ϕ changes fastest is along the line of greatest slope which is orthogonal (i.e. perpendicular) to the contours. Hence, at each point of a scalar field we can define a vector field giving the magnitude and direction of the greatest rate of change of ϕ locally.

A vector field, called the gradient, written grad ϕ, can be associated with a scalar field ϕ so that at every point the direction of the vector field is orthogonal to the scalar field contour. This vector field is the direction of the maximum rate of change of ϕ.

For a second example consider a metal plate heated at one corner and cooled by an ice bag at the opposite corner. All edges and surfaces are insulated. After a while a steady state situation exists in which the temperature ϕ at any point remains the same. Some temperature contours are shown in Figure 15.

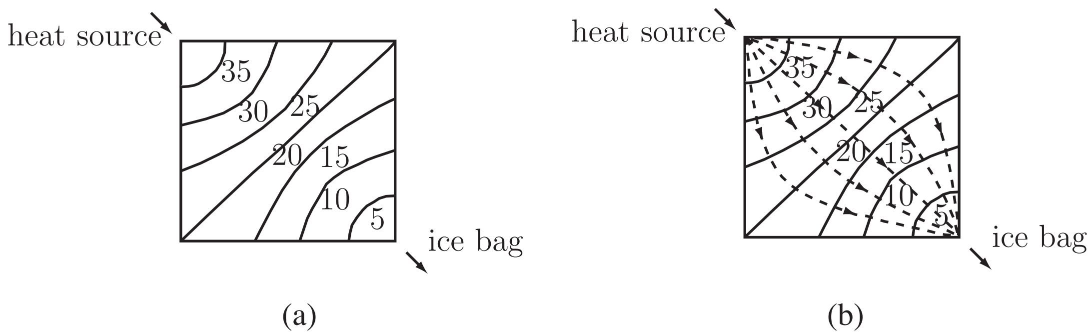


Figure 15: Temperature contours and heat flow lines for a metal plate

The direction of the heat flow is along the flow lines which are orthogonal to the contours (see the dashed lines in Figure 15(b)); this heat flow is proportional to the vector field grad ϕ.

Definition

The gradient of the scalar field $\phi = f(x, y, z)$ is $\quad$ grad $\phi = \underline{\nabla}\phi = \dfrac{\partial \phi}{\partial x}\underline{i} + \dfrac{\partial \phi}{\partial y}\underline{j} + \dfrac{\partial \phi}{\partial z}\underline{k}$

Often, instead of grad ϕ, the notation $\underline{\nabla}\phi$ is used. ($\underline{\nabla}$ is a vector differential operator called 'del' or 'nabla' defined by $\dfrac{\partial}{\partial x}\underline{i} + \dfrac{\partial}{\partial y}\underline{j} + \dfrac{\partial}{\partial z}\underline{k}$. As a vector differential operator, it retains the characteristics of a vector while also carrying out differentiation.)

The vector grad ϕ gives the magnitude and direction of the greatest rate of change of ϕ at any point, and is always orthogonal to the contours of ϕ. For example, in Figure 14, grad ϕ points in the direction of AB while the contour line is parallel to AD i.e. perpendicular to AB. Similarly, in Figure 15(b), the various intersections of the contours with the lines representing grad ϕ occur at right-angles.

For the hill considered earlier the direction and magnitude of grad ϕ are shown at various points in Figure 16. Note that the magnitude of grad ϕ is greatest (as indicated by the length of the arrow) when the hill is at its steepest (as indicated by the closeness of the contours).

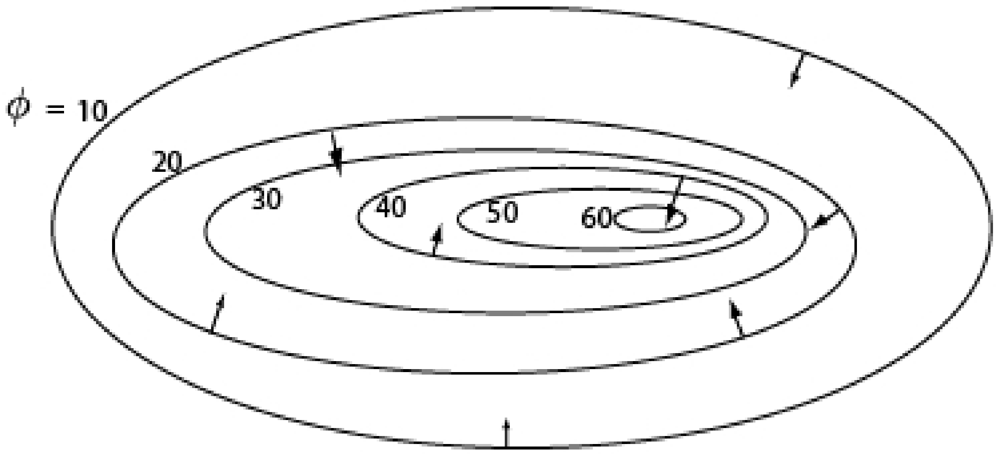


Figure 16: Grad ϕ and the steepest ascent direction for a hill

Key Point 3

ϕ is a scalar field but grad ϕ is a vector field.

Example 9

Find grad ϕ for

(a) $\phi = x^2 - 3y$ (b) $\phi = xy^2z^3$

Solution

(a) grad $\phi = \dfrac{\partial}{\partial x}(x^2-3y)\underline{i} + \dfrac{\partial}{\partial y}(x^2-3y)\underline{j} + \dfrac{\partial}{\partial z}(x^2-3y)\underline{k} = 2x\underline{i} + (-3)\underline{j} + 0\underline{k} = 2x\underline{i} - 3\underline{j}$

(b) grad $\phi = \dfrac{\partial}{\partial x}(xy^2z^3)\underline{i} + \dfrac{\partial}{\partial y}(xy^2z^3)\underline{j} + \dfrac{\partial}{\partial z}(xy^2z^3)\underline{k} = y^2z^3\underline{i} + 2xyz^3\underline{j} + 3xy^2z^2\underline{k}$

Example 10

For $f = x^2 + y^2$ find grad f at the point $A(1,2)$. Show that the direction of grad f is orthogonal to the contour at this point.

Solution

$$\text{grad } f = \frac{\partial f}{\partial x}\underline{i} + \frac{\partial f}{\partial y}\underline{j} + \frac{\partial f}{\partial z}\underline{k} = 2x\underline{i} + 2y\underline{j} + 0\underline{k} = 2x\underline{i} + 2y\underline{j}$$

and at $A(1,2)$ this equals $2 \times 1\underline{i} + 2 \times 2\underline{j} = 2\underline{i} + 4\underline{j}$.

Since $f = x^2 + y^2$ then the contours are defined by $x^2 + y^2 =$ constant, so the contours are circles centred at the origin. The vector grad f at $A(1,2)$ points directly away from the origin and hence grad f and the contour are orthogonal; see Figure 17. Note that $\underline{r}(A) = \underline{i} + 2\underline{j} = \dfrac{1}{2}\,\text{grad}\, f$.

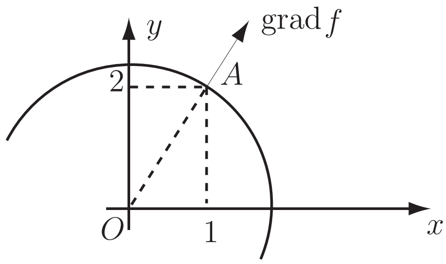


Figure 17: Grad f is perpendicular to the contour lines

The change in a function ϕ in a given direction (specified as a unit vector $\underline{a}$) is determined from the scalar product (grad ϕ) $\cdot\, \underline{a}$. This scalar quantity is called the directional derivative.
Note:

- $\underline{a}$ along a contour implies $\underline{a}$ is perpendicular to grad ϕ which implies $\underline{a} \cdot$ grad $\phi = 0$.
- $\underline{a}$ perpendicular to a contour implies $\underline{a} \cdot$ grad ϕ is a maximum.

Task

Given $\phi = x^2y^2z^2$, find

(a) grad ϕ

(b) grad ϕ at $(-1, 1, 1)$ and a unit vector in this direction.

(c) the derivative of ϕ at $(2, 1, -1)$ in the direction of

(i) $\underline{i}$ (ii) $\underline{d} = \frac{3}{5}\underline{i} + \frac{4}{5}\underline{k}$.

Your solution

Answer

(a) grad $\phi = \dfrac{\partial\phi}{\partial x}\underline{i} + \dfrac{\partial\phi}{\partial y}\underline{j} + \dfrac{\partial\phi}{\partial z}\underline{k} = 2xy^2z^2\underline{i} + 2x^2yz^2\underline{j} + 2x^2y^2z\underline{k}$

(b) At $(-1, 1, 1)$, grad $\phi = -2\underline{i} + 2\underline{j} + 2\underline{k}$
A unit vector in this direction is

$$\frac{\text{grad } \phi}{|\text{grad } \phi|} = \frac{-2\underline{i} + 2\underline{j} + 2\underline{k}}{\sqrt{(-2)^2 + 2^2 + 2^2}} = \frac{1}{2\sqrt{3}}(-2\underline{i} + 2\underline{j} + 2\underline{k}) = -\frac{1}{\sqrt{3}}\underline{i} + \frac{1}{\sqrt{3}}\underline{j} + \frac{1}{\sqrt{3}}\underline{k}$$

(c) At $(2, 1, -1)$, grad $\phi = 4\underline{i} + 8\underline{j} - 8\underline{k}$

(i) To find the derivative of ϕ in the direction of $\underline{i}$ take the scalar product $(4\underline{i} + 8\underline{j} - 8\underline{k}) \cdot \underline{i} = 4 \times 1 + 0 + 0 = 4.$ So the derivative in the direction of $\underline{d}$ is 4.

(ii) To find the derivative of ϕ in the direction of $\underline{d} = \dfrac{3}{5}\underline{i} + \dfrac{4}{5}\underline{k}$ take the scalar product

$$(4\underline{i} + 8\underline{j} - 8\underline{k}) \cdot (\frac{3}{5}\underline{i} + \frac{4}{5}\underline{k}) = 4 \times \frac{3}{5} + 0 + (-8) \times \frac{4}{5} = \frac{12}{5} - \frac{32}{5} = -4.$$

So the derivative in the direction of $\underline{d}$ is -4.

Exercises

1. Find grad ϕ for the following scalar fields

   (a) $\phi = y - x$. (b) $\phi = y - x^2$, (c) $\phi = x^2 + y^2 + z^2$.

2. Find grad ϕ for each of the following two-dimensional scalar fields given that $\underline{r} = x\underline{i} + y\underline{j}$ and $r = \sqrt{x^2 + y^2}$ (you should express your answer in terms of $\underline{r}$).

   (a) $\phi = r$, (b) $\phi = \ln r$, (c) $\phi = \dfrac{1}{r}$, (d) $\phi = r^n$.

3. If $\phi = x^3y^2z$, find,

   (a) $\underline{\nabla}\phi$

   (b) a unit vector normal to the contour at the point $(1, 1, 1)$.

   (c) the rate of change of ϕ at $(1, 1, 1)$ in the direction of $\underline{i}$.

   (d) the rate of change of ϕ at $(1, 1, 1)$ in the direction of the unit vector $\underline{n} = \frac{1}{\sqrt{3}}(\underline{i} + \underline{j} + \underline{k})$.

4. Find a unit vector which is normal to the sphere $x^2 + (y - 1)^2 + (z + 1)^2 = 2$ at the point $(0, 0, 0)$.

5. Find vectors normal to $\phi_1 = y - x^2$ and $\phi_2 = x + y - 2$. Hence find the angle between the curves $y = x^2$ and $y = 2 - x$ at their point of intersection in the first quadrant.

Answers

1. (a) $\dfrac{\partial}{\partial x}(y - x)\underline{i} + \dfrac{\partial}{\partial y}(y - x)\underline{j} = -\underline{i} + \underline{j}$

   (b) $-2x\underline{i} + \underline{j}$

   (c) $[\dfrac{\partial}{\partial x}(x^2 + y^2 + z^2)]\underline{i} + [\dfrac{\partial}{\partial y}(x^2 + y^2 + z^2)]\underline{j} + [\dfrac{\partial}{\partial z}(x^2 + y^2 + z^2)]\underline{k} = 2x\underline{i} + 2y\underline{j} + 2z\underline{k}$

2. (a) $\dfrac{\underline{r}}{r}$, (b) $\dfrac{\underline{r}}{r^2}$, (c) $-\dfrac{\underline{r}}{r^3}$, (d) $nr^{n-2}\underline{r}$

3. (a) $3x^2y^2z\underline{i} + 2x^3yz\underline{j} + x^3y^2\underline{k}$, (b) $\dfrac{1}{\sqrt{14}}(3\underline{i} + 2\underline{j} + \underline{k})$, (c) 3, (d) $2\sqrt{3}$

4. The vector field $\underline{\nabla}\phi$ where $\phi = x^2 + (y - 1)^2 + (z + 1)^2$ is $2x\underline{i} + 2(y - 1)\underline{j} + 2(z + 1)\underline{k}$
   The value that this vector field takes at the point $(0, 0, 0)$ is $-2\underline{j} + 2\underline{k}$ which is a vector normal to the sphere.
   Dividing this vector by its magnitude forms a unit vector: $\dfrac{1}{\sqrt{2}}(-\underline{j} + \underline{k})$

5. 108° or 72° (intersect at $(1, 1)$) [At intersection, grad $\phi_1 = -2\underline{i} + \underline{j}$ and grad $\phi_2 = \underline{i} + \underline{j}$.]

2. The divergence of a vector field

Consider the vector field $\underline{F} = F_1\underline{i} + F_2\underline{j} + F_3\underline{k}$.

In 3D cartesian coordinates the **divergence** of $\underline{F}$ is defined to be

$$\text{div}\ \underline{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

Note that $\underline{F}$ is a vector field but div $\underline{F}$ is a scalar.
In terms of the differential operator $\underline{\nabla}$, div $\underline{F} = \underline{\nabla} \cdot \underline{F}$ since

$$\underline{\nabla} \cdot \underline{F} = (\underline{i}\frac{\partial}{\partial x} + \underline{j}\frac{\partial}{\partial y} + \underline{k}\frac{\partial}{\partial z}) \cdot (F_1\underline{i} + F_2\underline{j} + F_3\underline{k}) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

Physical Significance of the Divergence

The meaning of the divergence is most easily understood by considering the behaviour of a fluid and hence is relevant to engineering topics such as thermodynamics. The divergence (of the vector field representing velocity) at a point in a fluid (liquid or gas) is a measure of the rate per unit volume at which the fluid is flowing away from the point. A negative divergence is a convergence indicating a flow towards the point. Physically positive divergence means that either the fluid is expanding or that fluid is being supplied by a source external to the field. Conversely convergence means a contraction or the presence of a sink through which fluid is removed from the field. The lines of flow diverge from a source and converge to a sink.

If there is no gain or loss of fluid anywhere then div $\underline{v} = 0$ which is the equation of continuity for an incompressible fluid.

The divergence also enters engineering topics such as electromagnetism. A magnetic field ($\underline{B}$) has the property $\underline{\nabla} \cdot \underline{B} = 0$, that is there are no isolated sources or sinks of magnetic field (no magnetic monopoles).

Key Point 4

$\underline{F}$ is a vector field but div $\underline{F}$ is a scalar field.

Example 11

Find the divergence of the following vector fields.

(a) $\underline{F} = x^2\underline{i} + y^2\underline{j} + z^2\underline{k}$

(b) $\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$

(c) $\underline{v} = -x\underline{i} + y\underline{j} + 2\underline{k}$

Solution

(a) $\text{div } \underline{F} = \dfrac{\partial}{\partial x}(x^2) + \dfrac{\partial}{\partial y}(y^2) + \dfrac{\partial}{\partial z}(z^2) = 2x + 2y + 2z$

(b) $\text{div } \underline{r} = \dfrac{\partial}{\partial x}(x) + \dfrac{\partial}{\partial y}(y) + \dfrac{\partial}{\partial z}(z) = 1 + 1 + 1 = 3$

(c) $\text{div } \underline{v} = \dfrac{\partial}{\partial x}(-x) + \dfrac{\partial}{\partial y}(y) + \dfrac{\partial}{\partial z}(2) = -1 + 1 + 0 = 0$

Example 12

Find the value of a for which $\underline{v} = (2x^2y + z^2)\underline{i} + (xy^2 - x^2z)\underline{j} + (axyz - 2x^2y^2)\underline{k}$ is the vector field of an incompressible fluid.

Solution

$\underline{v}$ is incompressible if div $\underline{v} = 0$.

$$\text{div } \underline{v} = \frac{\partial}{\partial x}(2x^2y + z^2) + \frac{\partial}{\partial y}(xy^2 - x^2z) + \frac{\partial}{\partial z}(axyz - 2x^2y^2) = 4xy + 2xy + axy$$

which is zero if $a = -6$.

Task

Find the divergence of the following vector field, in general terms and at the point $(1, 0, 3)$.

$$\underline{F}_1 = x^3\underline{i} + y^3\underline{j} + z^3\underline{k}$$

Your solution

Answer

(a) $3x^2 + 3y^2 + 3z^2$, 30

Task

Find the divergence of $\underline{F}_2 = x^2y\underline{i} - 2xy^2\underline{j}$, in general terms and at $(1,0,3)$.

Your solution

Answer

$-2xy$, 0,

Task

Find the divergence of $\underline{F}_3 = x^2z\underline{i} - 2y^3z^3\underline{j} + xyz^2\underline{k}$, in general terms and at the point $(1,0,3)$.

Your solution

Answer

$2xz - 6y^2z^3 + 2xyz,\ \ 6$

3. The curl of a vector field

The curl of the vector field given by $\underline{F} = F_1\underline{i} + F_2\underline{j} + F_3\underline{k}$ is defined as the vector field

$$\text{curl } \underline{F} = \underline{\nabla} \times \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\underline{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\underline{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\underline{k}$$

Physical significance of curl

The divergence of a vector field represents the outflow rate from a point; however the curl of a vector field represents the rotation at a point.

Consider the flow of water down a river (Figure 18). The surface velocity $\underline{v}$ of the water is revealed by watching a light floating object such as a leaf. You will notice two types of motion. First the leaf floats down the river following the streamlines of $\underline{v}$, but it may also rotate. This rotation may be quite fast near the bank, but slow or zero in midstream. Rotation occurs when the velocity, and hence the drag, is greater on one side of the leaf than the other.

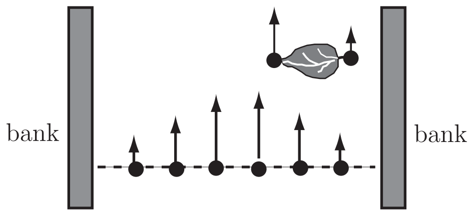


Figure 18: Rotation of a leaf in a stream

Note that for a two-dimensional vector field, such as $\underline{v}$ described here, curl $\underline{v}$ is perpendicular to the motion, and this is the direction of the axis about which the leaf rotates. The magnitude of curl $\underline{v}$ is related to the speed of rotation.

For motion in three dimensions a particle will tend to rotate about the axis that points in the direction of curl $\underline{v}$, with its magnitude measuring the speed of rotation.

If, at any point P, curl $\underline{v} = \underline{0}$ then there is no rotation at P and $\underline{v}$ is said to be **irrotational at P**. If curl $\underline{v} = \underline{0}$ at all points of the domain of $\underline{v}$ then the vector field is an **irrotational vector field**.

Key Point 5

Note that $\underline{F}$ is a vector field and that curl $\underline{F}$ is also a vector field.

Example 13

Find curl $\underline{v}$ for the following two-dimensional vector fields

(a) $\underline{v} = x\underline{i} + 2\underline{j}$ (b) $\underline{v} = -y\underline{i} + x\underline{j}$

If $\underline{v}$ represents the surface velocity of the flow of water, describe the motion of a floating leaf.

Solution

$$(a) \qquad \underline{\nabla} \times \underline{v} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & 2 & 0 \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(2)\right)\underline{i} + \left(\frac{\partial}{\partial z}(x) - \frac{\partial}{\partial x}(0)\right)\underline{j} + \left(\frac{\partial}{\partial x}(2) - \frac{\partial}{\partial y}(x)\right)\underline{k} = \underline{0}$$

A floating leaf will travel along the streamlines without rotating.

Solution (contd.)

(b)

$$\begin{aligned}
\underline{\nabla} \times \underline{v} &= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix} \\
&= \left(\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(x)\right)\underline{i} + \left(\frac{\partial}{\partial z}(-y) - \frac{\partial}{\partial x}(0)\right)\underline{j} + \left(\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y)\right)\underline{k} \\
&= 0\underline{i} + 0\underline{j} + 2\underline{k} = 2\underline{k}
\end{aligned}$$

A floating leaf will travel along the streamlines (anti-clockwise around the origin) and will rotate anticlockwise (as seen from above).
An analogy of the right-hand screw rule is that a positive (anti-clockwise) rotation in the xy plane represents a positive z-component of the curl. Similar results apply for the other components.

Example 14

(a) Find the curl of $\underline{u} = x^2\underline{i} + y^2\underline{j}$. When is $\underline{u}$ irrotational?

(b) Given $\underline{F} = (xy - xz)\underline{i} + 3x^2\underline{j} + yz\underline{k}$, find curl $\underline{F}$ at the origin $(0, 0, 0)$ and at the point $P = (1, 2, 3)$.

Solution

(a)

$$\begin{aligned}
\text{curl } \underline{u} &= \underline{\nabla} \times \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & y^2 & 0 \end{vmatrix} \\
&= \left(\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(y^2)\right)\underline{i} + \left(\frac{\partial}{\partial z}(x^2) - \frac{\partial}{\partial x}(0)\right)\underline{j} + \left(\frac{\partial}{\partial x}(y^2) - \frac{\partial}{\partial y}(x^2)\right)\underline{k} \\
&= 0\underline{i} + 0\underline{j} + 0\underline{k} = \underline{0}
\end{aligned}$$

curl $\underline{u} = \underline{0}$ so $\underline{u}$ is irrotational everywhere.

Solution (contd.)

(b)

$$\begin{aligned}
\text{curl } \underline{F} &= \underline{\nabla} \times \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy - xz & 3x^2 & yz \end{vmatrix} \\
&= \left(\frac{\partial}{\partial y}(yz) - \frac{\partial}{\partial z}(3x^2) \right) \underline{i} + \left(\frac{\partial}{\partial z}(xy - xz) - \frac{\partial}{\partial x}(yz) \right) \underline{j} \\
&\quad + \left(\frac{\partial}{\partial x}(3x^2) - \frac{\partial}{\partial y}(xy - xz) \right) \underline{k} \\
&= z\underline{i} - x\underline{j} + 5x\underline{k}
\end{aligned}$$

At the point $(0, 0, 0)$, curl $\underline{F} = \underline{0}$. At the point $(1, 2, 3)$, curl $\underline{F} = 3\underline{i} - \underline{j} + 5\underline{k}$.

Engineering Example 1

Current associated with a magnetic field

Introduction

In a magnetic field $\underline{B}$, an associated current is given by:

$$\underline{I} = \frac{1}{\mu_0}(\underline{\nabla} \times \underline{B})$$

Problem in words

Given the magnetic field $\underline{B} = B_0 x \underline{k}$ find the associated current $\underline{I}$.

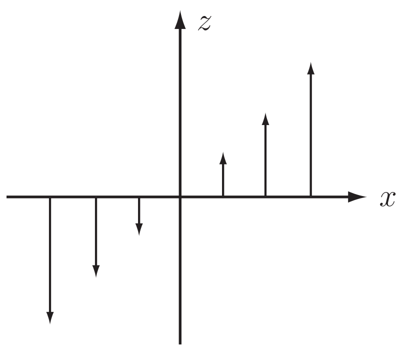


Figure 19: Magnetic field profile

Mathematical statement of problem

We need to evaluate the curl of $\underline{B}$.

Mathematical analysis

$$\underline{\nabla} \times \underline{B} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 0 & 0 & B_0 x \end{vmatrix}$$

$$= 0\underline{i} - B_0\underline{j} + 0\underline{k}$$

$$= -B_0\underline{j}$$

and so $\underline{I} = -\dfrac{B_0}{\mu_0}\underline{j}$.

Interpretation

The current is perpendicular to the field and to the direction of variation of the field.

Task

Find the curl of the following two-dimensional vector field (a) in general terms and (b) at the point $(1, 2)$.

$\underline{F}_2 = y^2\underline{i} + xy\underline{j}$

Your solution

Answer

(a) $\underline{\nabla} \times F_2 = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2 & xy & 0 \end{vmatrix} = 0\underline{i} + 0\underline{j} + (y - 2y)\underline{k} = -y\underline{k}$

(b) $-2\underline{k}$

Exercises

1. Find the curl of each of the following two-dimensional vector fields. Give each in general terms and also at the point $(1, 2)$.

   (a) $\underline{F}_1 = 2x\underline{i} + 2y\underline{j}$

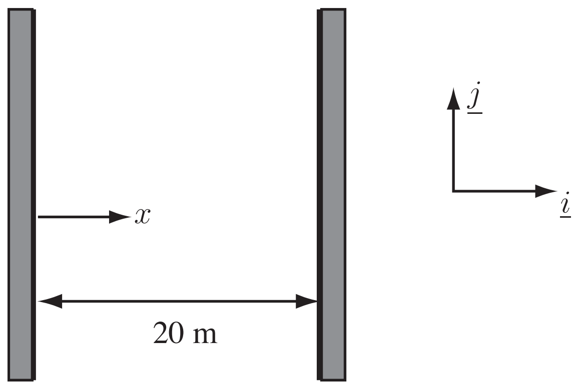
   (b) $\underline{F}_3 = x^2y^3\underline{i} - x^3y^2\underline{j}$

2. Find the curl of each of the following three-dimensional vector fields. Give each in general terms and also at the point $(2, 1, 3)$.

   (a) $\underline{F}_1 = y^2z^3\underline{i} + 2xyz^3\underline{j} + 3xy^2z^2\underline{k}$

   (b) $\underline{F}_2 = (xy + z^2)\underline{i} + x^2\underline{j} + (xz - 2)\underline{k}$

3. The surface water velocity on a straight uniform river 20 metres wide is modelled by the vector $\underline{v} = \frac{1}{50}x(20 - x)\underline{j}$ where x is the distance from the west bank (see diagram).



   (a) Find the velocity $\underline{v}$ at each bank and at midstream.

   (b) Find $\underline{\nabla} \times \underline{v}$ at each bank and at midstream.

4. The velocity field on the surface of an emptying bathroom sink can be modelled by two functions, the first describing the swirling vortex of radius a near the plughole and the second describing the more gently rotating fluid outside the vortex region. These functions are

$$\underline{u}(x, y) = w(-y\underline{i} + x\underline{j}), \qquad \left(\sqrt{x^2 + y^2} \leq a\right)$$

$$\underline{v}(x, y) = \frac{wa^2(-y\underline{i} + x\underline{j})}{x^2 + y^2} \qquad \left(\sqrt{x^2 + y^2} \geq a\right)$$

   Find (a) curl $\underline{u}$ and (b)curl $\underline{v}$.

Answers

1. (a) $\underline{0}$; $\underline{0}$ (b) $-6x^2y^2\underline{k}$, $-24\underline{k}$

2. (a) $\underline{0}$; $\underline{0}$ (b) $z\underline{j} + x\underline{k}$, $3\underline{j} + 2\underline{k}$

3. (a) $\underline{0}$; $\underline{0}$; $2\underline{j}$, (b) $+0.4\underline{k}$; $-0.4\underline{k}$; $\underline{0}$

4. (a) $2w\underline{k}$; (b) $\underline{0}$

4. The Laplacian

The Laplacian of a function ϕ is written as $\nabla^2\phi$ and is defined as: Laplacian $\phi =$ div grad ϕ, that is

$$\begin{aligned}\nabla^2\phi &= \underline{\nabla}\cdot\underline{\nabla}\phi \\ &= \underline{\nabla}\cdot\left(\frac{\partial\phi}{\partial x}\underline{i}+\frac{\partial\phi}{\partial y}\underline{j}+\frac{\partial\phi}{\partial z}\underline{k}\right) \\ &= \frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}\end{aligned}$$

The equation $\nabla^2\phi = 0$, that is $\dfrac{\partial^2\phi}{\partial x^2}+\dfrac{\partial^2\phi}{\partial y^2}+\dfrac{\partial^2\phi}{\partial z^2}=0$ is known as Laplace's equation and has applications in many branches of engineering including Heat Flow, Electrical and Magnetic Fields and Fluid Mechanics.

Example 15

Find the Laplacian of $u = x^2y^2z + 2xz$.

Solution

$\nabla^2 u = \dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}+\dfrac{\partial^2 u}{\partial z^2} = 2y^2z + 2x^2z + 0 = 2(x^2+y^2)z$

5. Examples involving grad, div, curl and the Laplacian

The vector differential operators can be combined in several ways as the following examples show.

Example 16

If $\underline{A} = 2yz\underline{i} - x^2y\underline{j} + xz^2\underline{k}$, $\underline{B} = x^2\underline{i} + yz\underline{j} - xy\underline{k}$ and $\phi = 2x^2yz^3$, find

(a) $(\underline{A}\cdot\underline{\nabla})\phi$ (b) $\underline{A}\cdot\underline{\nabla}\phi$ (c) $\underline{B}\times\underline{\nabla}\phi$ (d) $\nabla^2\phi$

Solution

(a)

$$\begin{aligned}(\underline{A}\cdot\underline{\nabla})\phi &= \left[(2yz\underline{i} - x^2y\underline{j} + xz^2\underline{k})\cdot(\frac{\partial}{\partial x}\underline{i} + \frac{\partial}{\partial y}\underline{j} + \frac{\partial}{\partial z}\underline{k})\right]\phi \\ &= \left[2yz\frac{\partial}{\partial x} - x^2y\frac{\partial}{\partial y} + xz^2\frac{\partial}{\partial z}\right]2x^2yz^3 \\ &= 2yz\frac{\partial}{\partial x}(2x^2yz^3) - x^2y\frac{\partial}{\partial y}(2x^2yz^3) + xz^2\frac{\partial}{\partial z}(2x^2yz^3) \\ &= 2yz(4xyz^3) - x^2y(2x^2z^3) + xz^2(6x^2yz^2) \\ &= 8xy^2z^4 - 2x^4yz^3 + 6x^3yz^4\end{aligned}$$

(b)

$$\begin{aligned}\underline{\nabla}\phi &= \frac{\partial}{\partial x}(2x^2yz^3)\underline{i} + \frac{\partial}{\partial y}(2x^2yz^3)\underline{j} + \frac{\partial}{\partial z}(2x^2yz^3)\underline{k} \\ &= 4xyz^3\underline{i} + 2x^2z^3\underline{j} + 6x^2yz^2\underline{k}\end{aligned}$$

So

$$\begin{aligned}\underline{A}\cdot\underline{\nabla}\phi &= \left(2yz\underline{i} - x^2y\underline{j} + xz^2\underline{k}\right)\cdot(4xyz^3\underline{i} + 2x^2z^3\underline{j} + 6x^2yz^2\underline{k}) \\ &= 8xy^2z^4 - 2x^4yz^3 + 6x^3yz^4\end{aligned}$$

(c) $\underline{\nabla}\phi = 4xyz^3\underline{i} + 2x^2z^3\underline{j} + 6x^2yz^2\underline{k}$ so

$$\begin{aligned}\underline{B}\times\underline{\nabla}\phi &= \begin{vmatrix}\underline{i} & \underline{j} & \underline{k} \\ x^2 & yz & -xy \\ 4xyz^3 & 2x^2z^3 & 6x^2yz^2\end{vmatrix} \\ &= \underline{i}(6x^2y^2z^3 + 2x^3yz^3) + \underline{j}(-4x^2y^2z^3 - 6x^4yz^2) + \underline{k}(2x^4z^3 - 4xy^2z^4)\end{aligned}$$

(d) $\nabla^2\phi = \dfrac{\partial^2}{\partial x^2}(2x^2yz^3) + \dfrac{\partial^2}{\partial y^2}(2x^2yz^3) + \dfrac{\partial^2}{\partial z^2}(2x^2yz^3) = 4yz^3 + 0 + 12x^2yz$

Example 17

For each of the expressions below determine whether the quantity can be formed and, if so, whether it is a scalar or a vector.

(a) grad(div $\underline{A}$)

(b) grad(grad ϕ)

(c) curl(div $\underline{F}$)

(d) div [curl ($\underline{A}\times$grad ϕ)]

Solution

(a) $\underline{A}$ is a vector and div$\underline{A}$ can be calculated and is a scalar. Hence, grad(div $\underline{A}$) can be formed and is a vector.

(b) ϕ is a scalar so grad ϕ can be formed and is a vector. As grad ϕ is a vector, it is not possible to take grad(grad ϕ).

(c) $\underline{F}$ is a vector and hence div $\underline{F}$ is a scalar. It is not possible to take the curl of a scalar so curl(div $\underline{F}$) does not exist.

(d) ϕ is a scalar so grad ϕ exists and is a vector. $\underline{A}\times$grad ϕ exists and is also a vector as is curl $\underline{A}\times$grad ϕ. The divergence can be taken of this last vector to give div [curl ($\underline{A}\times$grad ϕ)] which is a scalar.

6. Identities involving grad, div and curl

There are numerous identities involving the vector derivatives; a selection are given in Table 1.

Table 1

1	div($\phi\underline{A}$) = grad $\phi \cdot \underline{A} + \phi$ div $\underline{A}$	or	$\underline{\nabla} \cdot (\phi\underline{A}) = (\underline{\nabla}\phi) \cdot \underline{A} + \phi(\underline{\nabla} \cdot \underline{A})$
2	curl($\phi\underline{A}$) = grad $\phi \times \underline{A} + \phi$ curl $\underline{A}$	or	$\underline{\nabla} \times (\phi\underline{A}) = (\underline{\nabla}\phi) \times \underline{A} + \phi(\underline{\nabla} \times \underline{A})$
3	div ($\underline{A} \times \underline{B}$) = $\underline{B}\cdot$ curl $\underline{A} - \underline{A}\cdot$ curl $\underline{B}$	or	$\underline{\nabla} \cdot (\underline{A} \times \underline{B}) = \underline{B} \cdot (\underline{\nabla} \times \underline{A}) - \underline{A} \cdot (\underline{\nabla} \times \underline{B})$
4	curl ($\underline{A} \times \underline{B}$) = ($\underline{B}\cdot$ grad) $\underline{A}$ − ($\underline{A}\cdot$ grad) $\underline{B}$ $+\underline{A}$ div $\underline{B} - \underline{B}$ div $\underline{A}$	or	$\underline{\nabla} \times (\underline{A} \times \underline{B}) = (\underline{B} \cdot \underline{\nabla})\underline{A} - (\underline{A} \cdot \underline{\nabla})\underline{B}$ $+\underline{A}\ \underline{\nabla} \cdot \underline{B} - \underline{B}\ \underline{\nabla} \cdot \underline{A}$
5	grad ($\underline{A} \cdot \underline{B}$) = ($\underline{B}\cdot$ grad) $\underline{A}$ + ($\underline{A}\cdot$ grad) $\underline{B}$ $+\underline{A}\times$ curl $\underline{B} + \underline{B}\times$ curl $\underline{A}$	or	$\underline{\nabla}(\underline{A} \cdot \underline{B}) = (\underline{B} \cdot \underline{\nabla})\underline{A} + (\underline{A} \cdot \underline{\nabla})\underline{B}$ $+\underline{A} \times (\underline{\nabla} \times \underline{B}) + \underline{B} \times (\underline{\nabla} \times \underline{A})$
6	curl (grad ϕ) = $\underline{0}$	or	$\underline{\nabla} \times (\underline{\nabla}\phi) = \underline{0}$
7	div (curl $\underline{A}$) = 0	or	$\underline{\nabla} \cdot (\underline{\nabla} \times \underline{A}) = \underline{0}$

Example 18

Show for any vector field $\underline{A} = A_1\underline{i} + A_2\underline{j} + A_3\underline{k}$, that div curl $\underline{A} = \underline{0}$.

Solution

$$\begin{aligned}
\text{div curl } \underline{A} &= \text{div} \begin{vmatrix} \underline{i} & \underline{j} & \underline{j} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix} \\
&= \text{div} \left[\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \right) \underline{i} + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} \right) \underline{j} + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \right) \underline{k} \right] \\
&= \frac{\partial}{\partial x} \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \right) + \frac{\partial}{\partial y} \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} \right) + \frac{\partial}{\partial z} \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \right) \\
&= \frac{\partial^2 A_3}{\partial x \partial y} - \frac{\partial^2 A_2}{\partial z \partial x} + \frac{\partial^2 A_1}{\partial y \partial z} - \frac{\partial^2 A_3}{\partial y \partial x} + \frac{\partial^2 A_2}{\partial z \partial x} - \frac{\partial^2 A_1}{\partial z \partial y} \\
&= 0
\end{aligned}$$

N.B. This assumes $\dfrac{\partial^2 A_3}{\partial x \partial y} = \dfrac{\partial^2 A_3}{\partial y \partial x}$ etc.

Example 19

Verify identity 1 for the vector $\underline{A} = 2xy\underline{i} - 3z\underline{k}$ and the function $\phi = xy^2$.

Solution

$\phi\underline{A} = 2x^2y^3\underline{i} - 3xy^2z\underline{k}$ so

$\underline{\nabla} \cdot \phi\underline{A} = \underline{\nabla} \cdot \left(2x^2y^3\underline{i} - 3xy^2z\underline{k}\right) = \dfrac{\partial}{\partial x}(2x^2y^3) + \dfrac{\partial}{\partial z}(-3xy^2z) = 4xy^3 - 3xy^2$

So LHS $= 4xy^3 - 3xy^2$.

$\underline{\nabla}\phi = \dfrac{\partial}{\partial x}(xy^2)\underline{i} + \dfrac{\partial}{\partial y}(xy^2)\underline{j} + \dfrac{\partial}{\partial z}(xy^2)\underline{k} = y^2\underline{i} + 2xy\underline{j}$ so

$(\underline{\nabla}\phi) \cdot \underline{A} = (y^2\underline{i} + 2xy\underline{j}) \cdot (2xy\underline{i} - 3z\underline{k}) = 2xy^3$

$\underline{\nabla} \cdot \underline{A} = \underline{\nabla} \cdot (2xy\underline{i} - 3z\underline{k}) = 2y - 3$ so $\phi\underline{\nabla} \cdot \underline{A} = 2xy^3 - 3xy^2$ giving

$(\underline{\nabla}\phi) \cdot \underline{A} + \phi(\underline{\nabla} \cdot \underline{A}) = 2xy^3 + (2xy^3 - 3xy^2) = 4xy^3 - 3xy^2$

So RHS $= 4xy^3 - 3xy^2 =$ LHS.

So $\underline{\nabla} \cdot (\phi\underline{A}) = (\underline{\nabla}\phi) \cdot \underline{A} + \phi(\underline{\nabla} \cdot \underline{A})$ in this case.

Task

If $\underline{F} = x^2y\underline{i} - 2xz\underline{j} + 2yz\underline{k}$, find

(a) $\underline{\nabla} \cdot \underline{F}$

(b) $\underline{\nabla} \times \underline{F}$

(c) $\underline{\nabla}(\underline{\nabla} \cdot \underline{F})$

(d) $\underline{\nabla} \cdot (\underline{\nabla} \times \underline{F})$

(e) $\underline{\nabla} \times (\underline{\nabla} \times \underline{F})$

Your solution

Answer

(a) $2xy + 2y$,

(b) $(2x + 2z)\underline{i} - (x^2 + 2z)\underline{k}$,

(c) $2y\underline{i} + (2 + 2x)\underline{j}$ (using answer to (a)),

(d) 0 (using answer to (b)),

(e) $(2 + 2x)\underline{j}$ (using answer to (b))

Task

If $\phi = 2xz - y^2 z$, find

(a) $\underline{\nabla}\phi$

(b) $\nabla^2\phi = \underline{\nabla} \cdot (\underline{\nabla}\phi)$

(c) $\underline{\nabla} \times (\underline{\nabla}\phi)$

Your solution

Answer

(a) $2z\underline{i} - 2yz\underline{j} + (2x - y^2)\underline{k}$, (b) $-2z$, (c) $\underline{0}$ where (b) and (c) use the answer to (a).

Exercise

Which of the following combinations of grad, div and curl can be formed? If a quantity can be formed, state whether it is a scalar or a vector.

(a) div (grad ϕ)

(b) div (div $\underline{A}$)

(c) curl (curl $\underline{F}$)

(d) div (curl $\underline{F}$)

(e) curl (grad ϕ)

(f) curl (div $\underline{A}$)

(g) div ($\underline{A} \cdot \underline{B}$)

(h) grad ($\phi_1\phi_2$)

(i) curl (div ($\underline{A}\times$ grad ϕ))

Answers

(a), (d) are scalars;

(c), (e), (h) are vectors;

(b), (f), (g) and (i) are not defined.

Orthogonal Curvilinear Coordinates 28.3

Introduction

The derivatives div, grad and curl from Section 28.2 can be carried out using coordinate systems other than the rectangular Cartesian coordinates. This Section shows how to calculate these derivatives in other coordinate systems. Two coordinate systems - cylindrical polar coordinates and spherical polar coordinates - will be illustrated.

Prerequisites

Before starting this Section you should . . .

- be able to find the gradient, divergence and curl of a field in Cartesian coordinates
- be familiar with polar coordinates

Learning Outcomes

On completion you should be able to . . .

- find the divergence, gradient or curl of a vector or scalar field expressed in terms of orthogonal curvilinear coordinates

1. Orthogonal curvilinear coordinates

The results shown in Section 28.2 have been given in terms of the familiar Cartesian (x, y, z) coordinate system. However, other coordinate systems can be used to better describe some physical situations. A set of coordinates $u = u(x, y, z)$, $v = v(x, y, z)$ and $w = w(x, y, z)$ where the directions at any point indicated by u, v and w are orthogonal (perpendicular) to each other is referred to as a set of **orthogonal curvilinear coordinates**. With each coordinate is associated a **scale factor** h_u, h_v or h_w respectively where $h_u = \sqrt{\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2}$ (with similar expressions for h_v and h_w). The scale factor gives a measure of how a change in the coordinate changes the position of a point.

Two commonly-used sets of orthogonal curvilinear coordinates are **cylindrical polar coordinates** and **spherical polar coordinates**. These are similar to the plane polar coordinates introduced in HELM 17.2 but represent extensions to three dimensions.

Cylindrical polar coordinates

This corresponds to plane polar (ρ, ϕ) coordinates with an added z-coordinate directed out of the xy plane. Normally the variables ρ and ϕ are used instead of r and θ to give the three coordinates ρ, ϕ and z. A cylinder has equation $\rho =$ constant.
The relationship between the coordinate systems is given by

$$x = \rho\cos\phi \qquad y = \rho\sin\phi \qquad z = z$$

(i.e. the same z is used by the two coordinate systems). See Figure 20(a).

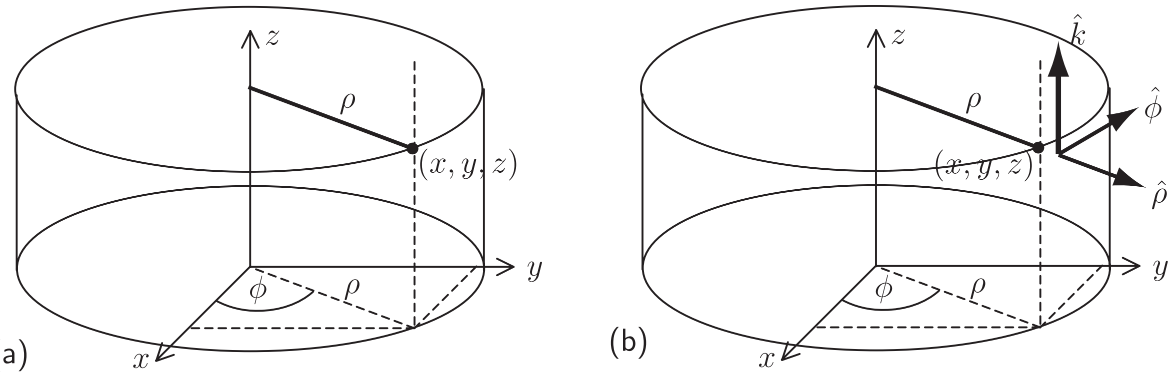


Figure 20: Cylindrical polar coordinates

The scale factors h_ρ, h_ϕ and h_z are given as follows

$$h_\rho = \sqrt{\left(\frac{\partial x}{\partial \rho}\right)^2 + \left(\frac{\partial y}{\partial \rho}\right)^2 + \left(\frac{\partial z}{\partial \rho}\right)^2} = \sqrt{(\cos\phi)^2 + (\sin\phi)^2 + 0} = 1$$

$$h_\phi = \sqrt{\left(\frac{\partial x}{\partial \phi}\right)^2 + \left(\frac{\partial y}{\partial \phi}\right)^2 + \left(\frac{\partial z}{\partial \phi}\right)^2} = \sqrt{(-\rho\sin\phi)^2 + (\rho\cos\phi)^2 + 0} = \rho$$

$$h_z = \sqrt{\left(\frac{\partial x}{\partial z}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2 + \left(\frac{\partial z}{\partial z}\right)^2} = \sqrt{(0^2 + 0^2 + 1^2)} = 1$$

Spherical polar coordinates

In this system a point is referred to by its distance from the origin r and two angles ϕ and θ. The angle θ is the angle between the positive z-axis and the line from the origin to the point. The angle ϕ is the angle from the x-axis to the projection of the point in the xy plane.

A useful analogy is of latitude, longitude and height on Earth.

- The variable r plays the role of height (but height measured above the centre of Earth rather than from the surface).
- The variable θ plays the role of latitude but is modified so that $\theta = 0$ represents the North Pole, $\theta = 90^\circ = \dfrac{\pi}{2}$ represents the equator and $\theta = 180^\circ = \pi$ represents the South Pole.
- The variable ϕ plays the role of longitude.

A sphere has equation $r =$ constant.
The relationship between the coordinate systems is given by

$x = r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi \qquad z = r\cos\theta.$ See Figure 21.

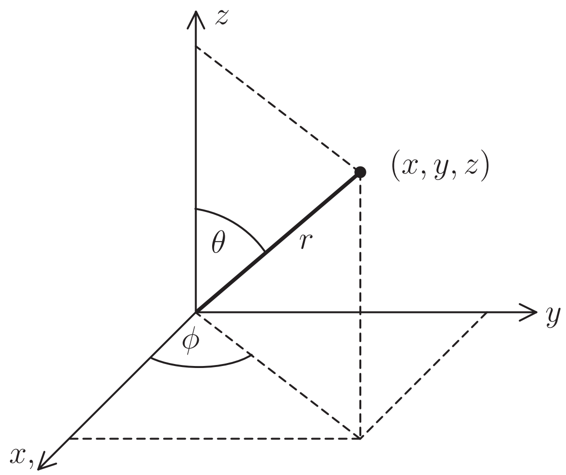


Figure 21: Spherical polar coordinates

The scale factors h_r, h_θ and h_ϕ are given by

$$h_r = \sqrt{\left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial r}\right)^2} = \sqrt{(\sin\theta\cos\phi)^2 + (\sin\theta\sin\phi)^2 + (\cos\theta)^2} = 1$$

$$h_\theta = \sqrt{\left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2} = \sqrt{(r\cos\theta\cos\phi)^2 + (r\cos\theta\sin\phi)^2 + (-r\sin\theta)^2} = r$$

$$h_\phi = \sqrt{\left(\frac{\partial x}{\partial \phi}\right)^2 + \left(\frac{\partial y}{\partial \phi}\right)^2 + \left(\frac{\partial z}{\partial \phi}\right)^2} = \sqrt{(-r\sin\theta\sin\phi)^2 + (r\sin\theta\cos\phi)^2 + 0} = r\sin\theta$$

2. Vector derivatives in orthogonal coordinates

Given an orthogonal coordinate system u, v, w with unit vectors $\underline{\hat{u}}$, $\underline{\hat{v}}$ and $\underline{\hat{w}}$ and scale factors, h_u, h_v and h_w, it is possible to find the derivatives $\underline{\nabla} f$, $\underline{\nabla} \cdot \underline{F}$ and $\underline{\nabla} \times \underline{F}$.

It is found that

$$\text{grad } f = \underline{\nabla} f = \frac{1}{h_u}\frac{\partial f}{\partial u}\underline{\hat{u}} + \frac{1}{h_v}\frac{\partial f}{\partial v}\underline{\hat{v}} + \frac{1}{h_w}\frac{\partial f}{\partial w}\underline{\hat{w}}$$

If $\underline{F} = F_u\underline{\hat{u}} + F_v\underline{\hat{v}} + F_w\underline{\hat{w}}$ then

$$\text{div } \underline{F} = \underline{\nabla} \cdot \underline{F} = \frac{1}{h_u h_v h_w}\left[\frac{\partial}{\partial u}(F_u h_v h_w) + \frac{\partial}{\partial v}(F_v h_u h_w) + \frac{\partial}{\partial w}(F_w h_u h_v)\right]$$

Also if $\underline{F} = F_u\underline{\hat{u}} + F_v\underline{\hat{v}} + F_w\underline{\hat{w}}$ then

$$\text{curl } \underline{F} = \underline{\nabla} \times \underline{F} = \frac{1}{h_u h_v h_w}\begin{vmatrix} h_u\underline{\hat{u}} & h_v\underline{\hat{v}} & h_w\underline{\hat{w}} \\ \dfrac{\partial}{\partial u} & \dfrac{\partial}{\partial v} & \dfrac{\partial}{\partial w} \\ h_u F_u & h_v F_v & h_w F_w \end{vmatrix}$$

Key Point 6

In orthogonal curvilinear coordinates, the vector derivatives $\underline{\nabla} f$, $\underline{\nabla} \cdot \underline{F}$ and $\underline{\nabla} \times \underline{F}$ include the scale factors h_u, h_v and h_w.

3. Cylindrical polar coordinates

In cylindrical polar coordinates (ρ, ϕ, z), the three unit vectors are $\underline{\hat{\rho}}$, $\underline{\hat{\phi}}$ and $\underline{\hat{z}}$ (see Figure 20(b) on page 38) with scale factors

$$h_\rho = 1,\ h_\phi = \rho,\ h_z = 1.$$

The quantities ρ and ϕ are related to x and y by $x = \rho\cos\phi$ and $y = \rho\sin\phi$. The unit vectors are $\underline{\hat{\rho}} = \cos\phi\underline{i} + \sin\phi\underline{j}$ and $\underline{\hat{\phi}} = -\sin\phi\underline{i} + \cos\phi\underline{j}$. In cylindrical polar coordinates,

$$\text{grad } f = \underline{\nabla} f = \frac{\partial f}{\partial \rho}\underline{\hat{\rho}} + \frac{1}{\rho}\frac{\partial f}{\partial \phi}\underline{\hat{\phi}} + \frac{\partial f}{\partial z}\underline{\hat{z}}$$

The scale factor ρ is necessary in the ϕ-component because the derivatives with respect to ϕ are distorted by the distance from the axis $\rho = 0$.

If $\underline{F} = F_\rho \underline{\hat{\rho}} + F_\phi \underline{\hat{\phi}} + F_z \underline{\hat{z}}$ then

$$\text{div } \underline{F} = \underline{\nabla} \cdot \underline{F} = \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(\rho F_\rho) + \frac{\partial}{\partial \phi}(F_\phi) + \frac{\partial}{\partial z}(\rho F_z)\right]$$

$$\text{curl } \underline{F} = \underline{\nabla} \times \underline{F} = \frac{1}{\rho}\begin{vmatrix} \underline{\hat{\rho}} & \rho\underline{\hat{\phi}} & \underline{\hat{z}} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ F_\rho & \rho F_\phi & F_z \end{vmatrix}.$$

Example 20

Working in cylindrical polar coordinates, find $\underline{\nabla} f$ for $f = \rho^2 + z^2$

Solution

If $f = \rho^2 + z^2$ then $\dfrac{\partial f}{\partial \rho} = 2\rho, \dfrac{\partial f}{\partial \phi} = 0$ and $\dfrac{\partial f}{\partial z} = 2z$ so $\underline{\nabla} f = 2\rho\underline{\hat{\rho}} + 2z\underline{\hat{z}}$.

Example 21

Working in cylindrical polar coordinates find

(a) $\underline{\nabla} f$ for $f = \rho^3 \sin\phi$

(b) Show that the result for (a) is consistent with that found working in Cartesian coordinates.

Solution

(a) If $f = \rho^3 \sin\phi$ then $\dfrac{\partial f}{\partial \rho} = 3\rho^2 \sin\phi$, $\dfrac{\partial f}{\partial \phi} = \rho^3 \cos\phi$ and $\dfrac{\partial f}{\partial z} = 0$ and hence,

$$\underline{\nabla} f = 3\rho^2 \sin\phi \underline{\hat{\rho}} + \rho^2 \cos\phi \underline{\hat{\phi}}.$$

(b) $f = \rho^3 \sin\phi = \rho^2 \rho \sin\phi = (x^2 + y^2)y = x^2 y + y^3$ so $\underline{\nabla} f = 2xy\underline{i} + (x^2 + 3y^2)\underline{j}$.
Using cylindrical polar coordinates, from (a) we have

$$\begin{aligned} \underline{\nabla} f &= 3\rho^2 \sin\phi \underline{\hat{\rho}} + \rho^2 \cos\phi \underline{\hat{\phi}} \\ &= 3\rho^2 \sin\phi(\cos\phi \underline{i} + \sin\phi \underline{j}) + \rho^2 \cos\phi(-\sin\phi \underline{i} + \cos\phi \underline{j}) \\ &= \left[3\rho^2 \sin\phi\cos\phi - \rho^2 \sin\phi\cos\phi\right]\underline{i} + \left[3\rho^2 \sin^2\phi + \rho^2\cos^2\phi\right]\underline{j} \\ &= \left[2\rho^2 \sin\phi\cos\phi\right]\underline{i} + \left[3\rho^2\sin^2\phi + \rho^2\cos^2\phi\right]\underline{j} = 2xy\underline{i} + (3y^2 + x^2)\underline{j} \end{aligned}$$

So the results using Cartesian and cylindrical polar coordinates are consistent.

Example 22

Find $\underline{\nabla} \cdot \underline{F}$ for $\underline{F} = F_\rho \hat{\underline{\rho}} + F_\phi \hat{\underline{\phi}} + F_z \hat{\underline{z}} = \rho^3 \hat{\underline{\rho}} + \rho z \hat{\underline{\phi}} + \rho z \sin\phi \hat{\underline{z}}$. Show that the results are consistent with those found using Cartesian coordinates.

Solution

Here, $F_\rho = \rho^3$, $F_\phi = \rho z$ and $F_z = \rho z \sin\phi$ so

$$\begin{aligned} \underline{\nabla} \cdot \underline{F} &= \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(\rho F_\rho) + \frac{\partial}{\partial \phi}(F_\phi) + \frac{\partial}{\partial z}(\rho F_z)\right] \\ &= \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(\rho^4) + \frac{\partial}{\partial \phi}(\rho z) + \frac{\partial}{\partial z}(\rho^2 z \sin\phi)\right] \\ &= \frac{1}{\rho}\left[4\rho^3 + 0 + \rho^2 \sin\phi\right] \\ &= 4\rho^2 + \rho \sin\phi \end{aligned}$$

Converting to Cartesian coordinates,

$$\begin{aligned} \underline{F} &= F_\rho \hat{\underline{\rho}} + F_\phi \hat{\underline{\phi}} + F_z \hat{\underline{z}} = \rho^3 \hat{\underline{\rho}} + \rho z \hat{\underline{\phi}} + \rho z \sin\phi \hat{\underline{z}} \\ &= \rho^3(\cos\phi \underline{i} + \sin\phi \underline{j}) + \rho z(-\sin\phi \underline{i} + \cos\phi \underline{j}) + \rho z \sin\phi \underline{k} \\ &= (\rho^3 \cos\phi - \rho z \sin\phi)\underline{i} + (\rho^3 \sin\phi + \rho z \cos\phi)\underline{j} + \rho z \sin\phi \underline{k} \\ &= \left[\rho^2(\rho\cos\phi) - \rho\sin\phi z\right]\underline{i} + \left[\rho^2(\rho\sin\phi) + \rho\cos\phi z\right]\underline{j} + \rho\sin\phi z \underline{k} \\ &= \left[(x^2+y^2)x - yz\right]\underline{i} + \left[(x^2+y^2)y + xz\right]\underline{j} + yz\underline{k} \\ &= (x^3 + xy^2 - yz)\underline{i} + (x^2 y + y^3 + xz)\underline{j} + yz\underline{k} \end{aligned}$$

So

$$\begin{aligned} \underline{\nabla} \cdot \underline{F} &= \frac{\partial}{\partial x}(x^3 + xy^2 - yz) + \frac{\partial}{\partial y}(x^2 y + y^3 + xz) + \frac{\partial}{\partial z}(yz) \\ &= (3x^2 + y^2) + (x^2 + 3y^2) + y = 4x^2 + 4y^2 + y \\ &= 4(x^2 + y^2) + y \\ &= 4\rho^2 + \rho\sin\phi \end{aligned}$$

So $\underline{\nabla} \cdot \underline{F}$ is the same in both coordinate systems.

Example 23

Find $\underline{\nabla} \times \underline{F}$ for $\underline{F} = \rho^2 \underline{\hat{\rho}} + z\sin\phi \underline{\hat{\phi}} + 2z\cos\phi \underline{\hat{z}}$.

Solution

$$\underline{\nabla} \times \underline{F} = \frac{1}{\rho}\begin{vmatrix} \underline{\hat{\rho}} & \rho\underline{\hat{\phi}} & \underline{\hat{z}} \\ \dfrac{\partial}{\partial\rho} & \dfrac{\partial}{\partial\phi} & \dfrac{\partial}{\partial z} \\ F_\rho & \rho F_\phi & F_z \end{vmatrix} = \frac{1}{\rho}\begin{vmatrix} \underline{\hat{\rho}} & \rho\underline{\hat{\phi}} & \underline{\hat{z}} \\ \dfrac{\partial}{\partial\rho} & \dfrac{\partial}{\partial\phi} & \dfrac{\partial}{\partial z} \\ \rho^2 & \rho z\sin\phi & 2z\cos\phi \end{vmatrix}$$

$$= \frac{1}{\rho}\left[\underline{\hat{\rho}}\left[\frac{\partial}{\partial\phi}(2z\cos\phi) - \frac{\partial}{\partial z}(\rho z\sin\phi)\right] + \rho\underline{\hat{\phi}}\left[\frac{\partial}{\partial z}\rho^2 - \frac{\partial}{\partial\rho}(2z\cos\phi)\right] + \underline{\hat{z}}\left[\frac{\partial}{\partial\rho}(\rho z\sin\phi) - \frac{\partial}{\partial\phi}\rho^2\right]\right]$$

$$= \frac{1}{\rho}\left[\underline{\hat{\rho}}(-2z\sin\phi - \rho\sin\phi) + \rho\underline{\hat{\phi}}(0) + \underline{\hat{z}}(z\sin\phi)\right]$$

$$= -\frac{(2z\sin\phi + \rho\sin\phi)}{\rho}\underline{\hat{\rho}} + \frac{z\sin\phi}{\rho}\underline{\hat{z}}$$

Engineering Example 2

Divergence of a magnetic field

Introduction

A magnetic field $\underline{B}$ must satisfy $\underline{\nabla} \cdot \underline{B} = 0$. An associated current is given by:

$$\underline{I} = \frac{1}{\mu_0}(\underline{\nabla} \times \underline{B})$$

Problem in words

For the magnetic field (in cylindrical polar coordinates ρ, ϕ, z)

$$\underline{B} = B_0 \frac{\rho}{1+\rho^2}\underline{\hat{\phi}} + \alpha\underline{\hat{z}}$$

show that the divergence of $\underline{B}$ is zero and find the associated current.

Mathematical statement of problem

We must

(a) show that $\underline{\nabla} \cdot \underline{B} = 0$ (b) find the current $\underline{I} = \dfrac{1}{\mu_0}(\underline{\nabla} \times \underline{B})$

Mathematical analysis

(a) Express $\underline{B}$ as (B_ρ, B_ϕ, B_z); then

$$\begin{aligned}
\underline{\nabla} \cdot \underline{B} &= \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(\rho B_\rho) + \frac{\partial}{\partial \phi}(B_\phi) + \frac{\partial}{\partial z}(\rho B_z)\right] \\
&= \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(0) + \frac{\partial}{\partial \phi}\left(B_0 \frac{\rho}{1+\rho^2}\right) + \rho\frac{\partial}{\partial z}(\alpha)\right] \\
&= \frac{1}{\rho}[0+0+0] = 0 \qquad \text{as required.}
\end{aligned}$$

(b) To find the current evaluate

$$\begin{aligned}
\underline{I} = \frac{1}{\mu_0}(\underline{\nabla} \times \underline{B}) &= \frac{1}{\mu_0}\frac{1}{\rho}\begin{vmatrix} \hat{\underline{\rho}} & \rho\hat{\underline{\phi}} & \hat{\underline{z}} \\ \dfrac{\partial}{\partial \rho} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ B_\rho & \rho B_\phi & B_z \end{vmatrix} = \begin{vmatrix} \hat{\underline{\rho}} & \rho\hat{\underline{\phi}} & \hat{\underline{z}} \\ \dfrac{\partial}{\partial \rho} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ 0 & B_0\dfrac{\rho^2}{1+\rho^2} & \alpha \end{vmatrix} \\
&= \frac{1}{\mu_0 \rho}\left[0\hat{\underline{\rho}} + 0\rho\hat{\underline{\phi}} + B_0\frac{\partial}{\partial \rho}\left(\frac{\rho^2}{1+\rho^2}\right)\hat{\underline{z}}\right] \\
&= \frac{1}{\mu_0 \rho}B_0\left[\frac{2\rho}{(1+\rho^2)^2}\right]\hat{\underline{z}} = \frac{2B_0}{\mu_0(1+\rho^2)^2}\,\hat{\underline{z}}
\end{aligned}$$

Interpretation

The magnetic field is in the form of a helix with the current pointing along its axis (Fig 22). Such an arrangement is often used for the magnetic containment of charged particles in a fusion reactor.

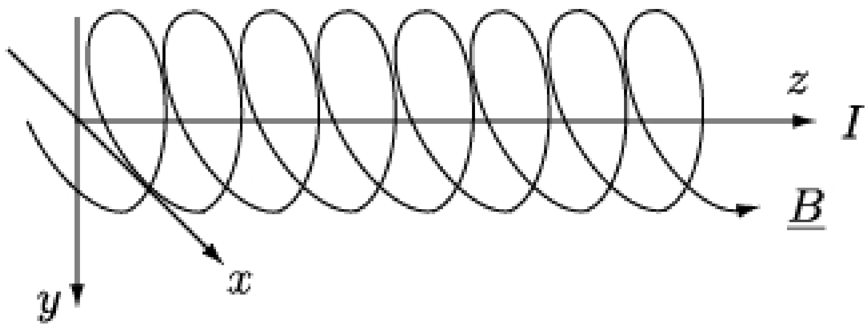


Figure 22: The magnetic field forms a helix

Example 24

A magnetic field $\underline{B}$ is given by $\underline{B} = \rho^{-2}\hat{\underline{\phi}} + k\hat{\underline{z}}$. Find $\underline{\nabla} \cdot \underline{B}$ and $\underline{\nabla} \times \underline{B}$.

Solution

$$\underline{\nabla} \cdot \underline{B} = \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(0) + \frac{\partial}{\partial \phi}(\rho^{-2}) + \frac{\partial}{\partial z}(k\rho)\right] = \frac{1}{\rho}[0+0+0] = 0$$

$$\underline{\nabla} \times \underline{B} = \frac{1}{\rho}\begin{vmatrix} \hat{\underline{\rho}} & \rho\hat{\underline{\phi}} & \hat{\underline{z}} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ B_\rho & \rho B_\phi & B_z \end{vmatrix} = \frac{1}{\rho}\begin{vmatrix} \hat{\underline{\rho}} & \rho\hat{\underline{\phi}} & \hat{\underline{z}} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ 0 & \rho^{-1} & k \end{vmatrix}$$

$$= -\frac{1}{\rho^3}\,\hat{\underline{z}}$$

All magnetic fields satisfy $\underline{\nabla} \cdot \underline{B} = 0$ i.e. an absence of magnetic monopoles.
Note that there is a class of magnetic fields known as potential fields that satisfy $\underline{\nabla} \times \underline{B} = \underline{0}$

Task

Using cylindrical polar coordinates, find $\underline{\nabla} f$ for $f = \rho^2 z \sin\phi$

Your solution

Answer

$$\frac{\partial}{\partial \rho}[\rho^2 z \sin\phi]\hat{\underline{\rho}} + \frac{1}{\rho}\frac{\partial}{\partial \phi}[\rho^2 z \sin\phi]\hat{\underline{\phi}} + \frac{\partial}{\partial z}[\rho^2 z \sin\phi]\hat{\underline{z}} = 2\rho z \sin\phi\hat{\underline{\rho}} + \rho z \cos\phi\hat{\underline{\phi}} + \rho^2 \sin\phi\hat{\underline{z}}$$

Task

Using cylindrical polar coordinates, find $\underline{\nabla} f$ for $f = z \sin 2\phi$

Your solution

Answer

$$\frac{\partial}{\partial \rho}[z \sin 2\phi]\underline{\hat{\rho}} + \frac{1}{\rho}\frac{\partial}{\partial \phi}[z \sin 2\phi]\underline{\hat{\phi}} + \frac{\partial}{\partial z}[z \sin 2\phi]\underline{\hat{z}} = \frac{2}{\rho} z \cos 2\phi \underline{\hat{\phi}} + \sin 2\phi \underline{\hat{z}}$$

Task

Find $\underline{\nabla} \cdot \underline{F}$ for $\underline{F} = \rho \cos \phi \underline{\hat{\rho}} - \rho \sin \phi \underline{\hat{\phi}} + \rho z \underline{\hat{z}}$

i.e. $F_\rho = \rho \cos \phi, F_\phi = -\rho \sin \phi, F_z = \rho z$

(a) First find the derivatives $\dfrac{\partial}{\partial \rho}[\rho F_\rho],\ \dfrac{\partial}{\partial \phi}[F_\phi],\ \dfrac{\partial}{\partial z}[\rho F_z]$:

Your solution

Answer

$2\rho \cos \phi, \quad -\rho \cos \phi, \quad \rho^2$

(b) Now combine these to find $\underline{\nabla} \cdot \underline{F}$:

Your solution

Answer

$$\begin{aligned}\underline{\nabla} \cdot \underline{F} &= \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(\rho F_\rho) + \frac{\partial}{\partial \phi}(F_\phi) + \frac{\partial}{\partial z}(\rho F_z)\right] \\ &= \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(\rho^2 \cos\phi) + \frac{\partial}{\partial \phi}(-\rho \sin\phi) + \frac{\partial}{\partial z}(\rho^2 z)\right] \\ &= \frac{1}{\rho}\left[2\rho\cos\phi - \rho\cos\phi + \rho^2\right] \\ &= \cos\phi + \rho\end{aligned}$$

Task

Find $\underline{\nabla} \times \underline{F}$ for $\underline{F} = F_\rho \underline{\hat{\rho}} + F_\phi \underline{\hat{\phi}} + F_z \underline{\hat{z}} = \rho^3 \underline{\hat{\rho}} + \rho z \underline{\hat{\phi}} + \rho z \sin\phi \underline{\hat{z}}$. Show that the results are consistent with those found using Cartesian coordinates.

(a) Find the curl $\underline{\nabla} \times \underline{F}$:

Your solution

Answer

$$\frac{1}{\rho}\begin{vmatrix} \underline{\hat{\rho}} & \rho\underline{\hat{\phi}} & \underline{\hat{z}} \\ \dfrac{\partial}{\partial \rho} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ \rho^3 & \rho^2 z & \rho z \sin\phi \end{vmatrix} = (z\cos\phi - \rho)\underline{\hat{\rho}} - z\sin\phi\underline{\hat{\phi}} + 2z\underline{\hat{z}}$$

(b) Find $\underline{F}$ in Cartesian coordinates:

Your solution

Answer

Use $\underline{\hat{\rho}} = \cos\phi \underline{i} + \sin\phi \underline{j}$, $\underline{\hat{\phi}} = -\sin\phi \underline{i} + \cos\phi \underline{j}$ to get $\underline{F} = (x^3 + xy^2 - yz)\underline{i} + (x^2 y + y^3 + xz)\underline{j} + yz\underline{k}$

(c) Hence find $\underline{\nabla} \times \underline{F}$ in Cartesian coordinates:

Your solution

Answer

$(z - x)\underline{i} - y\underline{j} + 2z\underline{k}$

(d) Using $\underline{\hat{\rho}} = \cos\phi \underline{i} + \sin\phi \underline{j}$ and $\underline{\hat{\phi}} = -\sin\phi \underline{i} + \cos\phi \underline{j}$, show that the solution to part (a) is equal to the solution for part (c):

Your solution

Answer

$(z\cos\phi - \rho)\ \underline{\hat{\rho}} - z\sin\phi\ \underline{\hat{\phi}} + 2z\ \underline{\hat{z}} = (z\cos\phi - \rho)(\cos\phi\ \underline{i} + \sin\phi\ \underline{j}) - z\sin\phi(-\sin\phi\ \underline{i} + \cos\phi\ \underline{j}) + 2z\ \underline{k}$
$= [z\cos^2\phi - \rho\cos\phi + z\sin^2\phi]\ \underline{i} + [z\cos\phi\sin\phi - \rho\sin\phi - z\sin\phi\cos\phi]\ \underline{j} + 2z\ \underline{k}$
$= [z - \rho\cos\phi]\ \underline{i} - \rho\sin\phi\ \underline{j} + 2z\ \underline{k} = (z - x)\ \underline{i} - y\ \underline{j} + 2z\ \underline{k}$

Exercises

1. For $\underline{F} = \rho\underline{\hat{\rho}} + (\rho\sin\phi + z)\underline{\hat{\phi}} + \rho z\underline{\hat{z}}$, find $\underline{\nabla} \cdot \underline{F}$ and $\underline{\nabla} \times \underline{F}$.
2. For $f = \rho^2 z^2 \cos 2\phi$, find $\underline{\nabla} \times (\underline{\nabla} f)$.

Answers

1. $2 + \cos\phi + \rho, \qquad -\underline{\hat{\rho}} - z\ \underline{\hat{\phi}} + (2\sin\phi + \dfrac{z}{\rho})\ \underline{\hat{z}}$

2. $\underline{0}$

4. Spherical polar coordinates

In spherical polar coordinates (r, θ, ϕ), the 3 unit vectors are $\underline{\hat{r}}$, $\underline{\hat{\theta}}$ and $\underline{\hat{\phi}}$ with scale factors $h_r = 1$, $h_\theta = r$, $h_\phi = r\sin\theta$. The quantities r, θ and ϕ are related to x, y and z by $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$ and $z = r\cos\theta$. In spherical polar coordinates,

$$\text{grad } f = \underline{\nabla} f = \frac{\partial f}{\partial r}\underline{\hat{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\underline{\hat{\theta}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\underline{\hat{\phi}}$$

If $\underline{F} = F_r\underline{\hat{r}} + F_\theta\underline{\hat{\theta}} + F_\phi\underline{\hat{\phi}}$

then

$$\text{div } \underline{F} = \underline{\nabla} \cdot \underline{F} = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(r^2\sin\theta F_r) + \frac{\partial}{\partial \theta}(r\sin\theta F_\theta) + \frac{\partial}{\partial \phi}(rF_\phi)\right]$$

$$\text{curl } \underline{F} = \underline{\nabla} \times \underline{F} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \underline{\hat{r}} & r\underline{\hat{\theta}} & r\sin\theta\underline{\hat{\phi}} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ F_r & rF_\theta & r\sin\theta F_\phi \end{vmatrix}$$

Example 25

In spherical polar coordinates, find $\underline{\nabla} f$ for

(a) $f = r$ (b) $f = \dfrac{1}{r}$ (c) $f = r^2 \sin(\phi + \theta)$

[Note: parts (a) and (b) relate to Exercises 2(a) and 2(c) on page 22.]

Solution

(a)
$$\begin{aligned}\underline{\nabla} f &= \frac{\partial f}{\partial r}\underline{\hat{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\underline{\hat{\theta}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\underline{\hat{\phi}} \\ &= \frac{\partial(r)}{\partial r}\underline{\hat{r}} + \frac{1}{r}\frac{\partial(r)}{\partial \theta}\underline{\hat{\theta}} + \frac{1}{r\sin\theta}\frac{\partial(r)}{\partial \phi}\underline{\hat{\phi}} \\ &= 1\underline{\hat{r}} = \underline{\hat{r}}\end{aligned}$$

(b)
$$\begin{aligned}\underline{\nabla} f &= \frac{\partial f}{\partial r}\underline{\hat{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\underline{\hat{\theta}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\underline{\hat{\phi}} \\ &= \frac{\partial(\frac{1}{r})}{\partial r}\underline{\hat{r}} + \frac{1}{r}\frac{\partial(\frac{1}{r})}{\partial \theta}\underline{\hat{\theta}} + \frac{1}{r\sin\theta}\frac{\partial(\frac{1}{r})}{\partial \phi}\underline{\hat{\phi}} \\ &= -\frac{1}{r^2}\underline{\hat{r}}\end{aligned}$$

(c)
$$\begin{aligned}\underline{\nabla} f &= \frac{\partial f}{\partial r}\underline{\hat{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\underline{\hat{\theta}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\underline{\hat{\phi}} \\ &= \frac{\partial(r\sin(\phi+\theta))}{\partial r}\underline{\hat{r}} + \frac{1}{r}\frac{\partial(r\sin(\phi+\theta))}{\partial \theta}\underline{\hat{\theta}} + \frac{1}{r\sin\theta}\frac{\partial(r^2\sin(\phi+\theta))}{\partial \phi}\underline{\hat{\phi}} \\ &= 2r\sin(\phi+\theta)\underline{\hat{r}} + \frac{1}{r}r^2\cos(\phi+\theta)\underline{\hat{\theta}} + \frac{1}{r\sin\theta}r^2\cos(\phi+\theta)\underline{\hat{\phi}} \\ &= 2r\sin(\phi+\theta)\underline{\hat{r}} + r\cos(\phi+\theta)\underline{\hat{\theta}} + \frac{r\cos(\phi+\theta)}{\sin\theta}\underline{\hat{\phi}}\end{aligned}$$

Engineering Example 3

Electric potential

Introduction

There is a scalar quantity V, called the electric potential, which satisfies

$\underline{\nabla}V = -\underline{E}$ where $\underline{E}$ is the electric field.

It is often easier to handle scalar fields rather than vector fields. It is therefore convenient to work with V and then derive $\underline{E}$ from it.

Problem in words

Given the electric potential, find the electric field.

Mathematical statement of problem

For a point charge, Q, the potential V is given by

$$V = \frac{Q}{4\pi\epsilon_0 r}$$

Verify, using spherical polar coordinates, that $\underline{E} = -\underline{\nabla}V = \dfrac{Q}{4\pi\epsilon_0 r^2}\underline{\hat{r}}$

Mathematical analysis

In spherical polar coordinates:

$$\begin{aligned}
\underline{\nabla}V &= \frac{\partial V}{\partial r}\underline{\hat{r}} + \frac{1}{r}\frac{\partial V}{\partial \theta}\underline{\hat{\theta}} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\underline{\hat{\phi}} \\
&= \frac{\partial V}{\partial r}\underline{\hat{r}} \quad \text{as the other partial derivatives are zero} \\
&= \frac{\partial}{\partial r}\left[\frac{Q}{4\pi\epsilon_0 r}\right]\underline{\hat{r}} \\
&= -\frac{Q}{4\pi\epsilon_0 r^2}\underline{\hat{r}}
\end{aligned}$$

Interpretation

So $\underline{E} = \dfrac{Q}{4\pi\epsilon_0 r^2}\underline{\hat{r}}$ as required.

This is a form of Coulomb's Law. A positive charge will experience a positive repulsion radially *outwards* in the field of another positive charge.

Example 26

Using spherical polar coordinates, find $\underline{\nabla} \cdot \underline{F}$ for the following vector functions.

(a) $\underline{F} = r\underline{\hat{r}}$ (b) $\underline{F} = r^2 \sin\theta \underline{\hat{r}}$ (c) $\underline{F} = r\sin\theta\,\underline{\hat{r}} + r^2\sin\phi\,\underline{\hat{\theta}} + r\cos\theta\,\underline{\hat{\phi}}$

Solution

(a)

$$\begin{aligned}
\underline{\nabla} \cdot \underline{F} &= \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(r^2\sin\theta F_r) + \frac{\partial}{\partial\theta}(r\sin\theta F_\theta) + \frac{\partial}{\partial\phi}(rF_\phi)\right] \\
&= \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(r^2\sin\theta \times r) + \frac{\partial}{\partial\theta}(r\sin\theta \times 0) + \frac{\partial}{\partial\phi}(r \times 0)\right] \\
&= \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(r^3\sin\theta) + \frac{\partial}{\partial\theta}(0) + \frac{\partial}{\partial\phi}(0)\right] = \frac{1}{r^2\sin\theta}\left[3r^2\sin\theta + 0 + 0\right] = 3
\end{aligned}$$

Note :- in Cartesian coordinates, the corresponding vector is $\underline{F} = x\underline{i} + y\underline{j} + z\underline{k}$ with $\underline{\nabla} \cdot \underline{F} = 1 + 1 + 1 = 3$ (hence consistency).

(b)

$$\begin{aligned}
\underline{\nabla} \cdot \underline{F} &= \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(r^2\sin\theta\ F_r) + \frac{\partial}{\partial\theta}(r\sin\theta\ F_\theta) + \frac{\partial}{\partial\phi}(rF_\phi)\right] \\
&= \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(r^2\sin\theta\ r^2\sin\theta) + \frac{\partial}{\partial\theta}(r\sin\theta \times 0) + \frac{\partial}{\partial\phi}(r \times 0)\right] \\
&= \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(r^4\sin^2\theta) + \frac{\partial}{\partial\theta}(0) + \frac{\partial}{\partial\phi}(0)\right] \\
&= \frac{1}{r^2\sin\theta}\left[4r^3\sin^2\theta + 0 + 0\right] = 4r\sin\theta
\end{aligned}$$

(c)

$$\begin{aligned}
\underline{\nabla} \cdot \underline{F} &= \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(r^2\sin\theta\ F_r) + \frac{\partial}{\partial\theta}(r\sin\theta\ F_\theta) + \frac{\partial}{\partial\phi}(rF_\phi)\right] \\
&= \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(r^2\sin\theta\ r\sin\theta) + \frac{\partial}{\partial\theta}(r\sin\theta \times r^2\sin\phi) + \frac{\partial}{\partial\phi}(r \times\ r\cos\theta)\right] \\
&= \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(r^3\sin^2\theta) + \frac{\partial}{\partial\theta}(r^3\sin\theta\sin\phi) + \frac{\partial}{\partial\phi}(r^2\cos\theta)\right] \\
&= \frac{1}{r^2\sin\theta}\left[3r^2\sin^2\theta + r^3\cos\theta\ \sin\phi + 0\right] = 3\sin\theta + r\cot\theta\ \sin\phi
\end{aligned}$$

Example 27

Using spherical polar coordinates, find $\underline{\nabla} \times \underline{F}$ for the following vector fields $\underline{F}$.

(a) $\underline{F} = r^k \hat{\underline{r}}$, where k is a constant (b) $\underline{F} = r^2 \cos\theta\, \hat{\underline{r}} + \sin\theta\, \hat{\underline{\theta}} + \sin^2\theta\, \hat{\underline{\phi}}$

Solution

(a)

$$\begin{aligned}
\underline{\nabla} \times \underline{F} &= \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{\underline{r}} & r\hat{\underline{\theta}} & r\sin\theta\, \hat{\underline{\phi}} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ F_r & rF_\theta & r\sin\theta F_\phi \end{vmatrix} \\
&= \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{\underline{r}} & r\hat{\underline{\theta}} & r\sin\theta\, \hat{\underline{\phi}} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ r^k & r \times 0 & r\sin\theta \times 0 \end{vmatrix} \\
&= \frac{1}{r^2 \sin\theta} \left[\left(\frac{\partial}{\partial \theta}(0) - \frac{\partial}{\partial \phi}(0) \right) \hat{\underline{r}} + \left(\frac{\partial}{\partial \phi}(r^k) - \frac{\partial}{\partial r}(0) \right) r\hat{\underline{\theta}} \right. \\
&\quad \left. + \left(\frac{\partial}{\partial r}(0) - \frac{\partial}{\partial \theta}(r^k) \right) r\sin\theta\, \hat{\underline{\phi}} \right] \\
&= 0\, \hat{\underline{r}} + 0\, \hat{\underline{\theta}} + 0\, \hat{\underline{\phi}} = \underline{0}
\end{aligned}$$

(b)

$$\begin{aligned}
\underline{\nabla} \times \underline{F} &= \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{\underline{r}} & r\hat{\underline{\theta}} & r\sin\theta\, \hat{\underline{\phi}} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ F_r & rF_\theta & r\sin\theta F_\phi \end{vmatrix} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{\underline{r}} & r\hat{\underline{\theta}} & r\sin\theta\, \hat{\underline{\phi}} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ r^2\cos\theta & r \times \sin\theta & r\sin\theta \times \sin^2\theta \end{vmatrix} \\
&= \frac{1}{r^2 \sin\theta} \left[\left(\frac{\partial}{\partial \theta}(r\sin^3\theta) - \frac{\partial}{\partial \phi}(r\sin\theta) \right) \hat{\underline{r}} + \left(\frac{\partial}{\partial \phi}(r^2\cos\theta) - \frac{\partial}{\partial r}(r\sin^3\theta) \right) r\hat{\underline{\theta}} \right. \\
&\quad \left. + \left(\frac{\partial}{\partial r}(r\sin\theta) - \frac{\partial}{\partial \theta}(r^2\cos\theta) \right) r\sin\theta\, \hat{\underline{\phi}} \right] \\
&= \frac{1}{r^2 \sin\theta} \left[\left(3r\sin^2\theta\cos\theta + 0 \right) \hat{\underline{r}} + \left(0 - \sin^3\theta \right) r\hat{\underline{\theta}} + \left(\sin\theta + r^2\sin\theta \right) r\sin\theta\, \hat{\underline{\phi}} \right] \\
&= \frac{3\sin\theta\cos\theta}{r}\, \hat{\underline{r}} - \frac{\sin^2\theta}{r} \hat{\underline{\theta}} + \frac{(1+r^2)}{r} \sin\theta\, \hat{\underline{\phi}}
\end{aligned}$$

Task

Using spherical polar coordinates, find $\underline{\nabla} f$ for

(a) $f = r^4$

(b) $f = \dfrac{r}{r^2+1}$

(c) $f = r^2 \sin 2\theta \cos\phi$

Your solution

Answer

(a) $4r^3\underline{\hat{r}}$,

(b) $\dfrac{1-r^2}{(1+r^2)^2}\underline{\hat{r}}$,

(c) $\dfrac{\partial}{\partial r}(r^2 \sin 2\theta \cos\phi)\underline{\hat{r}} + \dfrac{1}{r}\dfrac{\partial}{\partial \theta}(r^2 \sin 2\theta \cos\phi)\underline{\hat{\phi}} + \dfrac{1}{r\sin\theta}\dfrac{\partial}{\partial \phi}(r^2 \sin 2\theta \cos\phi)$

$$= 2r\sin 2\theta\cos\phi\ \underline{\hat{r}} + 2r\cos 2\theta\cos\phi\ \underline{\hat{\theta}} - 2r\cos\theta\sin\phi\ \underline{\hat{\phi}}$$

Exercises

1. For $\underline{F} = r\sin\theta\underline{\hat{r}} + r\cos\phi\underline{\hat{\theta}} + r\sin\phi\underline{\hat{\phi}}$, find $\underline{\nabla}\cdot\underline{F}$ and $\underline{\nabla}\times\underline{F}$.

2. For $\underline{F} = r^{-4}\cos\theta\underline{\hat{r}} + r^{-4}\sin\theta\underline{\hat{\theta}}$, find $\underline{\nabla}\cdot\underline{F}$ and $\underline{\nabla}\times\underline{F}$.

3. For $\underline{F} = r^2\cos\theta\underline{\hat{r}} + \cos\phi\underline{\hat{\theta}}$ find $\underline{\nabla}\cdot(\underline{\nabla}\times\underline{F})$.

Answers

1. $\cos\phi(\cot\theta + \text{cosec}\theta) + 3\sin\theta$, $\quad \cot\dfrac{\theta}{2}\sin\phi\underline{\hat{r}} - 2\sin\phi\underline{\hat{\theta}} + (2\cos\phi - \cos\theta)\underline{\hat{\phi}}$

2. 0, $\quad -2r^{-5}\sin\theta\underline{\hat{\phi}}$

3. 0