Contents 26

Functions of a Complex Variable

Learning outcomes

By studying the Workbook you will understand the concept of a complex function and its derivative and learn what is meant by an analytic function and why analytic functions are important.
You will learn about the Cauchy-Riemann equations and the concept of conformal mapping and be able to solve complex problems involving standard complex functions and evaluate simple complex integrals.
You will learn Cauchy's theorem and be able to use it to evaluate complex integrals.
You will learn how to develop simple Laurent series and classify singularities of a complex function.
Finally, you will learn about the residue theorem and how to use it to solve problems.

Complex Functions

26.1

Introduction

In this introduction to functions of a complex variable we shall show how the operations of taking a limit and of finding a derivative, which we are familiar with for functions of a real variable, extend in a natural way to the complex plane. In fact the **notation** used for functions of a complex variable and for complex operations is almost identical to that used for functions of a real variable. In effect, the **real variable x** is simply replaced by the **complex variable** z. However, it is the **interpretation** of functions of a complex variable and of complex operations that differs significantly from the real case. In effect, a function of a complex variable is equivalent to **two** functions of a real variable and our standard interpretation of a function of a real variable as being a curve on an xy plane no longer holds.

There are many situations in engineering which are described quite naturally by specifying two harmonic functions of a real variable: a harmonic function is one satisfying the two-dimensional Laplace equation:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

Fluids and heat flow in two dimensions are particular examples. It turns out that knowledge of functions of a complex variable can significantly ease the calculations involved in this area.

Prerequisites

Before starting this Section you should ...

- understand how to use the polar and exponential forms of a complex number
- be familiar with trigonometric relations, hyperbolic and logarithmic functions
- be able to form a partial derivative
- be able to take a limit

Learning Outcomes

On completion you should be able to ...

- explain the meaning of the term analytic function

1. Complex functions

Let the complex variable z be defined by $z = x + \mathrm{i}y$ where x and y are real variables and i is, as usual, given by $\mathrm{i}^2 = -1$. Now let a second complex variable w be defined by $w = u + \mathrm{i}v$ where u and v are real variables. If there is a relationship between w and z such that to each value of z in a given region of the $z-$plane there is assigned one, and only one, value of w then w is said to be a **function** of z, defined on the given region. In this case we write

$$w = f(z).$$

As a example consider $w = z^2 - z$, which is defined for all values of z (that is, the right-hand side can be computed for **every** value of z). Then, remembering that $z = x + \mathrm{i}y$,

$$w = u + \mathrm{i}v = (x + \mathrm{i}y)^2 - (x + \mathrm{i}y) = x^2 + 2\mathrm{i}xy - y^2 - x - \mathrm{i}y.$$

Hence, equating real and imaginary parts: $u = x^2 - x - y^2$ and $v = 2xy - y$.

If $z = 2 + 3\mathrm{i}$, for example, then $x = 2, y = 3$ so that $u = 4 - 2 - 9 = -7$ and $v = 12 - 3 = 9$, giving $w = -7 + 9\mathrm{i}$.

Example 1

(a) For which values of z is $w = \dfrac{1}{z}$ defined?

(b) For these values obtain u and v and evaluate w when $z = 2 - \mathrm{i}$.

Solution

(a) w is defined for all $z \neq 0$.

(b) $u + \mathrm{i}v = \dfrac{1}{x + \mathrm{i}y} = \dfrac{1}{x + iy} \cdot \dfrac{x - \mathrm{i}y}{x - \mathrm{i}y} = \dfrac{x - \mathrm{i}y}{x^2 + y^2}$. Hence $u = \dfrac{x}{x^2 + y^2}$ and $v = \dfrac{-y}{x^2 + y^2}$.

If $z = 2 - \mathrm{i}$ then $x = 2, y = -1$ so that $x^2 + y^2 = 5$. Then $u = \dfrac{2}{5}, v = -\dfrac{1}{5}$and $w = \dfrac{2}{5} - \dfrac{1}{5}\mathrm{i}$.

2. The limit of a function

The limit of $w = f(z)$ as $z \to z_0$ is a number ℓ such that $|f(z) - \ell|$ can be made as small as we wish by making $|z - z_0|$ sufficiently small. In some cases the limit is simply $f(z_0)$, as is the case for $w = z^2 - z$. For example, the limit of this function as $z \to \mathrm{i}$ is $f(\mathrm{i}) = \mathrm{i}^2 - \mathrm{i} = -1 - \mathrm{i}$.

There is a fundamental difference from functions of a real variable: there, we could approach a point on the curve $y = g(x)$ either from the left or from the right when considering limits of $g(x)$ at such points. With the function $f(z)$ we are allowed to approach the point $z = z_0$ along **any** path in the z-plane; we require merely that the distance $|z - z_0|$ decreases to zero.

Suppose that we want to find the limit of $f(z) = z^2 - z$ as $z \to 2 + \text{i}$ along each of the paths (a), (b) and (c) indicated in Figure 1.

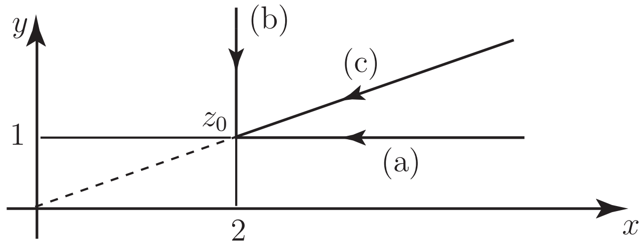


Figure 1

(a) Along this path $z = x + \text{i}$ (for any x) and $z^2 - z = x^2 + 2x\text{i} - 1 - x - i$

That is: $z^2 - z = x^2 - 1 - x + (2x - 1)\text{i}$.

As $z \to 2 + \text{i}$, then $x \to 2$ so that the limit of $z^2 - z$ is $2^2 - 1 - 2 + (4 - 1)\text{i} = 1 + 3\text{i}$.

(b) Here $z = 2 + y\text{i}$ (for any y) so that $z^2 - z = 4 - y^2 - 2 + (4y - y)\text{i}$.

As $z \to 2 + \text{i}, y \to 1$ so that the limit of $z^2 - z$ is $4 - 1 - 2 + (4 - 1)\text{i} = 1 + 3\text{i}$.

(c) Here $z = k(2 + \text{i})$ where k is a real number. Then

$$z^2 - z = k^2(4 + 4i - 1) - k(2 + \text{i}) = 3k^2 - 2k + (4k^2 - k)\text{i}.$$

As $z \to 2 + \text{i}, k \to 1$ so that the limit of $z^2 - z$ is $3 - 2 + (4 - 1)\text{i} = 1 + 3\text{i}$.

In each case the limit is the same.

Task

Evaluate the limit of $f(z) = z^2 + z + 1$ as $z \to 1 + 2\text{i}$ along the paths

(a) parallel to the x-axis coming from the right,

(b) parallel to the y-axis, coming from above,

(c) the line joining the point $1 + 2\text{i}$ to the origin, coming from the origin.

Your solution

Answer

(a) Along this path $z = x + 2\text{i}$ and $z^2 + z + 1 = x^2 - 4 + x + 1 + (4x + 2)\text{i}$. As $z \to 1 + 2\text{i}, x \to 1$ and $z^2 + z + 1 \to -1 + 6\text{i}$.

(b) Along this path $z = 1 + y\text{i}$
and $z^2 + z + 1 = 1 - y^2 + 1 + 1 + (2y + y)\text{i}$. As $z \to 1 + 2\text{i}, y \to 2$ and $z^2 + z + 1 \to -1 + 6\text{i}$.

(c) If $z = k(1 + 2\text{i})$ then $z^2 + z + 1 = k^2 + k + 1 - 4k^2 + (4k^2 + 2k)\text{i}$. As $z \to 1 + 2\text{i}, k \to 1$ and $z^2 + z + 1 \to -1 + 6\text{i}$.

Not all functions of a complex variable are as straightforward to analyse as the last two examples. Consider the function $f(z) = \dfrac{\bar{z}}{z}$. Along the x-axis moving towards the origin from the right

$$z = x \quad \text{and} \quad \bar{z} = x \quad \text{so that} \quad f(z) = 1 \quad \text{which is the limit as} \quad z \to 0 \textbf{ along this path}.$$

However, we can approach the origin along any path. If instead we approach the origin along the positive y-axis $z = \text{i}y$ then

$$\bar{z} = -\text{i}y \quad \text{and} \quad f(z) = \frac{\bar{z}}{z} = -1, \quad \text{which is the limit as } z \to 0 \textbf{ along this path}.$$

Since these two limits are distinct then $\lim_{z \to 0} \dfrac{\bar{z}}{z}$ **does not exist**.

We cannot assume that the limit of a function $f(z)$ as $z \to z_0$ is independent of the path chosen.

Definition of continuity

The function $f(z)$ is **continuous** as $z \to z_0$ if the following two statements are true:

(a) $f(z_0)$ exists;

(b) $\lim_{z \to z_0} f(z)$ exists and is equal to $f(z_0)$.

As an example consider $f(z) = \dfrac{z^2 + 4}{z^2 + 9}$. As $z \to \text{i}$, then $f(z) \to f(\text{i}) = \dfrac{\text{i}^2 + 4}{\text{i}^2 + 9} = \dfrac{3}{8}$. Thus $f(z)$ is continuous at $z = \text{i}$.

However, when $z^2 + 9 = 0$ then $z = \pm 3\text{i}$ and neither $f(3\text{i})$ nor $f(-3\text{i})$ exists. Thus $\dfrac{z^2 + 4}{z^2 + 9}$ is discontinuous at $z = \pm 3\text{i}$. It is easily shown that these are the *only* points of discontinuity.

Task

State where $f(z) = \dfrac{z}{z^2 + 4}$ is discontinuous. Find $\lim_{z \to \text{i}} f(z)$.

Your solution

Answer

$z^2 + 4 = 0$ where $z = \pm 2i$; at these points $f(z)$ is discontinuous as $f(\pm 2\text{i})$ does not exist.

$$\lim_{z \to \text{i}} f(z) = f(\text{i}) = \frac{\text{i}}{\text{i}^2 + 4} = \frac{1}{3}\text{i}.$$

It is easily shown that any polynomial in z is continuous everywhere whilst any rational function is continuous everywhere except at the zeroes of the denominator.

Exercises

1. For which values of z is $w = \dfrac{1}{z - \text{i}}$ defined? For these values obtain u and v and evaluate w when $z = 1 - 2\text{i}$.
2. Find the limit of $f(z) = z^3 + z$ as $z \to \text{i}$ along the paths (a) parallel to the x-axis coming from the right, (b) parallel to the y-axis coming from above.
3. Where is $f(z) = \dfrac{z}{z^2 + 9}$ discontinuous?. Find the $\lim\limits_{z \to -\text{i}} f(z)$.

Answers

1. w is defined for all $z \neq \text{i}$ $\quad w = \dfrac{1}{x + y\text{i} - \text{i}} = \dfrac{1}{x + (y-1)\text{i}} \times \dfrac{x - (y-1)\text{i}}{x - (y-1)\text{i}} = \dfrac{x - (y-1)\text{i}}{x^2 + (y-1)^2}.$

$$\therefore \qquad u = \frac{x}{x^2 + (y-1)^2}, \qquad v = \frac{-(y-1)}{x^2 + (y-1)^2}.$$

When $z = 1 - 2\text{i}$, $x = 1$, $y = -2$ so that $u = \dfrac{1}{1+9} = \dfrac{1}{10}$, $v = \dfrac{3}{10}$, $z = \dfrac{1}{10} + \dfrac{3}{10}\text{i}$

2. (a) $z = x + \text{i}$, $z^3 + z = x^3 + 3x^2\text{i} - 2x$. As $z \to \text{i}$, $x \to 0$ and $z^3 + z \to 0$

   (b) $z = y\text{i}$, $z^3 + z = -y^3\text{i} + y\text{i}$. As $z \to \text{i}$, $y \to 1$ and $z^3 + z \to -\text{i} + \text{i} = 0$.

3. $f(z)$ is discontinuous at $z = \pm 3\text{i}$. The limit is $f(-\text{i}) = \dfrac{-\text{i}}{-1+9} = -\dfrac{1}{8}\text{i}.$

3. Differentiating functions of a complex variable

The function $f(z)$ is said to be **differentiable** at $z = z_0$ if

$$\lim_{\Delta z \to 0} \left\{ \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \right\} \quad \text{exists.} \qquad \text{Here } \Delta z = \Delta x + \text{i}\Delta y.$$

Apart from a change of notation this is precisely the same as the definition of the derivative of a function of a real variable. Not surprisingly then, the rules of differentiation used in functions of a real variable can be used to differentiate functions of a complex variable. The value of the limit is the **derivative** of $f(z)$ at $z = z_0$ and is often denoted by $\frac{df}{dz}|_{z=z_0}$ or by $f'(z_0)$.

A point at which the derivative does not exist is called a **singular point** of the function.

A function $f(z)$ is said to be **analytic** at a point z_0 if it is differentiable throughout a neighbourhood of z_0, however small. (A neighbourhood of z_0 is the region contained within some circle $|x-z_0|=r$.)

For example, the function $f(z)=\dfrac{1}{z^2+1}$ has singular points where $z^2+1=0$, i.e. at $z=\pm\mathrm{i}$.

For all other points the usual rules for differentiation apply and hence

$$f'(z)=-\frac{2z}{(z^2+1)^2} \qquad \text{(quotient rule)}$$

So, for example, at $z=3\mathrm{i}$, $f'(z)=-\dfrac{6\mathrm{i}}{(-9+1)^2}=-\dfrac{3}{32}\mathrm{i}$.

Example 2

Find the singular point of the rational function $f(z)=\dfrac{z}{z+\mathrm{i}}$. Find $f'(z)$ at other points and evaluate $f'(2\mathrm{i})$.

Solution

$z+\mathrm{i}=0$ when $z=-\mathrm{i}$ and this is the singular point: $f(-\mathrm{i})$ does not exist. Elsewhere, differentiating using the quotient rule:

$f'(z)=\dfrac{(z+\mathrm{i})\cdot 1-z\cdot 1}{(z+\mathrm{i})^2}=\dfrac{\mathrm{i}}{(z+\mathrm{i})^2}$. Thus at $z=2\mathrm{i}$, we have $f'(z)=\dfrac{\mathrm{i}}{(3\mathrm{i})^2}=-\dfrac{1}{9}\mathrm{i}$.

The simple function $f(z)=\bar{z}=x-\mathrm{i}y$ is not analytic anywhere in the complex plane. To see this consider looking at the derivative at an *arbitrary* point z_0. We easily see that

$$\begin{aligned} R &= \frac{f(z_0+\Delta z)-f(z_0)}{\Delta z} \\ &= \frac{(x_0+\Delta x)-\mathrm{i}(y_0+\Delta y)-(x_0-\mathrm{i}y_0)}{\Delta x+\mathrm{i}\Delta y}=\frac{\Delta x-\mathrm{i}\,\Delta y}{\Delta x+\mathrm{i}\,\Delta y}\end{aligned}$$

Hence $f(z)$ will fail to have a derivative at z_0 if we can show that this expression has no limit. To do this we consider looking at the limit of the function along two distinct paths.

Along a path parallel to the x-axis:

$$\Delta y=0 \quad \text{so that} \quad R=\frac{\Delta x}{\Delta x}=1, \quad \text{and this is the limit as} \quad \Delta z=\Delta x\to 0.$$

Along a path parallel to the y-axis:

$$\Delta x=0 \quad \text{so that} \quad R=\frac{-\mathrm{i}\,\Delta y}{\mathrm{i}\Delta y}=-1, \quad \text{and this is the limit as } \Delta z=\Delta y\to 0.$$

As these two values of R are distinct, the limit of $\dfrac{f(z+\Delta z)-f(z)}{\Delta z}$ as $z\to z_0$ does not exist and so $f(z)$ fails to be differentiable at *any* point. Hence it is not analytic anywhere.

Cauchy-Riemann Equations and Conformal Mapping

26.2

Introduction

In this Section we consider two important features of complex functions. The Cauchy-Riemann equations provide a necessary and sufficient condition for a function $f(z)$ to be analytic in some region of the complex plane; this allows us to find $f'(z)$ in that region by the rules of the previous Section.

A mapping between the z-plane and the w-plane is said to be conformal if the angle between two intersecting curves in the z-plane is equal to the angle between their mappings in the w-plane. Such a mapping has widespread uses in solving problems in fluid flow and electromagnetics, for example, where the given problem geometry is somewhat complicated.

Prerequisites

Before starting this Section you should ...

- understand the idea of a complex function and its derivative

Learning Outcomes

On completion you should be able to ...

- use the Cauchy-Riemann equations to obtain the derivative of complex functions
- appreciate the idea of a conformal mapping

1. The Cauchy-Riemann equations

Remembering that $z = x + \mathsf{i}y$ and $w = u + \mathsf{i}v$, we note that there is a very useful test to determine whether a function $w = f(z)$ is analytic at a point. This is provided by the **Cauchy-Riemann** equations. These state that $w = f(z)$ is differentiable at a point $z = z_0$ if, and only if,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{at that point.}$$

When these equations hold then it can be shown that the complex derivative may be determined by using either $\dfrac{df}{dz} = \dfrac{\partial f}{\partial x}$ or $\dfrac{df}{dz} = -\mathsf{i}\dfrac{\partial f}{\partial y}$.
(The use of 'if, and only if,' means that if the equations are valid, then the function is differentiable **and vice versa**.)

If we consider $f(z) = z^2 = x^2 - y^2 + 2\mathsf{i}xy$ then $u = x^2 - y^2$ and $v = 2xy$ so that

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = -2y, \quad \frac{\partial v}{\partial x} = 2y, \quad \frac{\partial v}{\partial y} = 2x.$$

It should be clear that, for this example, the Cauchy-Riemann equations are always satisfied; therefore, the function is analytic everywhere. We find that

$$\frac{df}{dz} = \frac{\partial f}{\partial x} = 2x + 2\mathsf{i}y = 2z \quad \text{or, equivalently,} \quad \frac{df}{dz} = -\mathsf{i}\frac{\partial f}{\partial y} = -\mathsf{i}(-2y + 2\mathsf{i}x) = 2z$$

This is the result we would expect to get by simply differentiating $f(z)$ as if it was a real function. **For analytic functions this will always be the case** i.e. for an analytic function $f'(z)$ can be found using the rules for differentiating real functions.

Example 3

Show that the function $f(z) = z^3$ is analytic everwhere and hence obtain its derivative.

Solution

$$w = f(z) = (x + \mathsf{i}y)^3 = x^3 - 3xy^2 + (3x^2y - y^3)\mathsf{i}$$

Hence

$$u = x^3 - 3xy^2 \quad \text{and} \quad v = 3x^2y - y^3.$$

Then

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2, \qquad \frac{\partial u}{\partial y} = -6xy, \qquad \frac{\partial v}{\partial x} = 6xy, \qquad \frac{\partial v}{\partial y} = 3x^2 - 3y^2.$$

The Cauchy-Riemann equations are identically true and $f(z)$ is analytic everywhere.

Furthermore $\dfrac{df}{dz} = \dfrac{\partial f}{\partial x} = 3x^2 - 3y^2 + (6xy)\mathsf{i} = 3(x + \mathsf{i}y)^2 = 3z^2$ as we would expect.

We can easily find functions which are not analytic anywhere and others which are only analytic in a restricted region of the complex plane. Consider again the function $f(z) = \bar{z} = x - \mathsf{i}y$.

Here

$$u = x \quad \text{so that} \quad \frac{\partial u}{\partial x} = 1, \quad \text{and} \quad \frac{\partial u}{\partial y} = 0; \qquad v = -y \quad \text{so that} \quad \frac{\partial v}{\partial x} = 0, \quad \frac{\partial v}{\partial y} = -1.$$

The Cauchy-Riemann equations are never satisfied so that $\bar{z}$ is not differentiable anywhere and so is not analytic anywhere.

By contrast if we consider the function $f(z) = \dfrac{1}{z}$ we find that

$$u = \frac{x}{x^2+y^2}; \quad v = \frac{y}{x^2+y^2}.$$

As can readily be shown, the Cauchy-Riemann equations are satisfied everywhere except for $x^2+y^2 = 0$, i.e. $x = y = 0$ (or, equivalently, $z = 0$.) At all other points $f'(z) = -\dfrac{1}{z^2}$. This function is analytic everywhere except at the single point $z = 0$.

Analyticity is a very powerful property of a function of a complex variable. Such functions tend to behave like functions of a real variable.

Example 4

Show that if $f(z) = z\bar{z}$ then $f'(z)$ exists only at $z = 0$.

Solution

$f(z) = x^2 + y^2$ so that $u = x^2 + y^2, \quad v = 0.$ $\quad \dfrac{\partial u}{\partial x} = 2x, \quad \dfrac{\partial u}{\partial y} = 2y, \quad \dfrac{\partial v}{\partial x} = 0, \quad \dfrac{\partial v}{\partial y} = 0.$

Hence the Cauchy-Riemann equations are satisfied only where $x = 0$ and $y = 0$, i.e. where $z = 0$. Therefore this function is not analytic anywhere.

Analytic functions and harmonic functions

Using the Cauchy-Riemann equations in a region of the z-plane where $f(z)$ is analytic, gives

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left(-\frac{\partial v}{\partial x}\right) = -\frac{\partial^2 v}{\partial x^2}$$

and

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}\right) = \frac{\partial^2 v}{\partial y^2}.$$

If these differentiations are possible then $\dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{\partial^2 u}{\partial y \partial x}$ so that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{(Laplace's equation)}$$

In a similar way we find that

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \quad \text{(Can you show this?)}$$

When $f(z)$ is analytic the functions u and v are called **conjugate harmonic functions**.

Suppose $u = u(x, y) = xy$ then it is easy to verify that u satisfies Laplace's equation (try this). We now try to find the conjugate harmonic function $v = v(x, y)$.

First, using the Cauchy-Riemann equations:

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = y \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -x.$$

Integrating the first equation gives $v = \frac{1}{2}y^2 +$ a function of x. Integrating the second equation gives $v = -\frac{1}{2}x^2 +$ a function of y. Bearing in mind that an additive constant leaves no trace after differentiation, we pool the information above to obtain

$$v = \frac{1}{2}(y^2 - x^2) + C \qquad \text{where } C \text{ is a constant}$$

Note that $f(z) = u + \mathrm{i}v = xy + \frac{1}{2}(y^2 - x^2)\mathrm{i} + D$ where D is a constant (replacing $C\mathrm{i}$).

We can write $f(z) = -\frac{1}{2}\mathrm{i}z^2 + D$ (as you can verify). This function is analytic everywhere.

Task

Given the function $u = x^2 - x - y^2$

(a) Show that u is harmonic, (b) Find the conjugate harmonic function, v.

Your solution

(a)

Answer

$$\frac{\partial u}{\partial x} = 2x - 1, \quad \frac{\partial^2 u}{\partial x^2} = 2, \quad \frac{\partial u}{\partial y} = -2y, \quad \frac{\partial^2 u}{\partial y^2} = -2.$$

Hence $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ and u is harmonic.

Your solution

(b)

Answer

Integrating $\dfrac{\partial v}{\partial y} = 2x - 1$ gives $v = 2xy - y +$ function of x.

Integrating $\dfrac{\partial v}{\partial x} = +2y$ gives $v = 2xy +$ function of y.

Ignoring the duplication, $v = 2xy - y + C$, where C is a constant.

Task

Find $f(z)$ in terms of z, where $f(z) = u + \mathrm{i}v$, where u and v are those found in the previous Task.

Your solution

Answer

$f(z) = u + iv = x^2 - x - y^2 + 2xy\mathrm{i} - \mathrm{i}y + D, \quad D$ constant.

Now $\quad z^2 = x^2 - y^2 + 2\mathrm{i}xy \quad$ and $\quad z = x + \mathrm{i}y \quad$ thus $\qquad f(z) = z^2 - z + D$.

Exercises

1. Find the singular point of the rational function $f(z) = \dfrac{z}{z - 2\mathrm{i}}$. Find $f'(z)$ at other points and evaluate $f'(-\mathrm{i})$.
2. Show that the function $f(z) = z^2 + z$ is analytic everywhere and hence obtain its derivative.
3. Show that the function $u = x^2 - y^2 - 2y$ is harmonic, find the conjugate harmonic function v and hence find $f(z) = u + \mathrm{i}v$ in terms of z.

Answers

1. $f(z)$ is singular at $z = 2\mathrm{i}$. Elsewhere

$$f'(z) = \frac{(z - 2\mathrm{i}).1 - z.1}{(z - 2\mathrm{i})^2} = \frac{-2\mathrm{i}}{(z - 2\mathrm{i})^2} \qquad f'(-\mathrm{i}) = \frac{-2\mathrm{i}}{(-3\mathrm{i})^2} = \frac{-2\mathrm{i}}{-9} = \frac{2}{9}\mathrm{i}$$

2. $u = x^2 + x - y^2$ and $\quad v = 2xy + y$

$$\frac{\partial u}{\partial x} = 2x + 1, \quad \frac{\partial u}{\partial y} = -2y, \quad \frac{\partial v}{\partial x} = 2y, \quad \frac{\partial v}{\partial y} = 2x + 1$$

Here the Cauchy-Riemann equations are identically true and $f(z)$ is analytic everywhere.

$$\frac{df}{dz} = \frac{\partial f}{\partial x} = 2x + 1 + 2y\mathrm{i} = 2z + 1$$

Answer

3. $$\frac{\partial^2 u}{\partial x^2} = 2, \quad \frac{\partial^2 u}{\partial y^2} = -2 \qquad \text{therefore } u \text{ is harmonic.}$$

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 2x \qquad \text{therefore } v = 2xy + \text{ function of } y$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 2y + 2 \qquad \text{therefore } v = 2xy + 2x + \text{ function of } x$$

$$\therefore \quad v = 2xy + 2x + \text{ constant}$$

$$f(z) = x^2 + 2\mathrm{i}xy - y^2 + 2x\mathrm{i} - 2y = z^2 + 2\mathrm{i}z$$

2. Conformal mapping

In Section 26.1 we saw that the real and imaginary parts of an analytic function each satisfies Laplace's equation. We shall show now that the curves

$$u(x, y) = \text{constant} \qquad \text{and} \qquad v(x, y) = \text{constant}$$

intersect each other at right angles (i.e. are **orthogonal**). To see this we note that along the curve $u(x, y) =$ constant we have $du = 0$. Hence

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 0.$$

Thus, on these curves the gradient at a general point is given by

$$\frac{dy}{dx} = -\frac{\dfrac{\partial u}{\partial x}}{\dfrac{\partial u}{\partial y}}.$$

Similarly along the curve $v(x, y) =$ constant, we have

$$\frac{dy}{dx} = -\frac{\dfrac{\partial v}{\partial x}}{\dfrac{\partial v}{\partial y}}.$$

The product of these gradients is

$$\frac{(\dfrac{\partial u}{\partial x})(\dfrac{\partial v}{\partial x})}{(\dfrac{\partial u}{\partial y})(\dfrac{\partial v}{\partial y})} = -\frac{(\dfrac{\partial u}{\partial x})(\dfrac{\partial u}{\partial y})}{(\dfrac{\partial u}{\partial y})(\dfrac{\partial u}{\partial x})} = -1$$

where we have made use of the Cauchy-Riemann equations. We deduce that the curves are orthogonal.

As an example of the practical application of this work consider two-dimensional electrostatics. If $u =$ constant gives the **equipotential** curves then the curves $v =$ constant are the **electric lines of force**. Figure 2 shows some curves from each set in the case of oppositely-charged particles near to each other; the dashed curves are the lines of force and the solid curves are the equipotentials.

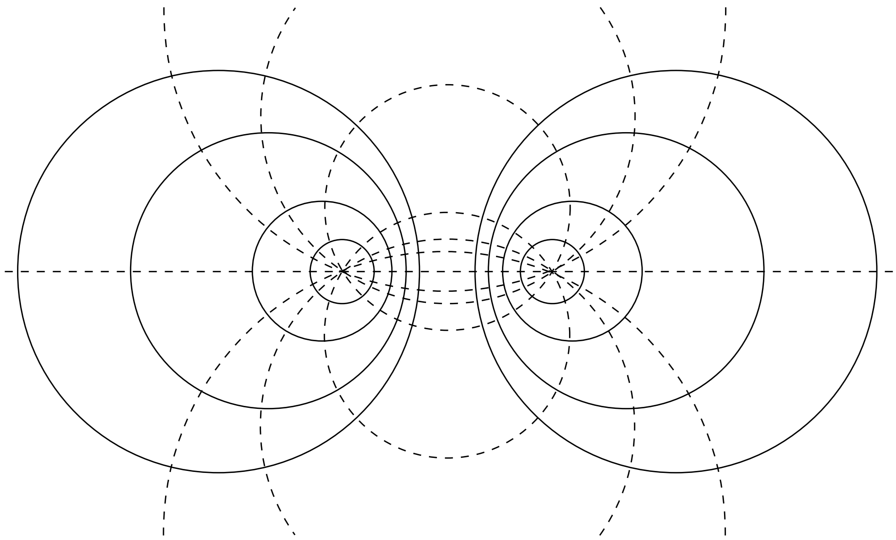


Figure 2

In ideal fluid flow the curves $v =$ constant are the **streamlines** of the flow.

In these situations the function $w = u + \mathrm{i}v$ is the **complex potential** of the field.

Function as mapping

A function $w = f(z)$ can be regarded as a mapping, which maps a point in the z-plane to a point in the w-plane. Curves in the z-plane will be mapped into curves in the w-plane.

Consider aerodynamics where we are interested in the fluid flow in a complicated geometry (say flow past an aerofoil). We first find the flow in a simple geometry that can be mapped to the aerofoil shape (the complex plane with a circular hole works here). Most of the calculations necessary to find physical characteristics such as lift and drag on the aerofoil can be performed in the simple geometry - the resulting integrals being much easier to evaluate than in the complicated geometry.

Consider the mapping

$$w = z^2.$$

The point $z = 2 + \mathrm{i}$ maps to $w = (2 + \mathrm{i})^2 = 3 + 4\mathrm{i}$. The point $z = 2 + \mathrm{i}$ lies on the intersection of the two lines $x = 2$ and $y = 1$. To what curves do these map? To answer this question we note that a point on the line $y = 1$ can be written as $z = x + \mathrm{i}$. Then

$$w = (x + \mathrm{i})^2 = x^2 - 1 + 2x\mathrm{i}$$

As usual, let $w = u + \mathrm{i}v$, then

$$u = x^2 - 1 \qquad \text{and} \qquad v = 2x$$

Eliminating x we obtain:

$$4u = 4x^2 - 4 = v^2 - 4 \quad \text{so} \quad v^2 = 4 + 4u \text{ is the curve to which } y = 1 \text{ maps.}$$

Example 5

Onto what curve does the line $x = 2$ map?

Solution

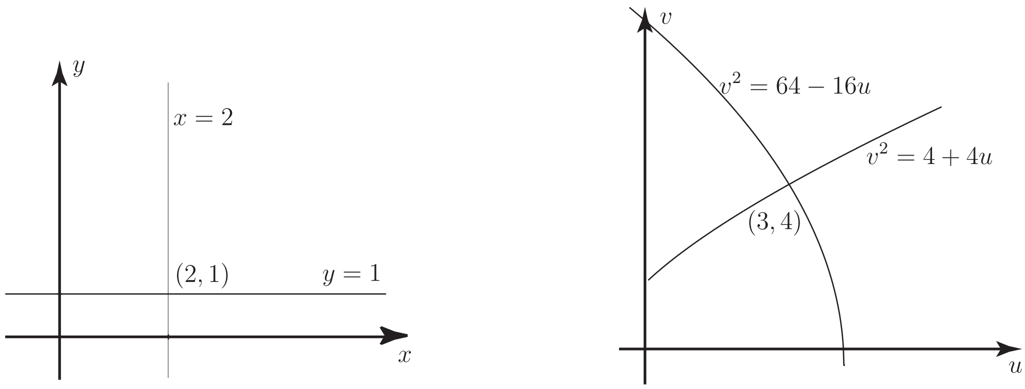
A point on the line is $z = 2 + y\text{i}$. Then

$$w = (2 + y\text{i})^2 = 4 - y^2 + 4y\text{i}$$

Hence $u = 4 - y^2$ and $v = 4y$ so that, eliminating y we obtain

$$16u = 64 - v^2 \quad \text{or} \quad v^2 = 64 - 16u$$

In Figure 3(a) we sketch the lines $x = 2$ and $y = 1$ and in Figure 3(b) we sketch the curves into which they map. Note these curves intersect at the point $(3, 4)$.



(a) (b)

Figure 3

The angle between the original lines in (a) is clearly 90^0; what is the angle between the curves in (b) at the point of intersection?

The curve $v^2 = 4 + 4u$ has a gradient $\dfrac{dv}{du}$. Differentiating the equation implicitly we obtain

$$2v\frac{dv}{du} = 4 \quad \text{or} \quad \frac{dv}{du} = \frac{2}{v}$$

At the point $(3, 4)$ $\dfrac{dv}{du} = \dfrac{1}{2}$.

Task Find $\dfrac{dv}{du}$ for the curve $v^2 = 64 - 16u$ and evaluate it at the point $(3, 4)$.

Your solution

Answer

$2v\dfrac{dv}{du} = -16 \qquad \therefore \quad \dfrac{dv}{du} = -\dfrac{8}{v}$. At $v = 4$ we obtain $\dfrac{dv}{du} = -2$.

Note that the product of the gradients at $(3, 4)$ is -1 and therefore the angle between the curves at their point of intersection is also 90^0. Since the angle between the lines and the angle between the curves is the same we say the **angle is preserved**.

In general, if two curves in the z-plane intersect at a point z_0, and their image curves under the mapping $w = f(z)$ intersect at $w_0 = f(z_0)$ and the angle between the two original curves at z_0 equals the angle between the image curves at w_0 we say that the mapping is **conformal** at z_0.

An analytic function is conformal everywhere except where $f'(z) = 0$.

Task At which points is $w = e^z$ not conformal?

Your solution

Answer

$f'(z) = e^z$. Since this is never zero the mapping is conformal everywhere.

Inversion

The mapping $\quad w = f(z) = \dfrac{1}{z}\quad$ is called an **inversion**. It maps the interior of the unit circle in the z-plane to the exterior of the unit circle in the w-plane, and vice-versa. Note that

$$w = u + \mathrm{i}v = \frac{x}{x^2+y^2} - \frac{y}{x^2+y^2}\mathrm{i} \quad \text{and similarly} \quad z = x + \mathrm{i}y = \frac{u}{u^2+v^2} - \frac{v}{u^2+v^2}\mathrm{i}$$

so that

$$u = \frac{x}{x^2+y^2} \qquad \text{and} \qquad v = -\frac{y}{x^2+y^2}.$$

A line through the origin in the z-plane will be mapped into a line through the origin in the w-plane. To see this, consider the line $y = mx$, for m constant. Then

$$u = \frac{x}{x^2 + m^2x^2} \qquad \text{and} \qquad v = -\frac{mx}{x^2 + m^2x^2}$$

so that $v = -mu$, which is a line through the origin in the w-plane.

Task

Consider the line $ax + by + c = 0$ where $c \neq 0$. This represents a line in the z-plane which does **not** pass through the origin. To what type of curve does it map in the w-plane?

Your solution

Answer

The mapped curve is

$$\frac{au}{u^2 + v^2} - \frac{bv}{u^2 + v^2} + c = 0$$

Hence $au - bv + c(u^2 + v^2) = 0$. Dividing by c we obtain the equation:

$$u^2 + v^2 + \frac{a}{c}u - \frac{b}{c}v = 0$$

which is the equation of a circle in the w-plane which passes through the origin.

Similarly, it can be shown that a circle in the z-plane passing through the origin maps to a line in the w-plane which does not pass through the origin. Also a circle in the z-plane which does not pass through the origin maps to a circle in the w-plane which does pass through the origin. The inversion mapping is an example of the **bilinear transformation**:

$$w = f(z) = \frac{az + b}{cz + d} \qquad \text{where we demand that} \quad ad - bc \neq 0$$

(If $ad - bc = 0$ the mapping reduces to $f(z) =$ constant.)

Task

Find the set of bilinear transformations $w = f(z) = \dfrac{az + b}{cz + d}$ which map $z = 2$ to $w = 1$.

Your solution

Answer

$1 = \dfrac{2a+b}{2c+d}$. Hence $2a+b = 2c+d$.

Any values of a, b, c, d satisfying this equation will do provided $ad - bc \neq 0$.

Task

Find the bilinear transformations for which $z = -1$ is mapped to $w = 3$.

Your solution

Answer

$3 = \dfrac{-a+b}{-c+d}$. Hence $-a+b = -3c+3d$.

Example 6

Find the bilinear transformation which maps

(a) $z = 2$ to $w = 1$, **and**

(b) $z = -1$ to $w = 3$, **and**

(c) $z = 0$ to $w = -5$

Solution

We have the answers to (a) and (b) from the previous two Tasks:

$$\begin{aligned} 2a+b &= 2c+d \\ -a+b &= -3c+3d \end{aligned}$$

If $z = 0$ is mapped to $w = -5$ then $-5 = \dfrac{b}{d}$ so that $b = -5d$. Substituting this last relation into the first two obtained we obtain

$$\begin{aligned} 2a-2c-6d &= 0 \\ -a+3c-8d &= 0 \end{aligned}$$

Solving these two in terms of d we find $2c = 11d$ and $2a = 17d$. Hence the transformation is:

$w = \dfrac{17z-10}{11z+2}$ (note that the d's cancel in the numerator and denominator).

Some other mappings are shown in Figure 4.

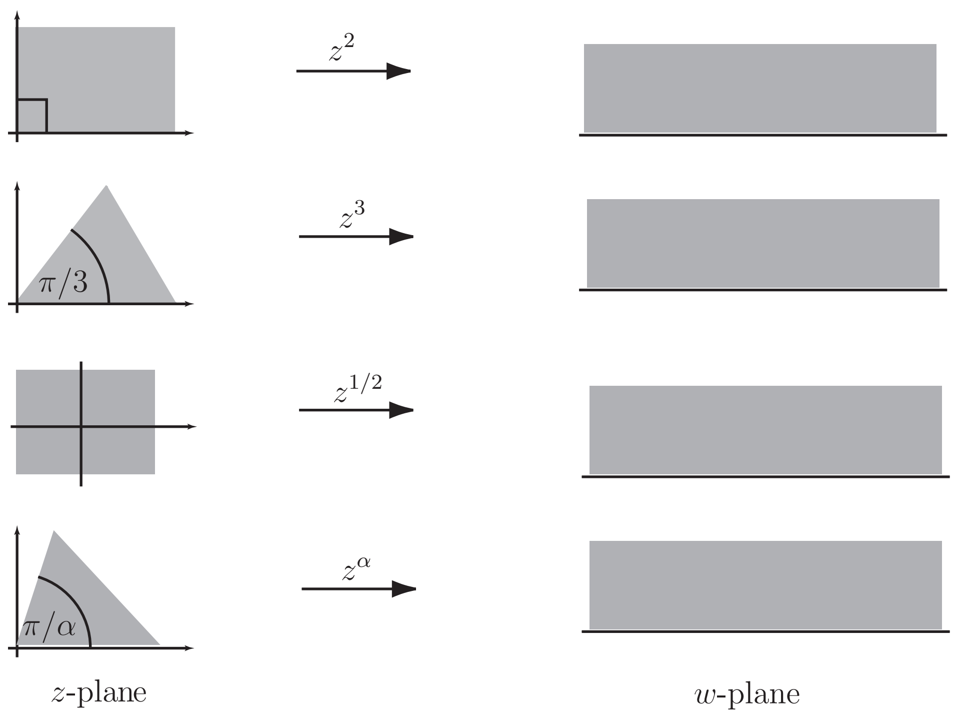


Figure 4

As an engineering application we consider the Joukowski transformation

$$w = z - \frac{\ell^2}{z} \qquad \text{where } \ell \text{ is a constant.}$$

It is used to map circles which contain $z = 1$ as an interior point and which pass through $z = -1$ into shapes resembling aerofoils. Figure 5 shows an example:

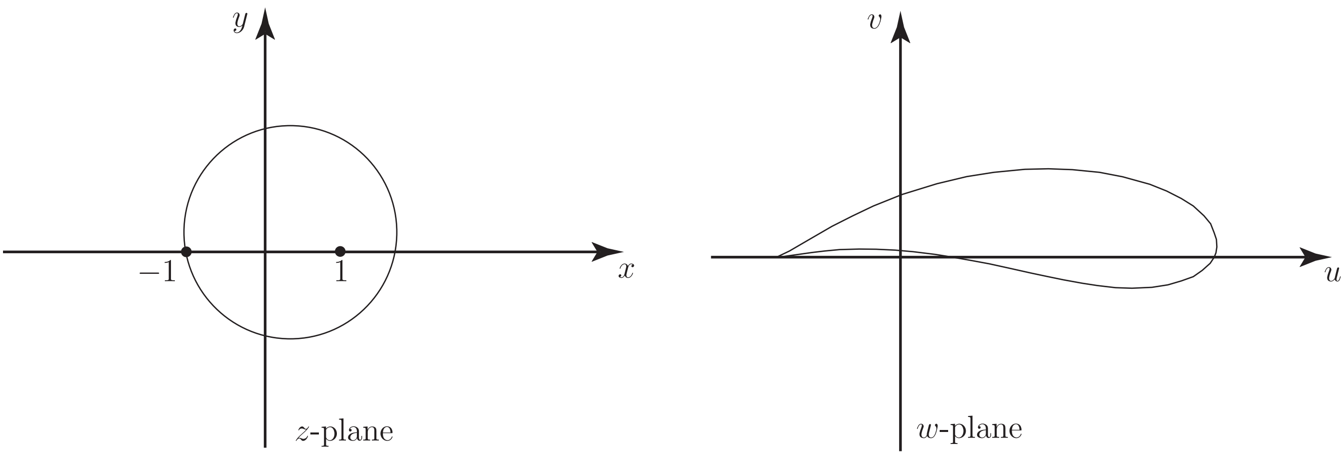


Figure 5

This creates a cusp at which the associated fluid velocity can be infinite. This can be avoided by adjusting the fluid flow in the z-plane. Eventually, this can be used to find the lift generated by such an aerofoil in terms of physical characteristics such as aerofoil shape and air density and speed.

Exercise

Find a bilinear transformation $w = \dfrac{az+b}{cz+d}$ which maps

(a) $z = 0$ into $w = \mathrm{i}$

(b) $z = -1$ into $w = 0$

(c) $z = -\mathrm{i}$ into $w = 1$

Answer

(a) $z = 0,\ w = \mathrm{i}$ gives $\mathrm{i} = \dfrac{b}{d}$ so that $b = d\mathrm{i}$

(b) $z = -1,\ w = 0$ gives $0 = \dfrac{-a+b}{-c+d}$ so $-a+b = 0$ so $a = b$.

(c) $z = -\mathrm{i},\ w = 1$ gives $1 = \dfrac{-a\mathrm{i}+b}{-c\mathrm{i}+d}$ so that $-c\mathrm{i}+d = -a\mathrm{i}+b = d+d\mathrm{i}$ (using (a) and (b))

We conclude from (c) that $-c = d$. We also know that $a = b = d\mathrm{i}$.

Hence $w = \dfrac{d\mathrm{i}z + d\mathrm{i}}{-dz+d} = \dfrac{\mathrm{i}z+\mathrm{i}}{-z+1}$

Standard Complex Functions

26.3

Introduction

In this Section we examine some of the standard functions of the calculus applied to functions of a complex variable. Note the similarities to and differences from their equivalents in real variable calculus.

Prerequisites

Before starting this Section you should ...

- understand the concept of a function of a complex variable and its derivative
- be familiar with the Cauchy-Riemann equations

Learning Outcomes

On completion you should be able to ...

- apply the standard functions of a complex variable discussed in this Section

1. Standard functions of a complex variable

The functions which we have considered so far have mostly been built from powers of z. We consider other functions here.

The exponential function

Using Euler's relation we are led to define

$$\mathrm{e}^z = \mathrm{e}^{x+\mathrm{i}y} = \mathrm{e}^x.\mathrm{e}^{\mathrm{i}y} = \mathrm{e}^x(\cos y + \mathrm{i}\sin y).$$

From this definition we can show readily that when $y = 0$ then e^z reduces to e^x, as it should.
If, as usual, we express w in real and imaginary parts then: $w = \mathrm{e}^z = u + \mathrm{i}v$ so that
$u = \mathrm{e}^x \cos y,\ \ v = \mathrm{e}^x \sin y$. Then

$$\frac{\partial u}{\partial x} = \mathrm{e}^x \cos y = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\mathrm{e}^x \sin y = -\frac{\partial v}{\partial x}.$$

Thus by the Cauchy-Riemann equations, **e^z is analytic everywhere**. It can be shown from the definition that if $f(z) = \mathrm{e}^z$ then $f'(z) = \mathrm{e}^z$, as expected.

Task

By calculating $|\mathrm{e}^z|^2$ show that $|\mathrm{e}^z| = \mathrm{e}^x$.

Your solution

Answer

$|\mathrm{e}^z|^2 = |\mathrm{e}^x \cos y + \mathrm{i}\mathrm{e}^x \sin y|^2 = (\mathrm{e}^x \cos y)^2 + (\mathrm{e}^x \sin y)^2 = (\mathrm{e}^x)^2(\cos^2 y + \sin^2 y) = (\mathrm{e}^x)^2.$

Therefore $|\mathrm{e}^z| = \mathrm{e}^x$.

Example 7

Find $\arg(\mathrm{e}^z)$.

Solution

If $\theta = \arg(\mathrm{e}^z) = \arg(\mathrm{e}^x(\cos y + \mathrm{i}\sin y))$ then $\tan\theta = \dfrac{\mathrm{e}^x \sin y}{\mathrm{e}^x \cos y} = \tan y$. Hence $\arg(\mathrm{e}^z) = y$.

Example 8

Find the solutions (for z) of the equation $\mathrm{e}^z = \mathrm{i}$

Solution

To find the solutions of the equation $\mathrm{e}^z = \mathrm{i}$ first write i as $0+1\mathrm{i}$ so that, equating real and imaginary parts of $\mathrm{e}^z = \mathrm{e}^x(\cos y + \mathrm{i}\sin y) = 0 + 1\mathrm{i}$ gives , $\mathrm{e}^x \cos y = 0$ and $\mathrm{e}^x \sin y = 1$.

Therefore $\cos y = 0$, which implies $y = \dfrac{\pi}{2} + k\pi$, where k is an integer. Then, using this we see that $\sin y = \pm 1$. But e^x must be positive, so that $\sin y = +1$ and $\mathrm{e}^x = 1$. This last equation has just one solution, $x = 0$. In order that $\sin y = 1$ we deduce that k must be even. Finally we have the complete solution to $\mathrm{e}^z = \mathrm{i}$, namely:

$$z = \left(\frac{\pi}{2} + k\pi\right)\mathrm{i},\ k \text{ an even integer.}$$

Task

Obtain all the solutions to $\mathrm{e}^z = -1$.

First find equations involving $\mathrm{e}^x \cos y$ and $\mathrm{e}^x \sin y$:

Your solution

Answer

As a first step to solving the equation $\mathrm{e}^z = -1$ obtain expressions for $\mathrm{e}^x \cos y$ and $\mathrm{e}^x \sin y$ from $\mathrm{e}^z = \mathrm{e}^x(\cos y + \mathrm{i}\sin y) = -1 + 0\mathrm{i}$. Hence $\mathrm{e}^x \cos y = -1$, $\mathrm{e}^x \sin y = 0$.

Now using the expression for $\sin y$ deduce possible values for y and hence from the first equation in $\cos y$ select the values of y satisfying both equations and deduce the form of the solutions for z:

Your solution

Answer

The two equations we have to solve are: $\mathrm{e}^x \cos y = -1$, $\mathrm{e}^x \sin y = 0$. Since $\mathrm{e}^x \neq 0$ we deduce $\sin y = 0$ so that $y = k\pi$, where k is an integer. Then $\cos y = \pm 1$ (depending as k is even or odd). But $\mathrm{e}^x \neq -1$ so $\mathrm{e}^x = 1$ leading to the only possible solution for x: $x = 0$. Then, from the second relation: $\cos y = -1$ so k must be an odd integer. Finally, $z = k\pi\mathrm{i}$ where k is an odd integer. Note the interesting result that if $z = 0 + \pi\mathrm{i}$ then $x = 0$, $y = \pi$ and $\mathrm{e}^z = 1(\cos\pi + \mathrm{i}\sin\pi) = -1$. Hence $\mathrm{e}^{\mathrm{i}\pi} = -1$, a remarkable equation relating fundamental numbers of mathematics in one relation.

Trigonometric functions

We denote the complex counterparts of the real trigonometric functions $\cos x$ and $\sin x$ by $\cos z$ and $\sin z$ and we define these functions by the relations:

$$\cos z \equiv \frac{1}{2}(\mathrm{e}^{\mathrm{i}z} + \mathrm{e}^{-\mathrm{i}z}), \quad \sin z \equiv \frac{1}{2\mathrm{i}}(\mathrm{e}^{\mathrm{i}z} - \mathrm{e}^{-\mathrm{i}z}).$$

These definitions are consistent with the definitions (Euler's relations) used for $\cos x$ and $\sin x$.

Other trigonometric functions can be defined in a way which parallels real variable functions. For example,

$$\tan z \equiv \frac{\sin z}{\cos z}.$$

Note that

$$\frac{d}{dz}(\sin z) = \frac{d}{dz}\left\{\frac{1}{2\mathrm{i}}(\mathrm{e}^{\mathrm{i}z} - \mathrm{e}^{-\mathrm{i}z})\right\} = \frac{1}{2\mathrm{i}}(\mathrm{i}\mathrm{e}^{\mathrm{i}z} + \mathrm{i}\mathrm{e}^{-\mathrm{i}z}) = \frac{1}{2}(\mathrm{e}^{\mathrm{i}z} + \mathrm{e}^{-\mathrm{i}z}) = \cos z.$$

Task

Show that $\dfrac{d}{dz}(\cos z) = -\sin z$.

Your solution

Answer

$$\begin{aligned}\frac{d}{dz}(\cos z) &= \frac{d}{dz}\left\{\frac{1}{2}(\mathrm{e}^{\mathrm{i}z} + \mathrm{e}^{-\mathrm{i}z})\right\}\\ &= \frac{\mathrm{i}}{2}(\mathrm{e}^{\mathrm{i}z} - \mathrm{e}^{-\mathrm{i}z}) = -\frac{1}{2\mathrm{i}}(\mathrm{e}^{\mathrm{i}z} - \mathrm{e}^{-\mathrm{i}z}) = -\sin z.\end{aligned}$$

Among other useful relationships are

$$\begin{aligned}\sin^2 z + \cos^2 z &= -\frac{1}{4}(\mathrm{e}^{\mathrm{i}z} - \mathrm{e}^{-\mathrm{i}z})^2 + \frac{1}{4}(\mathrm{e}^{\mathrm{i}z} + \mathrm{e}^{-\mathrm{i}z})^2\\ &= \frac{1}{4}(-\mathrm{e}^{2\mathrm{i}z} + 2 - \mathrm{e}^{-2\mathrm{i}z} + \mathrm{e}^{2\mathrm{i}z} + 2 + \mathrm{e}^{-2\mathrm{i}z}) = \frac{1}{4}\cdot 4 = 1.\end{aligned}$$

Also, using standard trigonometric expansions:

$$\begin{aligned}\sin z = \sin(x+\mathrm{i}y) = \sin x\cos \mathrm{i}y + \cos x \sin \mathrm{i}y &= \sin x\left(\frac{\mathrm{e}^{-y}+\mathrm{e}^{y}}{2}\right) + \cos x\left(\frac{\mathrm{e}^{-y}-\mathrm{e}^{y}}{2\mathrm{i}}\right)\\ &= \sin x\cosh y - \frac{1}{\mathrm{i}}\cos x\sinh y\\ &= \sin x\cosh y + \mathrm{i}\cos x\sinh y.\end{aligned}$$

Task

Show that $\cos z = \cos x\cosh y - \mathrm{i}\sin x\sinh y$.

Your solution

Answer

$$\begin{aligned}\cos z = \cos(x+\mathrm{i}y) = \cos x\cos \mathrm{i}y - \sin x \sin \mathrm{i}y &= \cos x\left(\frac{\mathrm{e}^{-y}+\mathrm{e}^{y}}{2}\right) - \sin x\left(\frac{\mathrm{e}^{-y}-\mathrm{e}^{y}}{2\mathrm{i}}\right)\\ &= \cos x\cosh y + \frac{1}{\mathrm{i}}\sin x\sinh y\\ &= \cos x\cosh y - \mathrm{i}\sin x\sinh y\end{aligned}$$

Hyperbolic functions

In an obvious extension from their real variable counterparts we define functions $\cosh z$ and $\sinh z$ by the relations:

$$\cosh z = \frac{1}{2}(\mathrm{e}^{z}+\mathrm{e}^{-z}), \qquad \sinh z = \frac{1}{2}(\mathrm{e}^{z}-\mathrm{e}^{-z}).$$

Note that $\dfrac{d}{dz}(\sinh z) = \dfrac{1}{2}\dfrac{d}{dz}(\mathrm{e}^{z}-\mathrm{e}^{-z}) = \dfrac{1}{2}(\mathrm{e}^{z}+\mathrm{e}^{-z}) = \cosh z.$

Task

Determine $\frac{d}{dz}(\cosh z)$.

Your solution

Answer

$\frac{d}{dz}(\cosh z) = \frac{1}{2}\frac{d}{dz}(\mathrm{e}^z + \mathrm{e}^{-z}) = \frac{1}{2}(\mathrm{e}^z - \mathrm{e}^{-z}) = \sinh z.$

Other relationships parallel those for trigonometric functions. For example it can be shown that

$\cosh z = \cosh x \cos y + \mathrm{i} \sinh x \sin y$ and $\sinh z = \sinh x \cos y + \mathrm{i} \cosh x \sin y$

These relationships can be deduced from the general relations between trigonometric and hyperbolic functions (can you prove these?):

$\cosh \mathrm{i}z = \cos z$ and $\sinh \mathrm{i}z = \mathrm{i} \sin z$

Example 9

Show that $\cosh^2 z - \sinh^2 z = 1$.

Solution

$$\begin{aligned} \cosh^2 z &= \frac{1}{4}(\mathrm{e}^z + \mathrm{e}^{-z})^2 = \frac{1}{4}(\mathrm{e}^{2z} + 2 + \mathrm{e}^{-2z}) \\ \sinh^2 z &= \frac{1}{4}(\mathrm{e}^z - \mathrm{e}^{-z})^2 = \frac{1}{4}(\mathrm{e}^{2z} - 2 + \mathrm{e}^{-2z}) \\ \therefore \quad \cosh^2 z &- \sinh^2 z = \frac{1}{4}(2+2) = 1. \end{aligned}$$

Alternatively since $\cosh \mathrm{i}z = \cos z$ then $\cosh z = \cos \mathrm{i}z$ and since $\sinh \mathrm{i}z = \mathrm{i} \sin z$ it follows that $\sinh z = -\mathrm{i} \sin \mathrm{i}z$ so that

$$\cosh^2 z - \sinh^2 z = \cos^2 \mathrm{i}z + \sin^2 \mathrm{i}z = 1$$

Logarithmic function

Since the exponential function is one-to-one it possesses an inverse function, which we call $\ln z$. If $w = u + \mathrm{i}v$ is a complex number such that $\mathrm{e}^w = z$ then the logarithm function is defined through the statement: $w = \ln z$. To see what this means it will be convenient to express the complex number z in exponential form as discussed in HELM 10.3: $z = r\mathrm{e}^{\mathrm{i}\theta}$ and so

$$w = u + \mathrm{i}v = \ln(r\mathrm{e}^{\mathrm{i}\theta}) = \ln r + \mathrm{i}\theta.$$

Therefore $u = \ln r = \ln |z|$ and $v = \theta$. However $\mathrm{e}^{\mathrm{i}(\theta+2k\pi)} = \mathrm{e}^{\mathrm{i}\theta}.\mathrm{e}^{2k\pi\mathrm{i}} = \mathrm{e}^{\mathrm{i}\theta}.1 = \mathrm{e}^{\mathrm{i}\theta}$ for integer k. This means that we must be more general and say that $v = \theta + 2k\pi$, k integer. If we take $k = 0$ and confine v to the interval $-\pi < v \leq \pi$, the corresponding value of w is called the **principal value** of $\ln z$ and is written $\mathrm{Ln}(z)$.

In general, to each value of $z \neq 0$ there are an infinite number of values of $\ln z$, each with the same real part. These values are partitioned into **branches** of range 2π by considering in turn $k = 0$, $k = \pm 1$, $k = \pm 2$ etc. Each branch is defined on the whole $z-$plane with the exception of the point $z = 0$. On each branch the function $\ln z$ is analytic with derivative $\dfrac{1}{z}$ **except** along the negative real axis (and at the origin). Figure 6 represents the situation schematically.

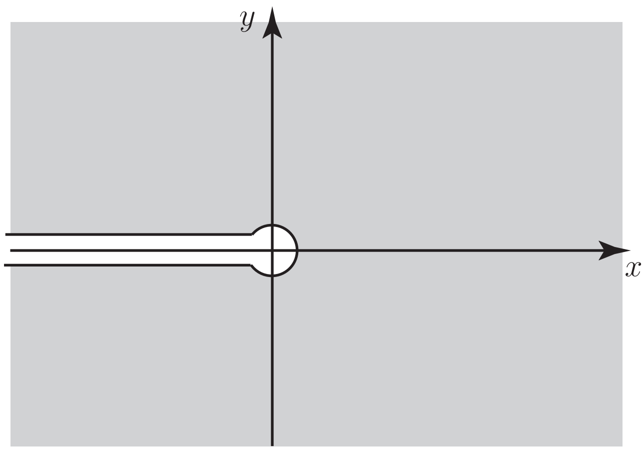


Figure 6

The familiar properties of a logarithm apply to $\ln z$, **except** that in the case of $\mathrm{Ln}(z)$ we have to adjust the argument by a multiple of 2π to comply with $-\pi < \arg(\mathrm{Ln}(z)) \leq \pi$
For example

(a) $\ln(1 + \mathrm{i}) = \ln\left(\sqrt{2}\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}\right) = \ln\sqrt{2} + \mathrm{i}\left(\frac{\pi}{4} + 2k\pi\right)$
$$= \frac{1}{2}\ln 2 + \mathrm{i}\left(\frac{\pi}{4} + 2k\pi\right).$$

(b) $\mathrm{Ln}(1 + \mathrm{i}) = \dfrac{1}{2}\ln 2 + \mathrm{i}\dfrac{\pi}{4}.$

(c) If $\ln z = 1 - \mathrm{i}\pi$ then $z = \mathrm{e}^{1-\mathrm{i}\pi} = \mathrm{e}^1.\mathrm{e}^{-\mathrm{i}\pi} = -e.$

Task

Find (a) $\ln(1-\mathrm{i})$ (b) $\mathrm{Ln}(1-\mathrm{i})$ (c) z when $\ln z = 1 + \mathrm{i}\pi$

Your solution

Answer

(a) $\ln(1-\mathrm{i}) = \ln\left(\sqrt{2}\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}}\right) = \ln\sqrt{2} + \mathrm{i}\left(-\dfrac{\pi}{4} + 2k\pi\right) = \dfrac{1}{2}\ln 2 + \left(-\dfrac{\pi}{4} + 2k\pi\right).$

(b) $\mathrm{Ln}(1-\mathrm{i}) = \dfrac{1}{2}\ln 2 - \mathrm{i}\dfrac{\pi}{4}.$

(c) $z = \mathrm{e}^{1+\mathrm{i}\pi} = \mathrm{e}^{1}.\mathrm{e}^{\mathrm{i}\pi} = -e.$

Exercises

1. Obtain all the solutions to $\mathrm{e}^z = 1$.
2. Show that $1 + \tan^2 z \equiv \sec^2 z$
3. Show that $\cosh^2 z + \sinh^2 z \equiv \cosh 2z$
4. Find $\ln(\sqrt{3} + \mathrm{i})$, $\mathrm{Ln}(\sqrt{3} + \mathrm{i})$.
5. Find z when $\ln z = 2 + \pi\mathrm{i}$

Answers

1. $\mathrm{e}^x \cos y = 1$ and $\mathrm{e}^x \sin y = 0$ $\quad\therefore\quad$ $\sin y = 0$ and $y = k\pi$ where k is an integer.

   Then $\cos y = \pm 1$ and since $\mathrm{e}^x > 0$ we take $\cos y = 1$ and $\mathrm{e}^x = 1$ so that $x = 0$. Then $\cos y = 1$ and k is an even integer. $\quad\therefore\quad$ $z = 2k\pi\mathrm{i}$ for k integer.

2. $\tan z = \dfrac{1}{\mathrm{i}}\left(\dfrac{\mathrm{e}^{\mathrm{i}z} - \mathrm{e}^{-\mathrm{i}z}}{\mathrm{e}^{\mathrm{i}z} + \mathrm{e}^{-\mathrm{i}z}}\right)$

   $$1 + \tan^2 z = 1 - \frac{\mathrm{e}^{2\mathrm{i}z} + \mathrm{e}^{-2\mathrm{i}z} - 2}{\mathrm{e}^{2\mathrm{i}z} + \mathrm{e}^{-2\mathrm{i}z} + 2} = \frac{4}{\mathrm{e}^{2\mathrm{i}z} + \mathrm{e}^{-2\mathrm{i}z} + 2} = \frac{2^2}{(\mathrm{e}^{\mathrm{i}z} + \mathrm{e}^{-\mathrm{i}z})^2} = \frac{1}{\cos^2 z} = \sec^2 z.$$

3. $\cosh^2 z + \sinh^2 z = \dfrac{1}{4}(\mathrm{e}^{2z} + 2 + \mathrm{e}^{-2z}) + \dfrac{1}{4}(\mathrm{e}^{2z} - 2 + \mathrm{e}^{-2z}) = \dfrac{1}{2}(\mathrm{e}^{2z} + \mathrm{e}^{-2z}) = \cosh 2z.$

4. $\ln(\sqrt{3} + 1) = \ln\sqrt{5} + \mathrm{i}(\frac{\pi}{6} + 2k\pi) = \frac{1}{2}\ln 5 + \mathrm{i}(\frac{\pi}{6} + 2k\pi).$ $\qquad \mathrm{Ln}(\sqrt{3} + \mathrm{i}) = \frac{1}{2}\ln 5 + \mathrm{i}\frac{\pi}{6}.$

5. If $\ln z = 2 + \pi\mathrm{i}$ then $z = \mathrm{e}^{2+\pi i} = \mathrm{e}^2\mathrm{e}^{\mathrm{i}\pi} = -\mathrm{e}^2.$

Basic Complex Integration

26.4

Introduction

Complex variable techniques have been used in a wide variety of areas of engineering. This has been particularly true in areas such as electromagnetic field theory, fluid dynamics, aerodynamics and elasticity. With the rapid developments in computer technology and the consequential use of sophisticated algorithms for analysis and design in engineering there has been, in recent years, less emphasis on the use of complex variable techniques and a shift towards numerical techniques applied directly to the underlying full partial differential equations which model the situation. However it is useful to have an analytical solution, possibly for an idealized model in order to develop a better understanding of the solution and to develop confidence in numerical estimates for the solution of more sophisticated models.

The design of aerofoil sections for aircraft is an area where the theory was developed using complex variable techniques. Throughout engineering, transforms defined as complex integrals in one form or another play a major role in analysis and design. The use of complex variable techniques allows us to develop criteria for the stability of systems.

Prerequisites

Before starting this Section you should ...

- be able to carry out integration of simple real-valued functions
- be familiar with the basic ideas of functions of a complex variable
- be familiar with line integrals

Learning Outcomes

On completion you should be able to ...

- understand the concept of complex integrals

1. Complex integrals

If $f(z)$ is a single-valued, continuous function in some region R in the complex plane then we define the **integral** of $f(z)$ **along a path** C in R (see Figure 7) as

$$\int_C f(z)\,dz = \int_C (u + \mathrm{i}v)(dx + \mathrm{i}\,dy).$$

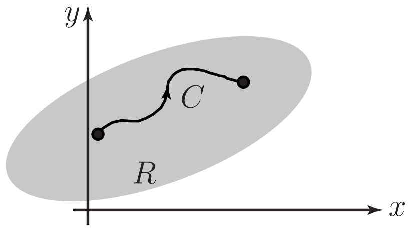


Figure 7

Here we have written $f(z)$ and dz in real and imaginary parts:

$$f(z) = u + \mathrm{i}v \quad \text{and} \quad dz = dx + \mathrm{i}\,dy.$$

Then we can separate the integral into real and imaginary parts as

$$\int_C f(z)\,dz = \int_C (u\,dx - v\,dy) + \mathrm{i}\int_C (v\,dx + u\,dy).$$

We often interpret real integrals in terms of area; now we define complex integrals in terms of line integrals over paths in the complex plane. The line integrals are evaluated as described in HELM 29.

Example 10

Obtain the complex integral:

$$\int_C z\,dz$$

where C is the straight line path from $z = 1 + \mathrm{i}$ to $z = 3 + \mathrm{i}$. See Figure 8.

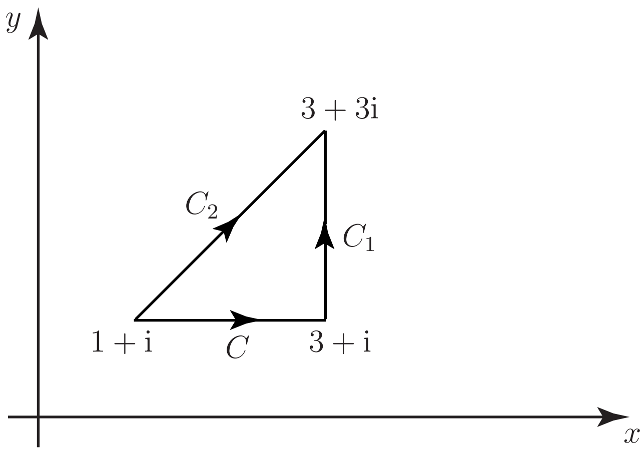


Figure 8

Solution

Here, since y is constant $(y = 1)$ along the given path then $z = x + \mathrm{i}$, implying that $u = x$ and $v = 1$. Also, as y is constant, $dy = 0$.

Therefore,

$$\begin{aligned}\int_C z\,dz &= \int_C (u\,dx - v\,dy) + \mathrm{i}\int_C (v\,dx + u\,dy)\\ &= \int_1^3 x\,dx + \mathrm{i}\int_1^3 1\,dx\\ &= \left[\frac{x^2}{2}\right]_1^3 + \mathrm{i}\left[x\right]_1^3\\ &= \left(\frac{9}{2} - \frac{1}{2}\right) + \mathrm{i}(3-1) = 4 + 2\mathrm{i}.\end{aligned}$$

Task

Evaluate $\displaystyle\int_{C_1} z\,dz$ where C_1 is the straight line path from $z = 3 + \mathrm{i}$ to $z = 3 + 3\mathrm{i}$.

First obtain expressions for u, v, dx and dy by finding an appropriate expression for z along the path:

Your solution

Answer

Along the path $z = 3 + \mathrm{i}y$, implying that $u = 3$ and $v = y$. Also $dz = 0 + \mathrm{i}dy$.

Now find limits on y:

Your solution

Answer

The limits on y are: $y = 1$ to $y = 3$.

Now evaluate the integral:

Your solution

Answer

$$\begin{aligned}\int_{C_1} z\,dz &= \int_{C_1}(u\,dx - v\,dy) + \mathsf{i}\int_{C_1}(v\,dx + u\,dy)\\ &= \int_1^3 -y\,dy + \mathsf{i}\int_1^3 3\,dy\\ &= \left[\frac{-y^2}{2}\right]_1^3 + \mathsf{i}\left[3y\right]_1^3 = \left(-\frac{9}{2}+\frac{1}{2}\right) + i(9-3)\end{aligned}$$

$= -4 + 6\mathsf{i}.$

Task

Evaluate $\displaystyle\int_{C_2} z\,dz$ where C_2 is the straight line path from $z = 1 + \mathsf{i}$ to $z = 3 + 3\mathsf{i}$.

Your solution

Answer

We first need to find the equation of the line C_2 in the Argand plane.

We note that both points lie on the line $y = x$ so the complex equation of the straight line is $z = x + \mathsf{i}x$ giving $u = x$ and $v = x$. Also $dz = dx + \mathsf{i}dx = (1 + \mathsf{i})dx$.

$$\therefore \qquad \int_{C_2} z\,dz = \int_{C_2}(x\,dx - x\,dx) + \mathsf{i}\int_{C_2}(x\,dx + x\,dx).$$

$$= \mathsf{i}\int_{C_2}(2x\,dx)$$

Next, we see that the limits on x are $x = 1$ to $x = 3$. We are now in a position to evaluate the integral:

$$\int_{C_2} z\,dz = \mathsf{i}\int_1^3 2x\,dx = \mathsf{i}\left[x^2\right]_1^3 = \mathsf{i}(9-1) = 8\mathsf{i}.$$

Note that this result is the sum of the integrals along C and C_1. You might have expected this.

A more intricate example now follows.

Example 11

Evaluate $\displaystyle\int_{C_1} z^2\,dz$ where C_1 is that part of the unit circle going anticlockwise from the point $z = 1$ to the point $z = \mathrm{i}$. See Figure 9.

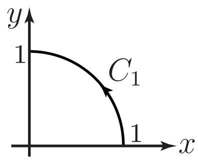


Figure 9

Solution

First, note that $\quad z^2 = (x + \mathrm{i}y)^2 = x^2 - y^2 + 2xy\mathrm{i}$ and $dz = dx + \mathrm{i}\,dy$ giving

$$\int_{C_1} z^2\,dx = \int_{C_1} \{(x^2 - y^2)\,dx - 2xy\,dy\} + \mathrm{i}\int_{C_1} \{2xy\,dx + (x^2 - y^2)dy\}.$$

This is obtained by simply expressing the integral in real and imaginary parts. These integrals cannot be evaluated in this form since y and x are related. Instead we re-write them in terms of the single variable θ.

Note that on the unit circle: $\quad x = \cos\theta,\ y = \sin\theta \quad$ so that $\quad dx = -\sin\theta\,d\theta$ and $dy = \cos\theta\,d\theta$.

The expressions $(x^2 - y^2)$ and $2xy$ can be expressed in terms of 2θ since

$$x^2 - y^2 = \cos^2\theta - \sin^2\theta \equiv \cos 2\theta \qquad 2xy = 2\cos\theta\sin\theta \equiv \sin 2\theta.$$

Now as the point z moves from $z = 1$ to $z = \mathrm{i}$ along the path C_1 the parameter θ changes from $\theta = 0$ to $\theta = \dfrac{\pi}{2}$. Hence,

$$\int_{C_1} f(z)\,dz = \int_0^{\frac{\pi}{2}} \{-\cos 2\theta\sin\theta\,d\theta - \sin 2\theta\cos\theta\,d\theta\} + \mathrm{i}\int_0^{\frac{\pi}{2}} \{-\sin 2\theta\sin\theta\,d\theta + \cos 2\theta\cos\theta\,d\theta\}.$$

We can simplify these daunting-looking integrals by using the trigonometric identities:

$$\sin(A + B) \equiv \sin A\cos B + \cos A\sin B \qquad \text{and} \qquad \cos(A + B) \equiv \cos A\cos B - \sin A\sin B.$$

We obtain (choosing $A = 2\theta$ and $B = \theta$ in both expressions):

$$-\cos 2\theta\sin\theta - \sin 2\theta\cos\theta \equiv -(\sin\theta\cos 2\theta + \cos\theta\sin 2\theta) \equiv -\sin 3\theta.$$

Also $\qquad -\sin 2\theta\sin\theta + \cos 2\theta\cos\theta \equiv \cos 3\theta.$

Now we can complete the evaluation of our integral:

$$\begin{aligned}\int_{C_1} f(z)\,dx &= \int_0^{\frac{\pi}{2}} (-\sin 3\theta)d\theta + \mathrm{i}\int_0^{\frac{\pi}{2}} \cos 3\theta\,d\theta \\ &= \left[\frac{1}{3}\cos 3\theta\right]_0^{\frac{\pi}{2}} + \mathrm{i}\left[\frac{1}{3}\sin 3\theta\right]_0^{\frac{\pi}{2}} = (0 - \frac{1}{3}) + \mathrm{i}\left(-\frac{1}{3} - 0\right) = -\frac{1}{3} - \frac{1}{3}\mathrm{i} \equiv -\frac{1}{3}(1 + \mathrm{i}).\end{aligned}$$

In the last Task we integrated z^2 over a given path. We had to perform some intricate mathematics to get the value. It would be convenient if there was a simpler way to obtain the value of such complex integrals. This is explored in the following Tasks.

Task

Evaluate $\left[\frac{1}{3}z^3\right]_1^{\mathsf{i}}$

Your solution

Answer

We obtain $-\frac{1}{3}(1+\mathsf{i})$ again, which is the same result as from the previous Task.

It would seem that, by carrying out an analogue of real integration (simply integrating the function and substituting in the limits) we can obtain the answer much more easily. Is this coincidence?

If you return to the first Task of this Section you will note:

$$\begin{aligned}\left[\frac{1}{2}z^2\right]_{1+\mathsf{i}}^{3+3\mathsf{i}} &= \frac{1}{2}\left\{(3+3\mathsf{i})^2-(1+\mathsf{i})^2\right\}\\ &= \frac{1}{2}\{9+18\mathsf{i}-9-1-2\mathsf{i}+1\}\\ &= \frac{1}{2}(16\mathsf{i})=8\mathsf{i},\end{aligned}$$

the result we obtained earlier.

We shall investigate these ‘coincidences’ in Section 26.5.

As a variation on this example, suppose that the path C_1 is the entire circumference of the unit circle travelled in an anti-clockwise direction. The limits are $\theta=0$ and $\theta=2\pi$. Hence

$$\begin{aligned}\int_{C_1} f(z)\,dz &= \int_0^{2\pi}(-\sin 3\theta)d\theta+\mathsf{i}\int_0^{2\pi}\cos 3\theta\,d\theta\\ &= \left[\frac{1}{3}\cos 3\theta\right]_0^{2\pi}+\mathsf{i}\left[\frac{1}{3}\sin 3\theta\right]_0^{2\pi}\\ &= (\frac{1}{3}-\frac{1}{3})+\mathsf{i}(0-0)=0.\end{aligned}$$

Is there an underlying reason for this result? (We shall see in Section 26.5.)

Another technique for evaluating integrals taken around the unit circle is shown in the next example, in which we need to evaluate

$$\oint_C \frac{1}{z}dz \qquad \text{where } C \text{ is the unit circle.}$$

Note the use of $\oint$ since we have a closed path; we could have used this notation earlier.

Task

Evaluate $\oint_C \frac{1}{z}dz$ where C is the unit circle.

First show that a point z on the unit circle can be written $z = e^{i\theta}$ and hence find dz in terms of θ:

Your solution

Answer

On the unit circle a point (x, y) is such that $x = \cos\theta$, $y = \sin\theta$ and hence $z = \cos\theta + i\sin\theta$ which, using De Moivre's theorem, can be seen to be $z = e^{i\theta}$.

Then $\frac{dz}{d\theta} = ie^{i\theta}$ so that $dz = ie^{i\theta}d\theta$.

Now evaluate the integral $\oint_C \frac{1}{z}dz$.

Your solution

Answer

$$\oint_C \frac{1}{z}dz = \int_0^{2\pi} \frac{1}{e^{i\theta}} ie^{i\theta} d\theta = \int_0^{2\pi} i d\theta = 2\pi i.$$

We now quote one of the most important results in complex integration which incorporates the last result.

Key Point 1

If n is an integer and C is the circle centre $z = z_0$ and radius r, that is, it has equation $|z - z_0| = r$ then

$$\oint_C \frac{dz}{(z - z_0)^n} = \begin{cases} 0, & n \neq 1; \\ 2\pi i, & n = 1. \end{cases}$$

Note that the result is independent of the value of r.

Engineering Example 1

Two-dimensional fluid flow

Introduction

Functions of a complex variable find a very elegant application in the mathematical treatment of two-dimensional fluid flow.

Problem in words

Find the forces and moments due to fluid flowing past a cylinder.

Mathematical statement of the problem

Figure 10 shows a cross section of a cylinder (not necessarily circular), whose boundary is C, placed in a steady non-viscous flow of an ideal fluid; the flow takes place in planes parallel to the xy plane. The cylinder is out of the plane of the paper. The flow of the fluid exerts forces and turning moments upon the cylinder. Let X, Y be the components, in the x and y directions respectively, of the force on the cylinder and let M be the anticlockwise moment (on the cylinder) about the orgin.

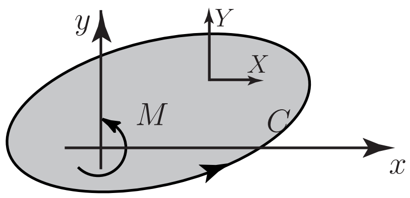


Figure 10

Blasius' theorem (which we shall not prove) states that

$$X - \mathrm{i}Y = \frac{1}{2}\mathrm{i}\rho \oint_C \left(\frac{dw}{dz}\right)^2 dz \qquad \text{and} \qquad M = \mathrm{Re}\left\{-\frac{1}{2}\rho \oint_C z\left(\frac{dw}{dz}\right)^2 dz\right\}$$

where Re denotes the real part, ρ is the (constant) density of the fluid and $w = u + iv$ is the complex potential (see Section 261) for the flow. Both ρ and ω are presumed known.

Mathematical analysis

We shall find X, Y and M if the cylinder has a circular cross section and the boundary is specified by $|z| = a$. Let the flow be a uniform stream with speed U.

Now, using a standard result, the complex potential describing this situation is:

$$w = U\left(z + \frac{a^2}{z}\right) \qquad \text{so that} \qquad \frac{dw}{dz} = U\left(1 - \frac{a^2}{z^2}\right) \text{ and } \left(\frac{dw}{dz}\right)^2 = U^2\left(1 - \frac{2a^2}{z^2} + \frac{a^4}{z^4}\right).$$

Using Key Point 1 with $z_0 = 0$:

$$X - \mathrm{i}Y = \frac{1}{2}\mathrm{i}\rho \oint_C \left(\frac{dw}{dz}\right)^2 dz = \frac{1}{2}\mathrm{i}\rho U^2 \oint \left(1 - \frac{2a^2}{z^2} + \frac{a^4}{z^4}\right) dz = 0 \quad \text{so} \quad X = Y = 0.$$

Also, $z\left(\dfrac{dw}{dz}\right)^2 = U^2\left(z - \dfrac{2a^2}{z} + \dfrac{a^4}{z^3}\right)$. The only term to contribute to M is $\dfrac{-2a^2U^2}{z}$.

Again using Key Point 1, this leads to $-4\pi a^2 U^2 \mathrm{i}$ and this has zero real part. Hence $M = 0$, also.

Interpretation

The implication is that no net force or moment acts on the cylinder. This is not so in practice. The discrepancy arises from neglecting the viscosity of the fluid.

Exercises

1. Obtain the integral $\displaystyle\int_C z\,dz$ along the straight-line paths

   (a) from $z = 2 + 2\mathrm{i}$ to $z = 5 + 2\mathrm{i}$

   (b) from $z = 5 + 2\mathrm{i}$ to $z = 5 + 5\mathrm{i}$

   (c) from $z = 2 + 2\mathrm{i}$ to $z = 5 + 5\mathrm{i}$

2. Find $\displaystyle\int_C (z^2 + z)\,dz$ where C is the part of the unit circle going anti-clockwise from the point $z = 1$ to the point $z = \mathrm{i}$.

3. Find $\displaystyle\oint_C f(z)\,dz$ where C is the circle $|z - z_0| = r$ for the cases

   (a) $f(z) = \dfrac{1}{z^2}$. $z_0 = 1$

   (b) $f(z) = \dfrac{1}{(z-1)^2}$, $z_0 = 1$

   (c) $f(z) = \dfrac{1}{z - 1 - \mathrm{i}}$, $z_0 = 1 + \mathrm{i}$

Answers

1. (a) Here y is constant along the given path $z = x + 2\mathsf{i}$ so that $u = x$ and $v = 2$. Also $dy = 0$. Thus

$$\begin{aligned} \int_C z\,dz &= \int_C (udx - vdy) + \mathsf{i}\int_C (vdx + udy) = \int_2^5 xdx + \mathsf{i}\int_2^5 2dx \\ &= \left[\frac{x^2}{2}\right]_2^5 + \mathsf{i}\left[2x\right]_2^5 = (\frac{25}{2} - \frac{4}{2}) + \mathsf{i}(10-4) = \frac{21}{2} + 6\mathsf{i}. \end{aligned}$$

(b) Here $dx = 0, \quad v = y, \quad u = 5$. Thus

$$\begin{aligned} \int_C z\,dz &= \int_2^5 (-y)dy + \mathsf{i}\int_2^5 5dy \\ &= \left[-\frac{y^2}{2}\right]_2^5 + \mathsf{i}\left[5y\right]_2^5 = (-\frac{25}{2} + \frac{4}{2}) + \mathsf{i}(25-10) = -\frac{21}{2} + 15\mathsf{i}. \end{aligned}$$

(c) $z = x + \mathsf{i}x, \quad u = x,\ v = x, \quad dz = (1+\mathsf{i})dx$, so

$$\int_C z\,dz = \int_C (xdx - xdx) + \mathsf{i}\int_C (xdx + xdx) = \mathsf{i}\int_C 2xdx = 2\mathsf{i}\left[\frac{x^2}{2}\right]_2^5 = 21\mathsf{i}.$$

Note that the result in (c) is the sum of the results in (a) and (b).

2. $\displaystyle\int_C (z^2 + z)\,dz = \left[\frac{z^3}{3} + \frac{z^2}{2}\right]_1^{\mathsf{i}} = (\frac{1}{3}\mathsf{i}^3 + \frac{\mathsf{i}^2}{2}) - (\frac{1}{3} + \frac{1}{2}) = -\frac{4}{3} - \frac{1}{3}\mathsf{i}.$

3. Using Key Point 1 we have (a) 0, (b) 0, (c) $2\pi\mathsf{i}$.

Note that in all cases the result is independent of r.

Cauchy's Theorem

26.5

Introduction

In this Section we introduce Cauchy's theorem which allows us to simplify the calculation of certain contour integrals. A second result, known as Cauchy's integral formula, allows us to evaluate some integrals of the form $\oint_C \dfrac{f(z)}{z - z_0}\, dz$ where z_0 lies inside C.

Prerequisites

Before starting this Section you should . . .

- be familiar with the basic ideas of functions of a complex variable
- be familiar with line integrals

Learning Outcomes

On completion you should be able to . . .

- state and use Cauchy's theorem
- state and use Cauchy's integral formula

1. Cauchy's theorem

Simply-connected regions

A region is said to be simply-connected if any closed curve in that region can be shrunk to a point without any part of it leaving a region. The interior of a square or a circle are examples of simply connected regions. In Figure 11 (a) and (b) the shaded grey area is the region and a typical closed curve is shown inside the region. In Figure 11 (c) the region contains a hole (the white area inside). The shaded region between the two circles is **not** simply-connected; curve C_1 can shrink to a point but curve C_2 cannot shrink to a point without leaving the region, due to the hole inside it.

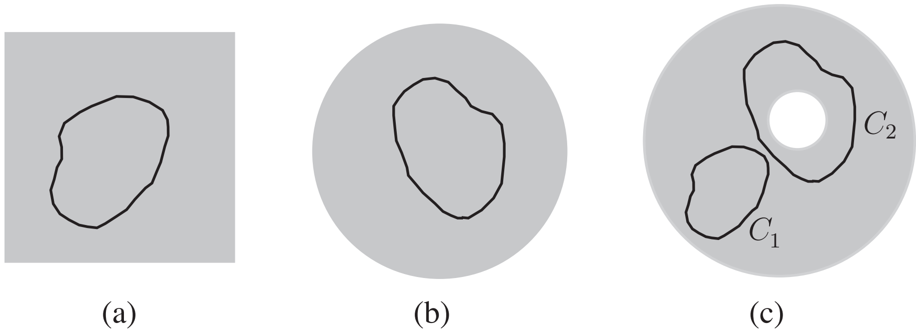


Figure 11

> **Key Point 2**
>
> **Cauchy's Theorem**
>
> The theorem states that if $f(z)$ is analytic everywhere within a simply-connected region then:
>
> $$\oint_C f(z)\,dz = 0$$
>
> for every simple closed path C lying in the region.
>
> This is perhaps the most important theorem in the area of complex analysis.

As a straightforward example note that $\oint_C z^2\,dz = 0$, where C is the unit circle, since z^2 is analytic everywhere (see Section 261). Indeed $\oint_C z^2\,dz = 0$ for *any* simple contour: it need not be circular.

Consider the contour shown in Figure 12 and assume $f(z)$ is analytic everywhere on and inside the contour C.

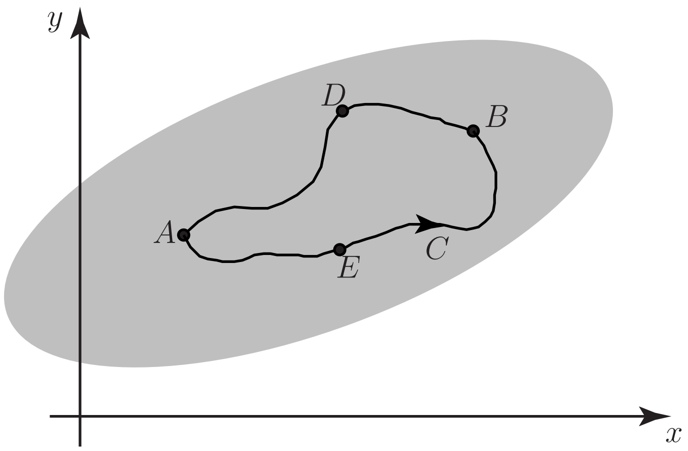


Figure 12

Then by analogy with real line integrals

$$\int_{AEB} f(z)\,dz + \int_{BDA} f(z)\,dz = \oint_C f(z)\,dz = 0 \qquad \text{by Cauchy's theorem.}$$

Therefore

$$\int_{AEB} f(z)\,dz = -\int_{BDA} f(z)\,dz = \int_{ADB} f(z)\,dz$$

(since reversing the direction of integration reverses the sign of the integral).

This implies that we may choose any path between A and B and the integral will have the same value **providing $f(z)$ is analytic in the region concerned.**

Integrals of analytic functions only depend on the positions of the points A and B, not on the path connecting them. This explains the 'coincidences' referred to previously in Section 26.4.

Using 'simple' integration evaluate $\int_{\mathrm{i}}^{1+2\mathrm{i}} \cos z\ dz$, and explain why this is valid.

Your solution

Answer

$$\int_{\mathrm{i}}^{1+2\mathrm{i}} \cos z\ dz = \Big[\ \sin z\Big]_{\mathrm{i}}^{1+2\mathrm{i}} = \sin(1+2\mathrm{i}) - \sin\ \mathrm{i}.$$

This way of determining the integral is legitimate because $\cos z$ is analytic (everywhere).

We now investigate what occurs when the closed path of integration does not necessarily lie within a simply-connected region. Consider the situation described in Figure 13.

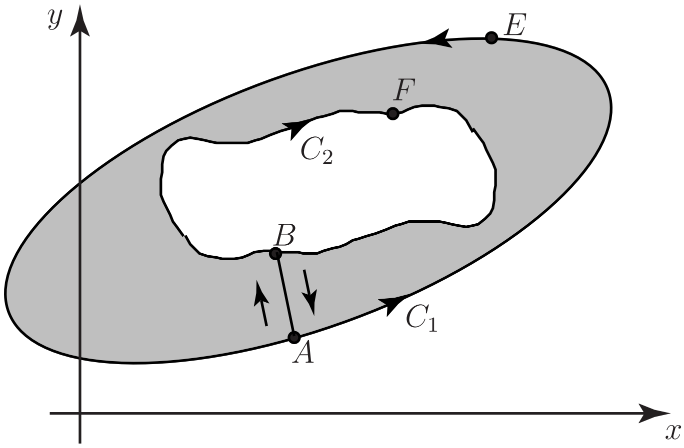


Figure 13

Let $f(z)$ be analytic in the region bounded by the closed curves C_1 and C_2. The region is cut by the line segment joining A and B.

Consider now the closed curve $AEABFBA$ travelling in the direction indicated by the arrows. No line can cross the cut AB and be regarded as remaining in the region. Because of the cut the shaded region is **simply connected**. Cauchy's theorem therefore applies (see Key Point 2).

Therefore

$$\oint_{AEABFBA} f(z)\,dz = 0 \quad \text{since } f(z) \text{ is analytic within and on the curve } AEABFBA.$$

Note that

$$\int_{AB} f(z)\,dz = -\int_{BA} f(z)\,dz, \qquad \text{being a simple change of direction.}$$

Also, we can divide the closed curve into smaller sections:

$$\begin{aligned}\oint_{AEABFBA} f(z)\,dz &= \int_{AEA} f(z)\,dz + \int_{AB} f(z)\,dz + \int_{BFB} f(z)\,dz + \int_{BA} f(z)\,dz \\ &= \int_{AEA} f(z)\,dz + \int_{BFB} f(z)\,dz = 0.\end{aligned}$$

i.e.

$$\oint_{C_1} f(z)\,dz - \oint_{C_2} f(z)\,dz = 0$$

(since we assume that closed paths are travelled anticlockwise).

Therefore $\displaystyle\oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz.$

This allows us to evaluate $\displaystyle\oint_{C_1} f(z)\,dz$ by replacing C_1 by any curve C_2 such that the region between them contains no singularities (see Section 261) of $f(z)$. Often we choose a circle for C_2.

Example 12

Determine $\oint_C \frac{6}{z(z-3)}\,dz$ where C is the curve $|z-3| = 5$ shown in Figure 14.

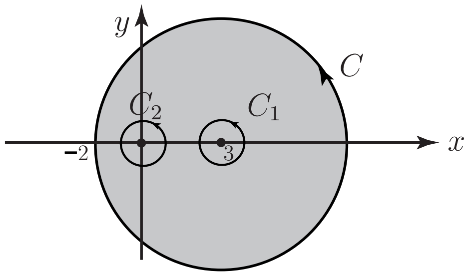


Figure 14

Solution

We observe that $f(z) = \frac{6}{z(z-3)}$ is analytic everywhere except at $z = 0$ and $z = 3$.

Let C_1 be the circle of unit radius centred at $z = 3$ and C_2 be the unit circle centered at the origin. By analogy with the previous example we state that

$$\oint_C \frac{6}{z(z-3)}\,dz = \oint_{C_1} \frac{6}{z(z-3)}\,dz + \oint_{C_2} \frac{6}{z(z-3)}\,dz.$$

(To show this you would need two cuts: from C to C_1 and from C to C_2.)

The remaining parts of this problem are presented as two Tasks.

Task

Expand $\frac{6}{z(z-3)}$ into partial functions.

Your solution

Answer

Let $\frac{6}{z(z-3)} \equiv \frac{A}{z} + \frac{B}{z-3} \equiv \frac{A(z-3)+Bz}{z(z-3)}$. Then $A(z-3) + Bz \equiv 6$.

If $z = 0$ $A(-3) = 6$ $\therefore$ $A = -2$. If $z = 3$ $B \times 3 = 6$ $\therefore$ $B = 2$.

$\therefore$ $\frac{6}{z(z-3)} \equiv -\frac{2}{z} + \frac{2}{z-3}$.

Thus:

$$\oint_C \frac{6}{z(z-3)}\,dz = \oint_{C_1} \frac{2}{z-3}\,dz - \oint_{C_1} \frac{2}{z}\,dz + \oint_{C_2} \frac{2}{z-3}\,dz - \oint_{C_2} \frac{2}{z}\,dz = I_1 - I_2 + I_3 - I_4.$$

Task

Find the values of I_1, I_2, I_3, I_4, using Key Point 1 (page 35):

(a) Find the value of I_1:

Your solution

Answer

Using Key Point 1 we find that $I_1 = 2 \times 2\pi\text{i} = 4\pi\text{i}$.

(b) Find the value of I_2:

Your solution

Answer

The function $\dfrac{1}{z}$ is analytic inside and on C_1 so that $I_2 = 0$.

(c) Find the value of I_3:

Your solution

Answer

The function $\dfrac{1}{z-3}$ is analytic inside and on C_2 so $I_3 = 0$.

(d) Find the value of I_4:

Your solution

Answer

$I_4 = 4\pi\text{i}$ again using Key Point 1.

(e) Finally, calculate $I = I_1 - I_2 + I_3 - I_4$:

Your solution

Answer

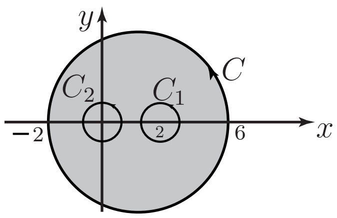
$$\oint_C \frac{6\,dz}{z(z-3)} = 4\pi\mathrm{i} - 0 + 0 - 4\pi\mathrm{i} = 0.$$

Exercises

1. Evaluate $\displaystyle\int_{1+\mathrm{i}}^{2+3\mathrm{i}} \sin z\, dz$.
2. Determine $\displaystyle\oint_C \frac{4}{z(z-2)}\, dz$ where C is the contour $|z-2| = 4$.

Answers

1. $\displaystyle\int_{1+\mathrm{i}}^{2+3\mathrm{i}} \sin z\, dz = \Big[-\cos z \Big]_{1+\mathrm{i}}^{2+3\mathrm{i}} = \cos(1+\mathrm{i}) - \cos(2+3\mathrm{i})$ since $\sin z$ is analytic everywhere.
2.



$f(z) = \dfrac{4}{z(z-2)}$ is analytic everywhere except at $z = 0$ and $z = 2$.

Call $\displaystyle I = \oint_C \frac{4}{z(z-2)}\, dz = \oint_{C_1} \frac{4}{z(z-2)}\, dz + \oint_{C_2} \frac{4}{z(z-2)}\, dz.$

Now $\quad \dfrac{4}{z(z-2)} \equiv -\dfrac{2}{z} + \dfrac{2}{z-2}$ so that

$$\begin{aligned} I &= \oint_{C_1} \frac{2}{z-2}\, dz - \oint_{C_1} \frac{2}{z}\, dz + \oint_{C_2} \frac{2}{z-2}\, dz - \oint_{C_2} \frac{2}{z}\, dz \\ &= I_1 + I_2 + I_3 + I_4 \end{aligned}$$

I_2 and I_3 are zero because of analyticity.

$I_1 = 2 \times 2\pi\mathrm{i} = 4\pi\mathrm{i}$, by Key Point 1 and $I_4 = -4\pi\mathrm{i}$ likewise.

Hence $I = 4\pi\mathrm{i} + 0 + 0 - 4\pi\mathrm{i} = 0$.

2. Cauchy's integral formula

This is a generalization of the result in Key Point 2:

Key Point 3

Cauchy's Integral Formula

If $f(z)$ is analytic inside and on the boundary C of a simply-connected region then for any point z_0 inside C,

$$\oint_C \frac{f(z)}{z - z_0}\, dz = 2\pi \mathrm{i}\ f(z_0).$$

Example 13

Evaluate $\displaystyle\oint_C \frac{z}{z^2+1}\, dz$ where C is the path shown in Figure 15:

C_1 : $|z - \mathrm{i}| = \frac{1}{2}$

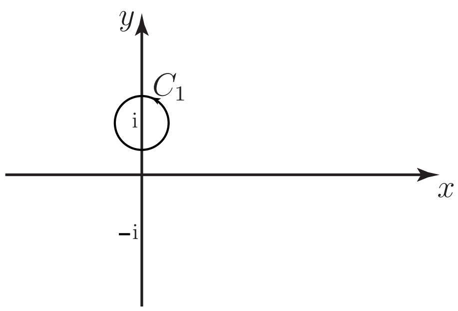


Figure 15

Solution

We note that $z^2 + 1 \equiv (z + \mathrm{i})(z - \mathrm{i})$.

Let $\displaystyle\frac{z}{z^2+1} = \frac{z}{(z+\mathrm{i})(z-\mathrm{i})} = \frac{z/(z+\mathrm{i})}{z - \mathrm{i}}$.

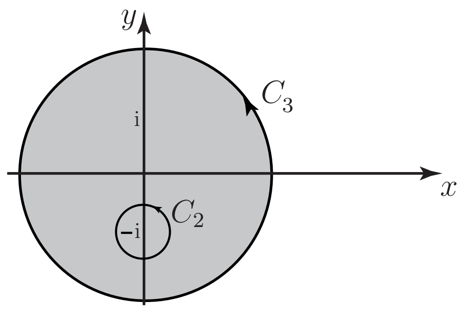
The numerator $z/(z + \mathrm{i})$ is analytic inside and on the path C_1 so putting $z_0 = \mathrm{i}$ in the Cauchy integral formula (Key Point 3)

$$\oint_{C_1} \frac{z}{z^2+1}\, dz = 2\pi\mathrm{i}\left[\frac{\mathrm{i}}{\mathrm{i}+\mathrm{i}}\right] = 2\pi\mathrm{i}.\frac{1}{2} = \pi\mathrm{i}.$$

Task

Evaluate $\oint_C \frac{z}{z^2+1}\,dz$ where C is the path (refer to the diagram)

(a) C_2 : $|z+\mathrm{i}| = \frac{1}{2}$ (b) C_3 : $|z| = 2$.



(a) Use the Cauchy integral formula to find an expression for $\oint_{C_2} \frac{z}{z^2+1}\,dz$:

Your solution

Answer

$\frac{z}{z^2+1} = \frac{z/(z-\mathrm{i})}{z+\mathrm{i}}$. The numerator is analytic inside and on the path C_2 so putting $z_0 = -\mathrm{i}$ in the Cauchy integral formula gives

$$\oint_{C_2} \frac{z}{z^2+1}\,dz = 2\pi\mathrm{i}\left[\frac{-\mathrm{i}}{-2\mathrm{i}}\right] = \pi\mathrm{i}.$$

(b) Now find $\oint_{C_3} \frac{z}{z^2+1}\,dz$:

Your solution

Answer

By analogy with the previous part,

$$\oint_{C_3} \frac{z}{z^2+1}\,dz = \oint_{C_1} \frac{z}{z^2+1}\,dz + \oint_{C_2} \frac{z}{z^2+1}\,dz = \pi\mathrm{i} + \pi\mathrm{i} = 2\pi\mathrm{i}.$$

The derivative of an analytic function

If $f(z)$ is analytic in a simply-connected region then at any interior point of the region, z_0 say, the derivatives of $f(z)$ of any order exist and are themselves analytic (which illustrates what a powerful property analyticity is!). The derivatives at the point z_0 are given by Cauchy's integral formula for derivatives:

$$f^{(n)}(z_0) = \frac{n!}{2\pi \mathrm{i}} \oint_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz$$

where C is any simple closed curve, in the region, which encloses z_0.
Note the case $n = 1$:

$$f'(z_0) = \frac{1}{2\pi \mathrm{i}} \oint_C \frac{f(z)}{(z-z_0)^2}\, dz.$$

Example 14

Evaluate the contour integral

$$\oint_C \frac{z^3}{(z-1)^2}\, dz$$

where C is a contour which encloses the point $z = 1$.

Solution

Since $f(z) = \dfrac{z^3}{(z-1)^2}$ has a pole of order 2 at $z = 1$ then $\displaystyle\oint_C f(z)\, dz = \oint_{C'} \frac{z^3}{(z-1)^2}\, dz$

where C' is a circle centered at $z = 1$.

If $g(z) = z^3$ then $\displaystyle\oint_C f(z)\, dz = \oint_{C'} \frac{g(z)}{(z-1)^2}\, dz$

Since $g(z)$ is analytic within and on the circle C' we use Cauchy's integral formula for derivatives to show that

$$\oint_C \frac{z^3}{(z-1)^2}\, dz = 2\pi \mathrm{i} \times \frac{1}{1!}\left[g'(z)\right]_{z=1} = 2\pi \mathrm{i}\left[3z^2\right]_{z=1} = 6\pi \mathrm{i}.$$

Exercise

Evaluate $\oint_C \frac{z}{z^2+9}\, dz$ where C is the path:

(a) $C_1: \quad |z - 3\text{i}| = 1$ (b) $C_2: \quad |z + 3\text{i}| = 1$ (c) $C_3: \quad |z| = 6.$

Answers

(a) We will use the fact that $\frac{z}{z^2+9} = \frac{z}{(z+3\text{i})(z-3\text{i})} = \frac{z/(z+3\text{i})}{z-3\text{i}}$

The numerator $\frac{z}{z+3\text{i}}$ is analytic inside and on the path C_1 so putting $z_0 = 3\text{i}$ in Cauchy's integral formula

$$\oint_{C_1} \frac{z}{z^2+9}\, dz = 2\pi\text{i} \left[\frac{3\text{i}}{3\text{i}+3\text{i}}\right] = 2\pi\text{i} \times \frac{1}{2} = \pi\text{i}.$$

(b) Here $\frac{z/(z-3\text{i})}{z+3\text{i}}$

The numerator is analytic inside and on the path C_2 so putting $z = -3\text{i}$ in Cauchy's integral formula:

$$\oint_{C_2} \frac{z}{z^2+9}\, dz = 2\pi\text{i} \left[\frac{-3\text{i}}{-3\text{i}-3\text{i}}\right] = \pi\text{i}.$$

(c) The integral is the sum of the two previous integrals and has value $2\pi\text{i}$.

Singularities and Residues

26.6

Introduction

Taylor's series for functions of a real variable is generalised here to the Laurent series for a function of a complex variable, which includes terms of the form $(z - z_0)^{-n}$.

The different types of singularity of a complex function $f(z)$ are discussed and the definition of a residue at a pole is given. The residue theorem is used to evaluate contour integrals where the only singularities of $f(z)$ inside the contour are poles.

Prerequisites

Before starting this Section you should . . .

- be familiar with binomial and Taylor series

Learning Outcomes

On completion you should be able to . . .

- understand the concept of a Laurent series
- find residues and use the residue theorem

1. Taylor and Laurent series

Many of the results in the area of series of real variables can be extended into complex variables. As an example, the concept of radius of convergence of a series is extended to the concept of a **circle of convergence**. If the circle of convergence of a series of complex numbers is $|z - z_0| = \rho$ then the series will converge if $|z - z_0| < \rho$.

For example, consider the function

$$f(z) = \frac{1}{1-z}$$

It has a singularity at $z = 1$. We can obtain the Maclaurin series, centered at $z = 0$, as

$$f(z) = 1 + z + z^2 + z^3 + \ldots$$

The circle of convergence is $|z| = 1$.

The radius of convergence for a series centred on $z = z_0$ is the distance between z_0 and the nearest singularity.

Laurent series

One of the shortcomings of Taylor series is that the circle of convergence is often only a part of the region in which $f(z)$ is analytic.

As an example, the series

$$1 + z + z^2 + z^3 + \ldots \quad \text{converges to} \quad f(z) = \frac{1}{1-z}$$

only inside the circle $|z| = 1$ even though $f(z)$ is analytic **everywhere except at $z = 1$**.

The **Laurent series** is an attempt to represent $f(z)$ as a series over as large a region as possible. We expand the series around a point of singularity up to, but not including, the singularity itself.

Figure 16 shows an **annulus of convergence** $r_1 < |z - z_0| < r_2$ within which the Laurent series (which is an extension of the Taylor series) will converge. The extension includes negative powers of $(z - z_0)$.

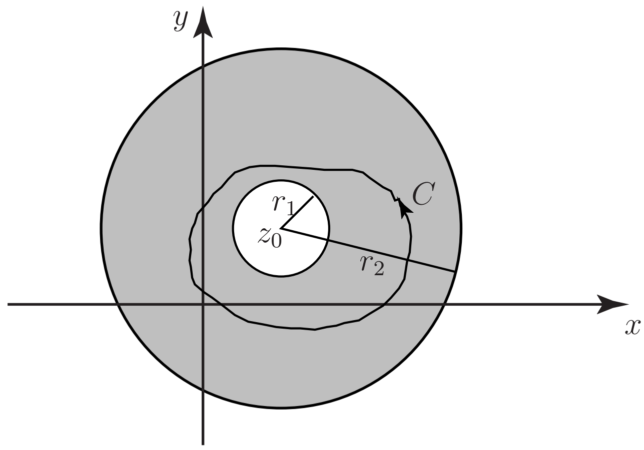


Figure 16

Now, we state Laurent's theorem in Key Point 4.

Key Point 4

Laurent's Theorem

If $f(z)$ is analytic through a closed annulus D centred at $z = z_0$ then at any point z inside D we can write

$$\begin{aligned} f(z) = a_0 &+ a_1(z-z_0) + a_2(z-z_0)^2 + \dots \\ &+ b_1(z-z_0)^{-1} + b_2(z-z_0)^{-2} + \dots \end{aligned}$$

where the coefficients a_n and b_n (for each n) is given by

$$a_n = \frac{1}{2\pi \mathrm{i}} \oint_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz, \qquad b_n = \frac{1}{2\pi \mathrm{i}} \oint_C \frac{f(z)}{(z-z_0)^{1-n}}\, dz,$$

the integral being taken around any simple closed path C lying inside D and encircling the inner boundary. (Refer to Figure 16.)

Example 15

Expand $f(z) = \dfrac{1}{1-z}$ in terms of negative powers of z which will be valid if $|z| > 1$.

Solution

First note that $1 - z = -z\left(1 - \dfrac{1}{z}\right)$ so that

$$\begin{aligned} f(z) &= -\frac{1}{z\left(1-\frac{1}{z}\right)} = -\frac{1}{z}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3}\right) \\ &= -\frac{1}{z} - \frac{1}{z^2} - \frac{1}{z^3} - \frac{1}{z^4} - \dots \end{aligned}$$

This is valid for $\left|\dfrac{1}{z}\right| < 1$, that is, $\dfrac{1}{|z|} < 1$ or $|z| > 1$. Note that we used a binomial expansion rather than the theorem itself. Also note that together with the earlier result we are now able to expand $f(z) = \dfrac{1}{1-z}$ everywhere, except for $|z| = 1$.

Task

This Task concerns $f(z) = \dfrac{1}{1+z}$.

(a) Using the binomial series, expand $f(z)$ in terms of non-negative power of z:

Your solution

Answer

$f(z) = (1+z)^{-1} = 1 - z + z^2 - z^3 + \ldots$

(b) State the values of z for which this expansion is valid:

Your solution

Answer

$|z| < 1$ (standard result for a GP).

(c) Using the identity $1 + z = z\left(1 + \dfrac{1}{z}\right)$ expand $f(z) = \dfrac{1}{1+z}$ in terms of negative powers of z and state the values of z for which your expansion is valid:

Your solution

Answer

$$f(z) = \frac{1}{z\left(1+\dfrac{1}{z}\right)} = \frac{1}{z}\left(1+\frac{1}{z}\right)^{-1} = \frac{1}{z}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \ldots\right) = \frac{1}{z} - \frac{1}{z^2} + \frac{1}{z^3} - \frac{1}{z^4} + \ldots$$

Valid for $\left|\dfrac{1}{z}\right| < 1$ i.e. $|z| > 1$.

2. Classifying singularities

If the function $f(z)$ has a singularity at $z = z_0$, and in a **neighbourhood** of z_0 (i.e. a region of the complex plane which contains z_0) there are no other singularities, then z_0 is an **isolated singularity** of $f(z)$.

The **principal part** of the Laurent series is the part containing negative powers of $(z - z_0)$. If the principal part has a finite number of terms say

$$\frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \ldots + \frac{b_m}{(z-z_0)^m} \quad \text{and } b_m \neq 0$$

then $f(z)$ has a **pole of order m** at $z = z_0$ (we have written b_1 for a_{-1}, b_2 for a_{-2} etc. for simplicity.) Note that if $b_1 = b_2 = \ldots = 0$ and $b_m \neq 0$, the pole is still of order m.

A pole of order 1 is called a **simple pole** whilst a pole of order 2 is called a **double pole**. If the principal part of the Laurent series has an infinite number of terms then $z = z_0$ is called an **isolated essential singularity** of $f(z)$.
The function

$$f(z) = \frac{\mathrm{i}}{z(z-\mathrm{i})} \equiv \frac{1}{z-\mathrm{i}} - \frac{1}{z}$$

has a simple pole at $z = 0$ and another simple pole at $z = \mathrm{i}$. The function $e^{\frac{1}{z-2}}$ has an isolated essential singularity at $z = 2$. Some complex functions have non-isolated singularities called **branch points**. An example of such a function is $\sqrt{z}$.

Task

Classify the singularities of the function $f(z) = \dfrac{2}{z} - \dfrac{1}{z^2} + \dfrac{1}{z+\mathrm{i}} + \dfrac{3}{(z-\mathrm{i})^4}$.

Your solution

Answer

A pole of order 2 at $z = 0$, a simple pole at $z = -\mathrm{i}$ and a pole of order 4 at $z = \mathrm{i}$.

Exercises

1. Expand $f(z) = \dfrac{1}{2-z}$ in terms of negative powers of z to give a series which will be valid if $|z| > 2$.

2. Classify the singularities of the function: $f(z) = \dfrac{1}{z^2} + \dfrac{1}{(z+\mathrm{i})^2} - \dfrac{2}{(z+\mathrm{i})^3}$.

Answers

1. $2 - z = -z(1 - \dfrac{2}{z})$ so that:

$$f(z) = \frac{-1}{z(1-\frac{2}{z})} = -\frac{1}{z}(1-\frac{2}{z})^{-1} = -\frac{1}{z}(1+\frac{2}{z}+\frac{4}{z^2}+\frac{8}{z^3}+\ldots) = -\frac{1}{z} - \frac{2}{z^2} - \frac{4}{z^3} - \frac{8}{z^3} - \ldots$$

This is valid for $\left|\dfrac{2}{z}\right| < 1$ or $|z| > 2$.

2. A double pole at $z = 0$ and a pole of order 3 at $z = -\mathrm{i}$.

3. The residue theorem

Suppose $f(z)$ is a function which is analytic inside and on a closed contour C, except for a pole of order m at $z = z_0$, which lies inside C. To evaluate $\oint_C f(z)\,dz$ we can expand $f(z)$ in a Laurent series in powers of $(z - z_0)$. If we let Γ be a circle of centre z_0 lying inside C then, as we saw in Section 262, $\quad \oint_C f(z)\,dz = \int_\Gamma f(z)\,dz.$

From Key Point 1 in Section 26.4 we know that the integral of each of the positive and negative powers of $(z - z_0)$ is zero with the exception of $\dfrac{b_1}{z - z_0}$ and this has value $2\pi b_1$. Since it is the only coefficient remaining after the integration, it is called the **residue** of $f(z)$ at $z = z_0$. It is given by

$$b_1 = \frac{1}{2\pi \mathrm{i}} \oint_C f(z)\,dz.$$

Calculating the residue, for any given function $f(z)$ is an important task and we examine some results concerning its determination for functions with simple poles, double poles and poles of order m.

Finding the residue

If $f(z)$ has a simple pole at $z = z_0$ then $\qquad f(z) = \dfrac{b_1}{z - z_0} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \dots$

so that $\qquad (z - z_0)f(z) = b_1 + a_0(z - z_0) + a_1(z - z_0)^2 + a_2(z - z_0)^3 + \dots$

Taking limits as $z \to z_0$, $\qquad \lim_{z \to z_0} \{(z - z_0)f(z)\} = b_1.$

For example, let $\qquad f(z) = \dfrac{1}{z^2 + 1} \equiv \dfrac{1}{(z + \mathrm{i})(z - \mathrm{i})} \equiv \dfrac{-\frac{1}{2i}}{z + \mathrm{i}} + \dfrac{\frac{1}{2\mathrm{i}}}{z - \mathrm{i}}.$

There are simple poles at $z = -\mathrm{i}$ and $z = \mathrm{i}$. The residue at $z = \mathrm{i}$ is

$$\lim_{z \to \mathrm{i}} \left\{ (z - \mathrm{i}) \frac{1}{(z + \mathrm{i})(z - \mathrm{i})} \right\} = \lim_{z \to \mathrm{i}} \left(\frac{1}{z + \mathrm{i}} \right) = \frac{1}{2\mathrm{i}}.$$

Similarly, the residue at $z = -\mathrm{i}$ is

$$\lim_{z \to \mathrm{i}} \left\{ (z + \mathrm{i}) \frac{1}{(z + \mathrm{i})(z - \mathrm{i})} \right\} = \lim_{z \to -\mathrm{i}} \left(\frac{1}{z - \mathrm{i}} \right) = \frac{-1}{2\mathrm{i}}.$$

Task

This Task concerns $f(z) = \dfrac{1}{z^2 + 4}$.

(a) Identify the singularities of $f(z)$:

Your solution

Answer

$$f(z) = \frac{1}{(z+2\text{i})(z-2\text{i})} = \frac{-\dfrac{1}{4\text{i}}}{z+2\text{i}} + \frac{\dfrac{1}{4\text{i}}}{z-2\text{i}}.$$ There are simple poles at $z = -2\text{i}$ and $z = 2\text{i}$.

(b) Now find the residues of $f(z)$ at $z = 2\text{i}$ and at $z = -2\text{i}$:

Your solution

Answer

$$\lim_{z\to 2\text{i}} \left\{ (z-2\text{i}) \frac{1}{(z+2\text{i})(z-2\text{i})} \right\} = \lim_{z\to 2\text{i}} \left(\frac{1}{z+2\text{i}} \right) = \frac{1}{4\text{i}}.$$

Similarly at $z = -2\text{i}$.

$$\lim_{z\to -2\text{i}} \left\{ (z+2\text{i}) \frac{1}{(z+2\text{i})(z-2\text{i})} \right\} = \lim_{z\to -2\text{i}} \left(\frac{1}{z-2\text{i}} \right) = -\frac{1}{4\text{i}}.$$

In general the residue at a pole of order m at $z = z_0$ is

$$b_1 = \frac{1}{(m-1)!} \lim_{z\to z_0} \left\{ \frac{d^{m-1}}{dz^{m-1}} \left[(z-z_0)^m f(z) \right] \right\}.$$

As an example, if $f(z) = \dfrac{z^2+1}{(z+1)^3}$, $f(z)$ has a pole of order 3 at $z = -1$ $(m = 3)$.

We need first

$$\frac{d^2}{dz^2} \left[(z+1)^3 \frac{(z^2+1)}{(z+1)^3} \right] = \frac{d^2}{dz^2}[z^2+1] = \frac{d}{dz}[2z] = 2.$$

Then $b_1 = \dfrac{1}{2!} \times 2 = 1.$

We have a useful result (Key Point 5) which allows us to evaluate contour integrals quickly when $f(z)$ has only poles inside the contour.

Key Point 5

The Residue Theorem

$$\oint_C f(z)\, dz = 2\pi\text{i} \times (\text{sum of the residues at the poles inside } C).$$

Example 16

Let $f(z) = \dfrac{1}{z^2+1}$. Find the integrals $\oint_{C_1} dz$, $\oint_{C_2} dz$ and $\oint_{C_3} dz$ in which C_1 is the circle $|z - \text{i}| = 1$, C_2 is the circle $|z + \text{i}| = 1$, and C_3 is any path enclosing both $z = \text{i}$ and $z = -\text{i}$. See Figure 17.

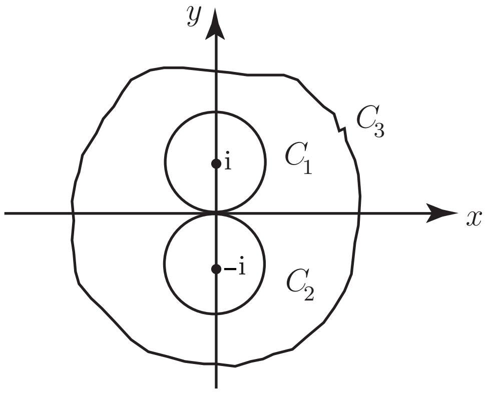


Figure 17

Solution

Figure 17 shows that only the pole at $z = \text{i}$ lies inside C_1. The residue at this pole is $\dfrac{1}{2\text{i}}$, as we found earlier. Hence $\oint_{C_1} f(z)\,dz = 2\pi\text{i} \times \dfrac{1}{2\text{i}} = \pi$.

Also, the residue at $z = -\text{i}$, the only pole inside C_2, is $-\dfrac{1}{2\text{i}}$. Hence

$$\oint_{C_2} f(z)\,dz = -2\pi\text{i} \times \frac{1}{2\text{i}} = -\pi.$$

Note that the contour C_3 encloses both poles so that $\oint_{C_3} f(z)\,dz = 2\pi\text{i}\left(\dfrac{1}{2\text{i}} - \dfrac{1}{2\text{i}}\right) = 0$.

Exercises

1. Identify the singularities of $f(z) = \dfrac{1}{z^2(z^2+9)}$ and find the residue at each of them.

2. Find the integral $\oint_C f(z)\,dz$ where $f(z) = \dfrac{1}{z^2+4}$ and C is

   (a) the circle $|z - 2\text{i}| = 1$;

   (b) the circle $|z + 2\text{i}| = 1$;

   (c) any closed path enclosing both $z = 2\text{i}$ and $z = -2\text{i}$.

Answers

1. Double pole at $z = 0$, simple poles at $z = 3\text{i}$ and $z = -3\text{i}$.

   Residue at $z = 3\text{i}$

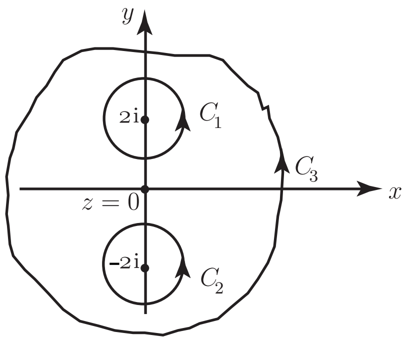
   $$= \lim_{z\to 3\text{i}} \left\{ (z-3\text{i}) \frac{1}{z^2(z+3\text{i})(z-3\text{i})} \right\} = \lim_{z\to 3\text{i}} \left\{ \frac{1}{z^2(z+3\text{i})} \right\} = \frac{1}{9\text{i}^2} \times \frac{1}{6\text{i}} = -\frac{1}{54\text{i}} = \frac{1}{54}\text{i}.$$

   Residue at $z = -3\text{i}$

   $$= \lim_{z\to -3\text{i}} \left\{ (z+3\text{i}) \frac{1}{z^2(z+3\text{i})(z-3\text{i})} \right\} = \lim_{z\to -3\text{i}} \left\{ \frac{1}{z^2(z-3\text{i})} \right\} = \frac{1}{9\text{i}^2} \times \frac{1}{-6\text{i}} = -\frac{1}{54}\text{i}.$$

   For the double pole at $z = 0$ we find $\dfrac{\text{d}}{dz}\left\{(z-0)^2 f(z)\right\} = \dfrac{\text{d}}{dz}\left(\dfrac{1}{z^2+9}\right) = \dfrac{-2z}{(z^2+9)^2}$.

   Then, $\displaystyle\lim_{z\to 0}\left(\frac{-2z}{(z^2+9)^2}\right) = 0.$

2. 

   $$f(z) = \frac{1}{(z+2\text{i})(z-2\text{i})}$$

   (a) Only the pole at $z = 2\text{i}$ lies inside C_1. The residue there is $\displaystyle\lim_{z\to 2\text{i}}\left(\frac{1}{z+2\text{i}}\right) = \frac{1}{4\text{i}}$.

   Hence $\displaystyle\oint_{C_1} f(z)\,dz = 2\pi\text{i} \times \frac{1}{4\text{i}} = \frac{\pi}{2}.$

   (b) Only the pole at $z = -2\text{i}$ lies inside C_2. The residue there is $\displaystyle\lim_{z\to -2\text{i}} \frac{1}{z-2\text{i}} = -\frac{1}{4\text{i}}$.

   Hence $\displaystyle\oint_{C_2} f(z)\,dz = 2\pi\text{i} \times (-\frac{1}{4\text{i}}) = -\frac{\pi}{2}.$

   (c) The contour C_3 encloses both poles so that

   $$\oint_{C_3} f(z)\,dz = 2\pi\text{i}\left(\frac{1}{4\text{i}} - \frac{1}{4\text{i}}\right) = 0.$$