Contents 25

Partial Differential Equations

Learning outcomes

By studying this Workbook you will learn to recognise the two-dimensional Laplace's equation and the one-dimensinal diffusion and wave equations.

You will learn how to verify solutions of these equations and how to find solutions by using the separation of variable method and by using Fourier series.

Partial Differential Equations

25.1

Introduction

A **partial differential equation** (PDE) is a differential equation involving partial derivatives of one dependent variable with respect to two or more independent variables. The independent variables may be space variables only or one or more space variables and time. Mathematical modelling of many situations involving natural phenomena leads to PDEs.

The subject of PDEs is a very large one. We shall discuss only a few special PDEs which model a wide range of applied problems.

Prerequisites

Before starting this Section you should ...

- be able to carry out partial differentiation
- be able to solve constant coefficient ordinary differential equations

Learning Outcomes

On completion you should be able to ...

- verify solutions of given partial differential equations arising in engineering and science

1. Introduction

You have already studied ordinary differential equations (ODEs) and have learnt how to obtain the solution of certain types. Since a knowledge of the solution of certain ODEs (i.e. those with constant coefficients) will be required in solving partial differential equations (PDEs), we will begin this unit reminding you of some important results.

Key Point 1

The first order ODE

$$\frac{dy}{dx} = ky$$

has general solution

$$y = Ae^{kx}$$

Here k is a constant which can be positive or negative and A is an arbitrary constant.

In Key Point 1 the quantity A in the general solution is a constant. To obtain the value of A we have to know the value of y at some value of x, perhaps $x = 0$. In other words, we need to know an **initial condition**.

Task

Find y as a function of x if

$$\frac{dy}{dx} = -2y$$

and the initial condition is $y(0) = 3$.

Your solution

Answer

From Key Point 1 with $k = -2$ we have the general solution

$$y = Ae^{-2x}$$

Putting $x = 0$ and $y = 3$ into this we obtain $3 = Ae^0$ i.e. $A = 3$ so the solution to the given initial value problem is

$$y = 3e^{-2x}$$

We shall also need to be familiar with solutions to second order, homogeneous, constant coefficient ODEs, summarised in Key Point 2.

Key Point 2

A second order ODE of the form

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0 \qquad (1)$$

where a, b, c are constants, has an auxiliary equation

$$\text{am}^2 + bm + c = 0 \qquad (2)$$

obtained by inserting the trial solution $y = e^{mx}$ in (1).

The general solution of (1) then depends on the solutions (or roots) of the quadratic equation (2).

(a) If (2) has real, distinct roots $m = m_1$ and $m = m_2$ then

$$y = Ae^{m_1x} + Be^{m_2x}$$

(b) If (2) has a repeated root $m = m_1$ then

$$y = (A + Bx)e^{m_1x}$$

(c) If (2) has complex roots (which will be a conjugate pair) $m = \alpha \pm j\beta$ then

$$y = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$$

Note that in each of these cases (a) to (c) the general solution is a linear combination of two particular solutions:

For (a) they are e^{m_1x} and e^{m_2x}.

For (b) they are e^{m_1x} and $x\mathrm{e}^{m_1x}$.

For (c) they are $\mathrm{e}^{\alpha x}\cos\beta x$ and $\mathrm{e}^{\alpha x}\sin\beta x$.

Task

Use Key Point 2 to find the general solution of $\dfrac{d^2y}{dx^2} - 4y = 0$.

First write down the auxiliary equation:

Your solution

Answer
$m^2 - 4 = 0$

Now find the roots of the auxiliary equation:

Your solution

Answer
$m = \pm 2$

Finally give the general solution to the ODE:

Your solution

Answer
$y = Ae^{2x} + Be^{-2x}$ (Since the roots of the auxiliary equation are real and distinct.)

Task

Find the general solution of $\dfrac{d^2y}{dx^2} + 9y = 0$

First write down the auxiliary equation:

Your solution

Answer
$m^2 + 9 = 0$

Now Find the roots of this auxiliary equation:

Your solution

Answer
$m = \pm 3\mathrm{i}$

Finally give the general solution to the ODE:

Your solution

Answer
$y = A\cos 3x + B\sin 3x$

(Since the roots of the auxiliary equation are complex conjugates with real part $\alpha = 0$ and imaginary part $\beta = 3$.)

The two Tasks above can be generalised as in Key Point 3.

Key Point 3

(1) The general solution to: $\dfrac{d^2y}{dx^2} - n^2y = 0$ is

$$y = Ae^{nx} + Be^{-nx}$$

or, equivalently using hyperbolic functions,

$$y = C\cosh nx + D\sinh nx$$

(2) The general solution to: $\dfrac{d^2y}{dx^2} + n^2y = 0$ is

$$y = A\cos nx + B\sin nx$$

Those of you who are familiar with elementary dynamics will recognise the second differential equation in Key Point 3 as modelling **simple harmonic motion**.

2. Partial differential equations

In all the above examples we had a function y of a single variable x, y being the solution of an **ordinary** differential equation.

In engineering and science ODEs arise as models for systems where there is **one** independent variable (often x) and **one** dependent variable (often y). Obvious examples are lumped electrical circuits where the current i is a function only of time t (and not of position in the circuit) and lumped mechanical systems (such as the simple harmonic oscillator referred to above) where the displacement of a moving particle depends only on t.

However, in problems where one variable, say u, depends on **more than one** independent variable, say both x and t, then any derivatives of u will be **partial derivatives** such as $\dfrac{\partial u}{\partial x}$ or $\dfrac{\partial^2 u}{\partial t^2}$ and any differential equation arising will be known as a **partial differential equation.** In particular, one-dimensional (1-D) time-dependent problems where u depends on a position coordinate x and the time t and two-dimensional (2-D) time-independent problems where u is a function of the two position coordinates x and y both give rise to PDEs involving **two** independent variables. This is the case we shall concentrate on. A two-dimensional time-dependent problem would involve 3 independent variables x, y, t as would a three-dimensional time-independent problem where x, y, z would be the independent variables.

Example 1

Show that $u = \sin x \cosh y$ satisfies the PDE $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0.$

This PDE is known as **Laplace's equation in two dimensions** and it arises in many applications e.g. electrostatics, fluid flow, heat conduction.

Solution

$u = \sin x \cosh y \qquad \Rightarrow \qquad \dfrac{\partial u}{\partial x} = \cos x \cosh y$ and $\dfrac{\partial u}{\partial y} = \sin x \sinh y$

Differentiating again gives $\dfrac{\partial^2 u}{\partial x^2} = -\sin x \cosh y$ and $\dfrac{\partial^2 u}{\partial y^2} = \sin x \cosh y$

Hence

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -\sin x \cosh y + \sin x \cosh y = 0$$

so the given function $u(x, y)$ is indeed a solution of the PDE.

Task

Show that $u = e^{-2\pi^2 t} \sin \pi x$ is a solution of the PDE $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{1}{2}\dfrac{\partial u}{\partial t}$

First find $\dfrac{\partial u}{\partial t}$ and $\dfrac{\partial u}{\partial x}$:

Your solution

Answer

$\dfrac{\partial u}{\partial t} = -2\pi^2 e^{-2\pi^2 t} \sin \pi x \qquad \dfrac{\partial u}{\partial x} = \pi e^{-2\pi^2 t} \cos \pi x$

Now find $\dfrac{\partial^2 u}{\partial x^2}$ and complete the Task:

Your solution

Answer

$\dfrac{\partial^2 u}{\partial x^2} = -\pi^2 e^{-2\pi^2 t} \sin \pi x$, and we see that $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{1}{2}\dfrac{\partial u}{\partial t}$ as required.

The PDE in the above Task has the general form

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t}$$

where k is a positive constant. This equation is referred to as the **one-dimensional heat conduction equation** (or sometimes as the **diffusion equation**). In a heat conduction context the dependent variable u represents the temperature $u(x,t)$.

The third important PDE involving two independent variables is known as the **one-dimensional wave equation**. This has the general form

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

(Note that both partial derivatives in the wave equation are second-order in contrast to the heat conduction equation where the time derivative is first order.)

Example 2

(a) Verify that $u(x,t) = u_0 \sin\left(\frac{\pi x}{\ell}\right)\ \cos\left(\frac{\pi c t}{\ell}\right)$ (where $u_0,\ \ell$ and c are constants) satisfies the one-dimensional wave equation.

(b) Verify the boundary conditions i.e. $u(0,t) = u(\ell,t) = 0$.

(c) Verify the initial conditions i.e. $\frac{\partial u}{\partial t}(x,0) = 0$ and $u(x,0) = u_0 \sin\left(\frac{\pi x}{\ell}\right)$.

(d) Give a physical interpretation of this problem.

Solution

(a) By straightforward partial differentiation of the given function $u(x,t)$:

$$\frac{\partial u}{\partial x} = u_0 \frac{\pi}{\ell}\cos\left(\frac{\pi x}{\ell}\right)\cos\left(\frac{\pi c t}{\ell}\right) \qquad \frac{\partial^2 u}{\partial x^2} = -u_0\left(\frac{\pi}{\ell}\right)^2 \sin\left(\frac{\pi x}{\ell}\right)\cos\left(\frac{\pi c t}{\ell}\right)$$

$$\frac{\partial u}{\partial t} = -u_0\left(\frac{\pi c}{\ell}\right)\sin\left(\frac{\pi x}{\ell}\right)\sin\left(\frac{\pi c t}{\ell}\right) \qquad \frac{\partial^2 u}{\partial t^2} = -u_0\left(\frac{\pi c}{\ell}\right)^2 \sin\left(\frac{\pi x}{\ell}\right)\cos\left(\frac{\pi c t}{\ell}\right)$$

We see that $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$ which completes the verification.

(b) Putting $x = 0$, and leaving t arbitrary, in the given solution for $u(x,t)$ gives

$$u(x,0) = u_0 \sin 0\ \cos\left(\frac{\pi c t}{\ell}\right) = 0 \text{ for all } t$$

Similarly putting $x = \ell,\ t$ arbitrary: $\quad u(\ell,0) = u_0 \sin\pi\ \cos\left(\frac{\pi c t}{\ell}\right) = 0$ for all t

Solution

(c) Evaluating $\dfrac{\partial u}{\partial t}$ firstly for general x and t

$$\frac{\partial u}{\partial t} = -u_0 \left(\frac{\pi c}{\ell}\right) \sin\left(\frac{\pi x}{\ell}\right) \sin\left(\frac{\pi c t}{\ell}\right)$$

Now putting $t = 0$ leaving x arbitrary

$$\frac{\partial u}{\partial t}(x, 0) = -u_0 \left(\frac{\pi c}{\ell}\right) \sin\left(\frac{\pi x}{\ell}\right) \sin 0 = 0.$$

Also, putting $t = 0$ in the expression for $u(x, t)$ gives

$$u(x, 0) = u_0 \sin\left(\frac{\pi x}{\ell}\right) \cos 0 = u_0 \sin\left(\frac{\pi x}{\ell}\right).$$

(d) Mathematically we have now proved that the given function $u(x, t)$ satisfies the 1-D wave equation specified in (a), the two **boundary conditions** specified in (b) and the two **initial conditions** specified in (c).

One possible physical interpretation of this problem is that $u(x, t)$ represents the displacement of a string stretched between two points at $x = 0$ and $x = \ell$. Clearly the position of any point P on the vibrating string will depend upon its distance x from one end and on the time t.

The boundary conditions (b) represent the fact that the string is fixed at these end-points.

The initial condition $u(x, 0) = u_0 \sin\left(\dfrac{\pi x}{\ell}\right)$ represents the displacement of the string at $t = 0$.

The initial condition $\dfrac{\partial u}{\partial t}(x, 0) = 0$ tells us that the string is at rest at $t = 0$.

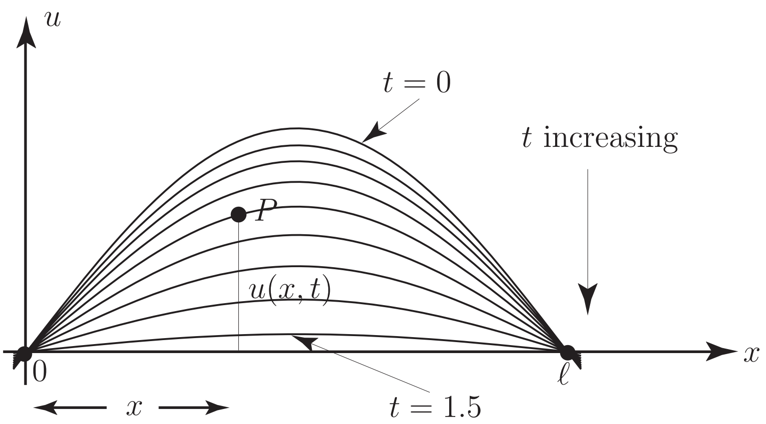


Figure 1

Note that it can be proved formally that if T is the tension in the string and if ρ is the mass per unit length of the string then u does, under certain conditions, satisfy the 1-D wave equation with $c^2 = \dfrac{T}{\rho}$.

Key Point 4

The three PDEs of greatest general interest involving two independent variables are:

(a) The two-dimensional Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

(b) The one-dimensional heat conduction equation:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t}$$

(c) The one-dimensional wave equation:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

Applications of PDEs 25.2

Introduction

In this Section we discuss briefly some of the most important PDEs that arise in various branches of science and engineering. We shall see that some equations can be used to describe a variety of different situations.

Prerequisites

Before starting this Section you should ...

- have a knowledge of partial differentiation

Learning Outcomes

On completion you should be able to ...

- recognise the heat conduction equation and the wave equation and have some knowledge of their applicability

Key Point 4 by no means exhausts the types of PDE which are important in applications. In this Section we will discuss those three PDEs in Key Point 4 in more detail and briefly discuss other PDEs over a wide range of applications. We will omit detailed derivations.

1. Wave equation

The simplest situation to give rise to the one-dimensional wave equation is the motion of a stretched string - specifically the **transverse vibrations** of a string such as the string of a musical instrument. Assume that a string is placed along the x-axis, is stretched and then fixed at ends $x = 0$ and $x = L$; it is then deflected and at some instant, which we call $t = 0$, is released and allowed to vibrate. The quantity of interest is the deflection u of the string at any point x, $0 \leq x \leq L$, and at any time $t > 0$. We write $u = u(x, t)$. Figure 2 shows a possible displacement of the string at a fixed time t.

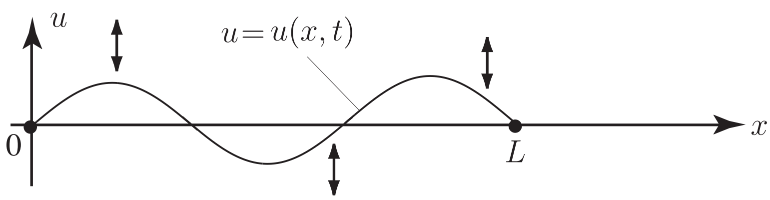


Figure 2

Subject to various assumptions $\cdots$

1. damping forces such as air resistance are negligible
2. the weight of the string is negligible
3. the tension in the string is tangential to the curve of the string at any point
4. the string performs small transverse oscillations i.e. every particle of the string moves strictly vertically and such that its deflection and slope as every point on the string is small.

$\cdots$ it can be shown, by applying Newton's law of motion to a small segment of the string, that u satisfies the PDE

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad (1)$$

where $c^2 = \dfrac{T}{\rho}$, ρ being the mass per unit length of the string and T being the (constant) horizontal component of the tension in the string. To determine $u(x, t)$ uniquely, we must also know

1. the initial definition of the string at the time $t = 0$ at which it is released
2. the initial velocity of the string.

Thus we must be given **initial conditions**

$$u\,(x, 0) = f(x) \qquad 0 \leq x \leq L \qquad \text{(initial position)}$$

$$\frac{\partial u}{\partial t}(x, 0) = g(x) \qquad 0 \leq x \leq L \qquad \text{(initial velocity)}$$

where $f(x)$ and $g(x)$ are known.

These two initial conditions are in addition to the two **boundary conditions**

$$u(0,t)=u(L,t)=0 \qquad \text{for } t\geq 0$$

which indicate that the string is fixed at each end. In Example 2 discussed in Section 25.1 we had

$$f(x)=u_0\ \sin\left(\frac{\pi x}{\ell}\right)$$

$$g(x)=0 \qquad \text{(string initially at rest).}$$

The PDE (1) is the (undamped) wave equation. We will discuss solutions of it for various initial conditions later. More complicated forms of the wave equation would arise if some of the assumptions were modified. For example:

(a) $\dfrac{\partial^2 u}{\partial t^2}=c^2\dfrac{\partial^2 u}{\partial x^2}-g$ if the weight of the string was allowed for,

(b) $\dfrac{\partial^2 u}{\partial t^2}=c^2\dfrac{\partial^2 u}{\partial x^2}-\alpha\dfrac{\partial u}{\partial t}$ if a damping force proportional to the velocity of the string (with damping constant α) was included.

Equation (1) is referred to as the one-dimensional wave equation because only one space variable, x, is present. The two-dimensional (undamped) wave equation is, in Cartesian coordinates,

$$\frac{\partial^2 u}{\partial t^2}=c^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) \qquad (2)$$

This arises for example when we model the transverse vibrations of a membrane. See Figures 3(a), 3(b). Here $u(x,y,t)$ is the definition of a point (x,y) on the membrane at time t. Again, a boundary condition must be specified: commonly

$$u=0 \qquad t\geq 0$$

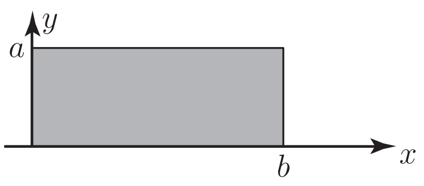
on the boundary of the membrane, if this is fixed (clamped). Also initial conditions must be given

$$u\,(x,y,0)=f(x,y) \quad \text{(initial position)} \qquad \frac{\partial u}{\partial t}(x,y,0)=g(x,y) \quad \text{(initial velocity)}$$

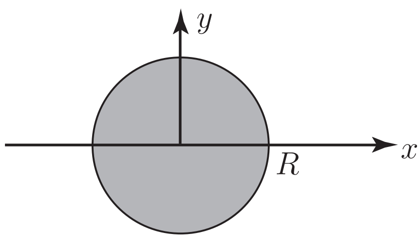
For a **circular** membrane, such as a drumhead, polar coordinates defined by $x=r\cos\theta,\ y=r\sin\theta$ would be more convenient than Cartesian. In this case (2) becomes

$$\frac{\partial^2 u}{\partial t^2}=c^2\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right) \qquad 0\leq r\leq R, \qquad 0\leq\theta\leq 2\pi$$

for a circular membrane of radius R.



(a)



(b)

Figure 3

2. Heat conduction equation

Consider a long thin bar, or wire, of constant cross-section and of homogeneous material oriented along the x-axis (see Figure 4).

Figure 4

Imagine that the bar is thermally insulated laterally and is sufficiently thin that heat flows (by conduction) only in the x-direction. Then the temperature u at any point in the bar depends only on the x-coordinate of the point and the time t. By applying the principle of conservation of energy it can be shown that $u(x,t)$ satisfies the PDE

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \qquad \begin{array}{c} 0 \leq x \leq L \\ t > 0 \end{array} \tag{3}$$

where k is a positive constant. In fact k, sometimes called the **thermal diffusivity** of the bar, is given by

$$k = \frac{\kappa}{s\rho}$$

where $\kappa =$ thermal conductivity of the material of the bar

$s =$ specific heat capacity of the material of the bar

$\rho =$ density of the material of the bar.

The PDE (3) is called the one-dimensional heat conduction equation (or, in other contexts where it arises, the **diffusion equation**).

Task

What is the obvious difference between the wave equation (1) and the heat conduction equation (3)?

Your solution

Answer

Both equations involve second derivatives in the space variable x but whereas the wave equation has a *second* derivative in the time variable t the heat conduction equation has only a *first* derivative in t. This means that the solutions of (3) are quite different in form from those of (1) and we shall study them separately later.

The fact that (3) is first order in t means that only one initial condition at $t = 0$ is needed, together with two boundary conditions, to obtain a unique solution. The usual initial condition specifies the initial temperature distribution in the bar

$$u(x, 0) = f(x)$$

where $f(x)$ is known. Various types of boundary conditions at $x = 0$ and $x = L$ are possible. For example:

(a) $u(0, t) = T_1$ and $u(L, T) = T_2$ (ends of the bar are at constant temperatures T_1 and T_2).

(b) $\dfrac{\partial u}{\partial x}(0, t) = \dfrac{\partial u}{\partial x}(L, t) = 0$ which are **insulation** conditions since they tell us that there is no heat flow through the ends of the bar.

As you would expect, there are two-dimensional and three-dimensional forms of the heat conduction equation. The two dimensional version of (3) is

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \qquad (4)$$

where $u(x, y, t)$ is the temperature in a flat plate. The plate is assumed to be thin and insulated on its top and bottom surface so that no heat flow occurs other than in the Oxy plane. Boundary conditions and an initial condition are needed to give unique solutions of (4). For example if the plate is rectangular as in Figure 5:

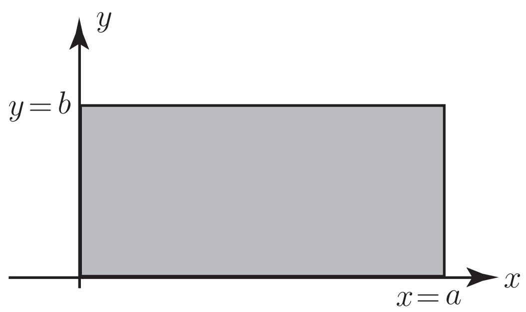


Figure 5

typical boundary conditions might be

$u(x, 0) = T_1 \qquad 0 \leq x \leq a$ (bottom side at fixed temperature)

$\dfrac{\partial u}{\partial x}(a, y) = 0 \qquad 0 \leq y \leq b$ (right-hand side insulated)

$u(x, b) = T_2 \qquad 0 \leq x \leq a$ (top side at fixed temperature)

$u(0, y) = 0 \qquad 0 \leq y \leq b$ (left hand side at zero fixed temperature).

An initial condition would have the form $u(x, y, 0) = f(x, y)$, where f is a given function.

3. Transmission line equations

In a long electrical cable or a telephone wire both the current and voltage depend upon position along the wire as well as the time (see Figure 6).

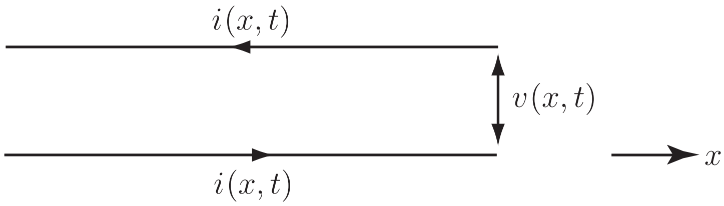


Figure 6

It is possible to show, using basic laws of electrical circuit theory, that the electrical current $i(x,t)$ satisfies the PDE

$$\frac{\partial^2 i}{\partial x^2} = LC\frac{\partial^2 i}{\partial t^2} + (RC+GL)\frac{\partial i}{\partial t} + RGi \qquad (5)$$

where the constants R, L, C and G are, for unit length of cable, respectively the resistance, inductance, capacitance and leakage conductance. The voltage $v(x,t)$ also satisfies (5). Special cases of (5) arise in particular situations. For a submarine cable G is negligible and frequencies are low so inductive effects can also be neglected. In this case (5) becomes

$$\frac{\partial^2 i}{\partial x^2} = RC\frac{\partial i}{\partial t} \qquad (6)$$

which is called the **submarine equation** or **telegraph equation**. For high frequency alternating currents, again with negligible leakage, (5) can be approximated by

$$\frac{\partial^2 i}{\partial x^2} = LC\frac{\partial^2 i}{\partial t^2} \qquad (7)$$

which is called the **high frequency line equation**.

Task

What PDEs, already discussed, have the same form as equations (6) or (7)?

Your solution

Answer

(6) has the same form as the one-dimensional heat conduction equation.

(7) has the same form as the one-dimensional wave equation.

4. Laplace's equation

If you look back at the two-dimensional heat conduction equation (4):

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

it is clear that if the heat flow is steady, i.e. time independent, then $\dfrac{\partial u}{\partial t} = 0$ so the temperature $u(x, y)$ is a solution of

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad (8)$$

(8) is the **two-dimensional Laplace equation**. Both this, and its three-dimensional counterpart

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \qquad (9)$$

arise in a wide variety of applications, quite apart from steady state heat conduction theory. Since time does not arise in (8) or (9) it is evident that Laplace's equation is always a model for **equilibrium** situations. In any problem involving Laplace's equation we are interested in solving it in a specific region R for given boundary conditions. Since conditions may involve

(a) u specified on the boundary curve C (two dimensions) or boundary surface S (three dimensions) of the region R. Such boundary conditions are called **Dirichlet conditions**.

(b) The derivative of u normal to the boundary, written $\dfrac{\partial u}{\partial n}$, specified on C or S. These are referred to as **Neumann boundary conditions**.

(c) A mixture of (a) and (b).

Some areas in which Laplace's equation arises are

(a) electrostatics (u being the electrostatic potential in a charge free region)

(b) gravitation (u being the gravitational potential in free space)

(c) steady state flow of inviscid fluids

(d) steady state heat conduction (as already discussed)

5. Other important PDEs in science and engineering

1. **Poisson's equation**

   $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y) \qquad \text{(two-dimensional form)}$$

   where $f(x, y)$ is a given function. This equation arises in electrostatics, elasticity theory and elsewhere.

2. **Helmholtz's equation**

   $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + k^2 u = 0 \qquad \text{(two dimensional form)}$$

   which arises in wave theory.

3. **Schrödinger's equation**

   $$-\frac{h^2}{8\pi^2 m}\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2}\right) = E\psi$$

   which arises in quantum mechanics. (h is Planck's constant)

4. **Transverse vibrations equation**

   $$a^2\frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial t^2} = 0$$

   for a homogeneous rod, where $u(x, t)$ is the displacement at time t of the cross section through x.

All the PDEs we have discussed are **second order** (because the highest order derivatives that arise are second order) apart from the last example which is **fourth order**.

Solution Using Separation of Variables 25.3

Introduction

The main topic of this Section is the solution of PDEs using the method of separation of variables. In this method a PDE involving n independent variables is converted into n ordinary differential equations. (In this introductory account n will always be 2.)

You should be aware that other analytical methods and also numerical methods are available for solving PDEs. However, the separation of variables technique does give some useful solutions to important PDEs.

Prerequisites

Before starting this Section you should ...

- be able to solve first and second order constant coefficient ordinary differential equations

Learning Outcomes

On completion you should be able to ...

- apply the separation of variables method to obtain solutions of the heat conduction equation, the wave equation and the 2-D Laplace equation for specified boundary or initial conditions

1. Solution of important PDEs

We shall just consider two analytic solution techniques for PDEs:

(a) Direct integration

(b) Separation of variables

The method of direct integration is a straightforward extension of solving very simple ODEs by integration, and will be considered first. The method of separation of variables is more important and we will study it in detail shortly.

You should note that many practical problems involving PDEs have to be solved by **numerical** methods but that is another story (introduced in HELM 32 and HELM 33).

Task

Solve the ODE

$$\frac{d^2y}{dx^2} = x^2 + 2$$

given that $y = 1$ when $x = 0$ and $\dfrac{dy}{dx} = 2$ when $x = 0$.

First find $\dfrac{dy}{dx}$ by integrating once, not forgetting the arbitrary constant of integration:

Your solution

Answer

$$\frac{dy}{dx} = \frac{x^3}{3} + 2x + A$$

Now find y by integrating again, not forgetting to include another arbitrary constant:

Your solution

Answer

$$y = \frac{x^4}{12} + x^2 + Ax + B$$

Now find A and B by inserting the two given initial conditions and so find the solution:

Your solution

Answer

$y(0) = 1$ gives $B = 1$ $\qquad$ $y'(0) = 2$ gives $A = 2$

so the required solution is

$$y = \frac{x^4}{12} + x^2 + 2x + 1$$

Consider now a similar type of PDE i.e. one that can also be solved by direct integration.
Suppose we require the general solution of

$$\frac{\partial^2 u}{\partial x^2} = 2xe^t$$

where u is a function of x and t.
Integrating with respect to x gives us

$$\frac{\partial u}{\partial x} = x^2e^t + f(t)$$

where the arbitrary function $f(t)$ replaces the normal "arbitrary constant" of ordinary integration. This function of t only is needed because we are integrating "partially" with respect to x i.e. we are reversing a partial differentiation with respect to x at constant t.
Integrating again with respect to x gives the general solution:

$$u = \frac{x^3}{3}e^t + x\ f(t) + g(t)$$

where $g(t)$ is a second arbitrary function. We have now obtained the general solution of the given PDE but to find the arbitrary function we must know two initial conditions.
Suppose, for the sake of example, that these conditions are

$$u(0,t) = t\ , \qquad \frac{\partial u}{\partial x}(0,t) = e^t$$

Inserting the first of these conditions into the general solution gives $g(t) = t$.

Inserting the second condition into the general solution gives $f(t) = e^t$.

So the final solution is $u = \dfrac{x^3}{3}e^t + xe^t + t$.

Task

Solve the PDE

$$\frac{\partial^2 u}{\partial x \partial y} = \sin x\ \cos y$$

subject to the conditions

$$\frac{\partial u}{\partial x} = 2x \text{ at } y = \frac{\pi}{2}, \quad u = 2\sin y \text{ at } x = \pi.$$

First integrate the PDE with respect to y: (it is equally valid to integrate first with respect to x). Don't forget the appropriate arbitrary function.

Your solution

Answer

Recall that $\dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial u}{\partial x}\right)$

Hence integration with respect to y gives $\quad \dfrac{\partial u}{\partial x} = \sin x \sin y + f(x)$

Since one of the given conditions is on $\dfrac{\partial u}{\partial x}$, impose this condition to determine the arbitrary function $f(x)$:

Your solution

Answer

At $y = \pi/2$ the condition gives $\quad \sin x \sin \pi/2 + f(x) = 2x \quad$ i.e. $\quad f(x) = 2x - \sin x$

So $\dfrac{\partial u}{\partial x} = \sin x \sin y + 2x - \sin x$

Now integrate again to determine u:

Your solution

Answer

Integrating now with respect to x gives $\quad u = -\cos x \sin y + x^2 + \cos x + g(y)$

Next, obtain the arbitrary function $g(y)$:

Your solution

Answer

The condition $u(\pi, y) = 2\sin y$ gives $\quad -\cos \pi \sin y + \pi^2 + \cos \pi + g(y) = 2\sin y$

$\therefore \qquad \sin y + \pi^2 - 1 + g(y) = 2\sin y$

$\therefore \qquad g(y) = \sin y + 1 - \pi^2$

Now write down the final answer for $u(x, y)$:

Your solution

Answer

$$u(x, y) = x^2 + \cos x(1 - \sin y) + \sin y + 1 - \pi^2$$

2. Method of separation of variables - general approach

In Section 25.2 we showed that

(a) $u(x,y) = \sin x \cosh y$

is **a** solution of the two-dimensional Laplace equation

(b) $u(x,t) = e^{-2\pi^2 t} \sin \pi x$

is **a** solution of the one-dimensional heat conduction equation

(c) $u(x,t) = u_0 \sin\left(\dfrac{\pi x}{\ell}\right) \cos\left(\dfrac{\pi c t}{\ell}\right)$

is **a** solution of the one-dimensional wave equation.

All three solutions here have a specific form: in (a) $u(x,y)$ is a product of a function of x alone, $\sin x$, and a function of y alone, $\cosh y$. Similarly, in both (b) and (c), $u(x,t)$ is a product of a function of x alone and a function of t alone.

The method of separation of variables involves finding solutions of PDEs which are of this product form. In the method we assume that a solution to a PDE has the form.

$$u(x,t) = X(x)T(t) \qquad \text{(or} \qquad u(x,y) = X(x)Y(y))$$

where $X(x)$ is a function of x only, $T(t)$ is a function of t only and $Y(y)$ is a function y only.

You should note that not all solutions to PDEs are of this type; for example, it is easy to verify that

$$u(x,y) = x^2 - y^2$$

(which is **not** of the form $u(x,y) = X(x)Y(y)$) is a solution of the Laplace equation.

However, many interesting and useful solutions of PDEs are obtainable which are of the product form. We shall firstly consider the types of solution obtainable for our three basic PDEs using trial solutions of the product form.

Heat conduction equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t} \qquad k > 0 \tag{1}$$

Assuming that

$$\begin{aligned} u &= X(x)T(t) \\ \text{then} \quad &= XT \quad \text{for short} \end{aligned} \tag{2}$$

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{dX}{dx}T = X'T \ \ \text{for short} \\ \frac{\partial^2 u}{\partial x^2} &= \frac{d^2X}{dx^2}T = X''T \ \ \text{for short} \\ \frac{\partial u}{\partial t} &= X\frac{dT}{dt} = XT' \ \ \text{for short} \end{aligned}$$

Substituting into the original PDE (1)

$$X''T = \frac{1}{k}XT'$$

which can be re-arranged as

$$\frac{X''}{X} = \frac{1}{k}\frac{T'}{T} \qquad (3)$$

Now the left-hand side of (3) involves functions of x only and the right-hand side expression contain functions of t only. Thus altering the value of t cannot change the left-hand side of (3) i.e. it stays constant. Hence so must the right-hand side be constant. We conclude that $T(t)$ is a function such that

$$\frac{1}{k}\frac{T'}{T} = K \qquad (4)$$

where K is a constant whose sign is yet to be determined.

By a similar argument, altering the value of x cannot change the right-hand side of (3) and consequently the left-hand side must be a constant, i.e.

$$\frac{X''}{X} = K \qquad (5)$$

We see that the effect of assuming a product trial solution of the form (2) converts the PDE (1) into the two ODEs (4) and (5).

Both these ODEs are types whose solution we revised at the beginning of this Workbook but we shall not attempt to solve them yet. In particular the solution of (5) depends on whether the constant K is positive or negative.

Wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \qquad (6)$$

Task

By following a similar procedure to the above, assume a product solution

$$u(x,t) = X(x)T(t)$$

for the wave equation and find the two ODEs satisfied by $X(x)$ and $T(t)$.

First obtain $\dfrac{\partial^2 u}{\partial x^2}$ and $\dfrac{\partial^2 u}{\partial t^2}$:

Your solution

Answer

$u = X(x)T(t)$ gives $\dfrac{\partial^2 u}{\partial x^2} = X''T$ and $\dfrac{\partial^2 u}{\partial t^2} = XT''$

Now substitute these results into (6) and transpose so the variables are separated i.e. all functions of x are on the left-hand side, all funtions of t on the right-hand side:

Your solution

Answer

We get $X''T = \dfrac{1}{c^2}XT''$ and, transposing, $\dfrac{X''}{X} = \dfrac{1}{c^2}\dfrac{T''}{T}$

Finally, write down the required ordinary differential equations:

Your solution

Answer

Equating both sides to the same constant K gives

$$\frac{X''}{X} = K \quad \text{or} \quad \frac{d^2X}{dx^2} - KX = 0 \tag{7}$$

and

$$\frac{1}{c^2}\frac{T''}{T} = K \qquad \text{or} \qquad \frac{d^2T}{dt^2} - Kc^2T = 0 \tag{8}$$

The solution of the ODEs (7) and (8) has been obtained earlier, and will depend on the sign of K.

Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{9}$$

Task

Separating the variables for Laplace's equation follows similar lines to the previous Task. Obtain the ODEs satisfied by $X(x)$ and $Y(y)$.

Your solution

Answer

Assuming $u(x,y) = X(x)Y(y)$ leads to: $\dfrac{\partial^2 u}{\partial x^2} = X''Y \qquad \dfrac{\partial^2 u}{\partial y^2} = XY''$ so

$$X''Y + XY'' = 0 \qquad \text{or} \qquad \frac{X''}{X} = -\frac{Y''}{Y}$$

Equating each side to a constant K

$$\frac{X''}{X} = K \quad \text{or} \quad \frac{d^2X}{dx^2} - KX = 0 \tag{10a}$$

$$\frac{Y''}{Y} = -K \quad \text{or} \quad \frac{d^2Y}{dy^2} + KY = 0 \tag{10b}$$

(Note carefully the different signs in the two ODEs. Yet again the sign of the "separation constant" K will determine the solutions.)

3. Method of separation of variables - specific solutions

We shall now study some specific problems which can be fully solved by the separation of variables method.

Example 3

Solve the heat conduction equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{2}\frac{\partial u}{\partial t}$$

over $\quad 0 < x < 3, \quad t > 0 \quad$ for the boundary conditions

$$u(0,t) = u(3,t) = 0$$

and the initial condition

$$u(x,0) = 5\sin 4\pi x.$$

Solution

Assuming $u(x,t) = X(x)T(t)$ gives rise to the differential equations (4) and (5) with the parameter $k = 2$:

$$\frac{dT}{dt} = 2KT \qquad\qquad \frac{d^2X}{dx^2} = KX$$

The T equation has general solution

$$T = Ae^{2Kt}$$

which will increase exponentially with increasing t if K is positive and decrease with t if K is negative. In any physical problem the latter is the meaningful situation. To emphasise that K is being taken as negative we put

$$K = -\lambda^2$$

so

$$T = Ae^{-2\lambda^2 t}.$$

The X equation then becomes

$$\frac{d^2X}{dx^2} = -\lambda^2 X$$

which has solution

$$X(x) = B\cos\lambda x + C\sin\lambda x.$$

Hence

$$u(x,t) = X(x)T(t) = (D\cos\lambda x + E\sin\lambda x)e^{-2\lambda^2 t} \qquad (11)$$

where $D = AB$ and $E = AC$.

(You should always try to keep the number of arbitrary constants down to an absolute minimum by multiplying them together in this way.)

We now insert the initial and boundary conditions to obtain the constant D and$E$ and also the separation constant λ.
The initial condition $u(0,t) = 0$ gives

$$(D\cos 0 + E\sin 0)e^{-2\lambda^2 t} = 0 \qquad \text{for all } t.$$

Since $\sin 0 = 0$ and $\cos 0 = 1$ this must imply that $D = 0$.
The other initial condition $u(3,t) = 0$ then gives

$$E\sin(3\lambda)e^{-2\lambda^2 t} = 0 \qquad \text{for all } t.$$

We cannot deduce that the constant E has to be zero because then the solution (11) would be the trivial solution $u \equiv 0$. The only sensible deduction is that

$\sin 3\lambda = 0$ i.e. $3\lambda = n\pi$ (where n is some integer).
Hence solutions of the form (11) satisfying the 2 boundary conditions have the form

$$u(x,t) = E_n \sin\left(\frac{n\pi x}{3}\right) e^{-\frac{2n^2\pi^2 t}{9}}$$

where we have written E_n for E to allow for the possibility of a different value for the constant for each different value of n.

We obtain the value of n by using the initial condition $u(x,0) = 5\sin 4\pi x$ and forcing this solution to agree with it. That is,

$$u(x,0) = E_n \sin\left(\frac{n\pi x}{3}\right) = 5\sin 4\pi x$$

so we must choose $n = 12$ with $E_{12} = 5$.
Hence, finally,

$$u(x,t) = 5\sin\left(\frac{12\pi x}{3}\right) e^{-\frac{2}{9}(12)^2\pi^2 t} = 5\sin(4\pi x)\ e^{-32\pi^2 t}.$$

Task

Solve the 1-dimensional wave equation $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{1}{16}\dfrac{\partial^2 u}{\partial t^2}$ for $0 < x < 2, \quad t > 0$

The boundary conditions are

$$u(0,t) = u(2,t) = 0$$

The initial conditions are

(i) $u(x,0) = 6\sin\pi x - 3\sin 4\pi x$ (ii) $\dfrac{\partial u}{\partial t}(x,0) = 0$

Firstly, either using (7) and (8) or by working from first principles assuming the product solution

$$u(x,t) = X(x)T(t),$$

write down the ODEs satisfied by $X(x)$ and $T(t)$:

Your solution

Answer

$$\frac{X''}{X} = K \qquad \frac{T''}{16T} = K$$

Now decide on the appropriate sign for K and then write down the solution to these equations:

Your solution

Answer

Choosing K as negative (say $K = -\lambda^2$) will produce Sinusoidal solutions for X and T which are appropriate in the context of the wave equation where oscillatory solutions can be expected.

Then $X'' = -\lambda^2 X$ gives

$$X = A\cos\lambda x + B\sin\lambda x$$

Similarly $T'' = -16\lambda^2 T$ gives

$$T = C\cos 4\lambda t + D\sin 4\lambda t$$

Now obtain the general solution $u(x,t)$ by multiplying $X(x)$ by $T(t)$ and insert the two boundary conditions to obtain information about two of the constants:

Your solution

Answer

$u(x,t) = (A\cos\lambda x + B\sin\lambda x)(C\cos 4\lambda t + D\sin 4\lambda t)$

$u(0,t) = 0$ for all t gives

$$A(C\cos 4\lambda t + D\sin 4\lambda t) = 0$$

which implies that $A = 0$.

$u(2,t) = 0$ for all t gives

$$B\sin 2\lambda(C\cos 4\lambda t + D\sin 4\lambda t) = 0$$

so, for a non-trivial solution,

$$\sin 2\lambda = 0 \quad \text{i.e.} \quad \lambda = \frac{n\pi}{2} \quad \text{for some integer } n.$$

At this stage we write the solution as

$$u(x,t) = \sin\left(\frac{n\pi x}{2}\right)\ (E\cos 2n\pi t + F\sin 2n\pi t)$$

where we have multiplied constants and put $E = BC$ and $F = BD$.

Now insert the initial condition

$$\frac{\partial u}{\partial t}(x,0) = 0 \ \text{ for all } x \qquad 0 < x < 2.$$

and deduce the value of F:

Your solution

Answer
Differentiating partially with respect to t

$$\frac{\partial u}{\partial t} = \sin\left(\frac{n\pi x}{2}\right)(-2n\pi E \sin 2n\pi t + 2n\pi F \cos 2n\pi t)$$

so at $t = 0$

$$\frac{\partial u}{\partial t}(x,0) = \sin\left(\frac{n\pi x}{2}\right) 2n\pi F = 0$$

from which we must have that $F = 0$.

Finally using the other the initial condition $u(x,0) = 6\sin(\pi x) - 3\sin(4\pi x)$ deduce the form of $u(x,t)$:

Your solution

Answer
At this stage the solution reads

$$u(x,t) = E\sin\left(\frac{n\pi x}{2}\right)\cos(2n\pi t) \tag{12}$$

We now have to insert the last condition i.e. the initial condition

$$u(x,0) = 6\sin\pi x - 3\sin 4\pi x \tag{13}$$

This seems strange because, putting $t = 0$ in our solution (12) suggests

$$u(x,0) = E\sin\left(\frac{n\pi x}{2}\right)$$

At this point we seem to have incompatability because no single value of n will enable us to satisfy (13). However, in the solution (12), **any** positive integer value of n is acceptable and we can in fact, **superpose** solutions of the form (12) and still have a valid solution to the PDE Hence we first write, instead of (12)

$$u(x,t) = \sum_{n=1}^{\infty} E_n \sin\left(\frac{n\pi x}{2}\right)\cos(2n\pi t) \tag{14}$$

from which

$$u(x,0) = \sum_{n=1}^{\infty} E_n \sin\left(\frac{n\pi x}{2}\right) \tag{15}$$

(which looks very much like, and indeed is, a Fourier series.)

To make the solution (15) fit the initial condition (13) we do not require all the terms in the infinite Fourier series. We need only the terms with $n = 2$ with coefficient $E_2 = 6$ and the term for which $n = 8$ with $E_8 = -3$. All the other coefficients E_n have to be chosen as zero.
Using these results in (14) we obtain the solution

$$u(x,t) = 6\sin\pi x\cos 4\pi t - 3\sin 4\pi x\cos 16\pi t$$

The above solution perhaps seems rather involved but there is a definite sequence of logical steps which can be readily applied to other similar problems.

Engineering Example 1

Heat conduction through a furnace wall

Introduction

Conduction is a mode of heat transfer through molecular collision inside a material without any motion of the material as a whole. If one end of a solid material is at a higher temperature, then heat will be transferred towards the colder end because of the relative movement of the particles. They will collide with the each other with a net transfer of energy.

Energy flows through heat conductive materials by a thermal process generally known as 'gradient heat transport'. Gradient heat transport depends on three quantities: the heat conductivity of the material, the cross-sectional area of the material which is available for heat transfer and the spatial gradient of temperature (driving force for the process). The larger the conductivity, the gradient, and the cross section, the faster the heat flows.

The temperature profile within a body depends upon the rate of heat transfer to the atmosphere, its capacity to store some of this heat, and its rate of thermal conduction to its boundaries (where the heat is transferred to the surrounding environment). Mathematically this is stated by the heat equation

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) = \rho c\frac{\partial T}{\partial x} \qquad (1)$$

The thermal diffusivity α is related to the thermal conductivity k, the specific heat c, and the density of solid material ρ, by

$$\alpha = \frac{k}{\rho c}$$

Problem in Words

The wall (thickness L) of a furnace, with inside temperature 800° C, is comprised of brick material [thermal conductivity $= 0.02$ W m^{-1} K^{-1})]. Given that the wall thickness is 12 cm, the atmospheric temperature is 0° C, the density and heat capacity of the brick material are 1.9 gm cm^{-3} and 6.0 J kg^{-1} K^{-1} respectively, estimate the temperature profile within the brick wall after 2 hours.

Mathematical statement of problem

Solve the partial differential equation

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) = \rho c\frac{\partial T}{\partial t} \qquad (2)$$

subject to the initial condition

$$T(x,0) = 800\sin\frac{\pi x}{2L} \qquad (3)$$

and boundary conditions at the inner ($x = L$) and outer ($x = 0$) walls of

$$T = 0 \qquad \text{at} \qquad x = 0 \tag{4}$$

and

$$\frac{\partial T}{\partial x} = 0 \qquad \text{at} \qquad x = L \tag{4b}$$

Find the temperature profile at $T = 7200$ seconds $= 2$ hours.

Mathematical analysis

Using separation of variables

$$T(x,t) = X(x) \times Y(t) \tag{5}$$

so Equation (2) becomes

$$\frac{Y'}{Y} = \alpha \frac{X''}{X} = K \tag{6}$$

Using values of K which are zero or positive does not allow a solution which satisfies the initial and boundary conditions. Thus, K is assumed to be negative i.e. $K = -\lambda^2$. Equation (6) separates into the two ordinary differential equations

$$\begin{aligned} \frac{dY}{dt} &= -\lambda^2 Y \\ \frac{d^2X}{dx^2} &= -\lambda^2 \alpha X \end{aligned}$$

with solutions

$$\begin{aligned} Y &= Ce^{-\lambda^2 t} \\ X &= A^* \cos \frac{\lambda}{\sqrt{\alpha}} x + B^* \sin \frac{\lambda}{\sqrt{\alpha}} x \end{aligned}$$

and

$$T = X \times Y = e^{-\lambda^2 t} \left\{ A \cos \frac{\lambda}{\sqrt{\alpha}} x + B \sin \frac{\lambda}{\sqrt{\alpha}} x \right\} \tag{8}$$

where $A = A^* \times C$ and $B = B^* \times C$.

Setting $T = 0$ where $x = 0$ (Equation (4a)) gives $A = 0$ i.e.

$$T = Be^{-\lambda^2 t} \sin \frac{\lambda}{\sqrt{\alpha}} x \tag{9}$$

and hence

$$\frac{dT}{dx} = B \frac{\lambda}{\sqrt{\alpha}} e^{-\lambda^2 t} \cos \frac{\lambda}{\sqrt{\alpha}} x \tag{10}$$

Setting $\dfrac{dT}{dx} = 0$ where $x = L$ (and for all t), Equation (4b) gives one of the conclusions,

$$\begin{aligned} B &= 0 \\ \lambda &= 0 \\ \cos \frac{\lambda}{\sqrt{\alpha}} L &= 0 \end{aligned}$$

The first two possibilities ($B = 0$ and $\lambda = 0$) can be discounted as they leave $T = 0$ for all x and t and it is not possible to satisfy the initial condition (3). Hence $\cos \dfrac{\lambda}{\sqrt{\alpha}} L = 0$ so $\dfrac{\lambda}{\sqrt{\alpha}} L = (n + \frac{1}{2})\pi$ and we deduce that

$$\lambda = \frac{\sqrt{\alpha}}{L}(n + \frac{1}{2})\pi \tag{11}$$

and so the temperature T satisfies

$$T = Be^{-\dfrac{\alpha}{L^2}(n+\frac{1}{2})^2 t} \sin\left\{(n + \frac{1}{2})\frac{\pi x}{L}\right\} \tag{12}$$

However, this must also satisfy Equation (2) i.e.

$$800 \sin \frac{\pi x}{2L} = B \sin\left\{(n + \frac{1}{2})\frac{\pi x}{L}\right\} \tag{13}$$

Equating the arguments of the sine terms

$$\frac{\pi x}{2L} = (n + \frac{1}{2})\frac{\pi x}{L} \qquad \text{so} \quad n = 0$$

Equating the coefficients of the sine terms

$$800 = B$$

So the temperature profile is

$$T = 800e^{-\frac{\alpha t}{2L^2}} \sin \frac{\pi x}{2L} \tag{14}$$

where $\alpha = \dfrac{k}{\rho c} = \dfrac{0.02}{1900 \times 6} = 1.764 \times 10^{-6} m^2 s^{-1}$.

After two hours, $t = 7200$ so $-\dfrac{\alpha t}{2L^2} = -0.438$ so

$$T = 800 \times e^{-0.438} \sin \frac{\pi x}{2L} = 516 \sin \frac{\pi x}{2L} \tag{15}$$

so the inner wall of the furnace has cooled from 800°C to 516°C.

Interpretation

The boundary conditions (2) and (3) represent approximations to the true boundary conditions, approximations made to enable solution by separation of variables. More realistic conditions would be

$$\begin{aligned}
-k\frac{\partial T(0,t)}{\partial x} &= h_{outside}\left\{T_\infty - T(0,t)\right\} \\
-k\frac{\partial T(L,t)}{\partial x} &= h_{inside}\left\{T(0,t) - T_s\right\}
\end{aligned}$$

Solution Using Fourier Series

25.4

Introduction

In this Section we continue to use the separation of variables method for solving PDEs but you will find that, to be able to fit certain boundary conditions, Fourier series methods have to be used leading to the final solution being in the (rather complicated) form of an infinite series. The techniques will be illustrated using the two-dimensional Laplace equation but similar situations often arise in connection with other important PDEs.

Prerequisites

Before starting this Section you should ...

- be familiar with the separation of variables method
- be familiar with trigonometric Fourier series

Learning Outcomes

On completion you should be able to ...

- solve the 2-D Laplace equation for given boundary conditions and utilize Fourier series in the solution when necessary

1. Solutions involving infinite Fourier series

We shall illustrate this situation using Laplace's equation but infinite Fourier series can also be necessary for the heat conduction and wave equations.

We recall from the previous Section that using a product solution

$$u(x,t) = X(x)Y(y)$$

in Laplace's equation gives rise to the ODEs:

$$\frac{X''}{X} = K \qquad \frac{Y''}{Y} = -K$$

To determine the sign of K and hence the appropriate solutions for $X(x)$ and $Y(y)$ we must impose appropriate boundary conditions. We will investigate solving Laplace's equation in the square

$$0 \leq x \leq \ell \qquad 0 \leq y \leq \ell$$

for the boundary conditions $u(x,0) = 0 \quad u(0,y) = 0 \quad u(\ell,y) = 0 \quad u(x,\ell) = U_0$, a constant.
See Figure 7.

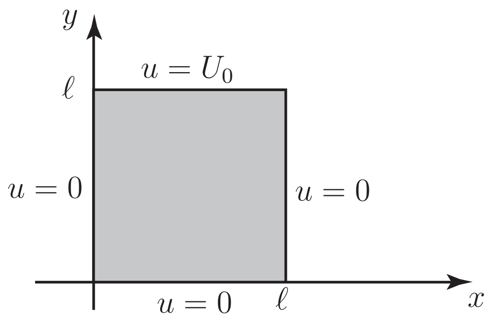


Figure 7

(a) We must first deduce the sign of the separation constant K:
if K is chosen to be positive say $K = \lambda^2$, then the X equation is

$$X'' = \lambda^2 X$$

with general solution

$$X = Ae^{\lambda x} + Be^{-\lambda x}$$

while the Y equation becomes

$$Y'' = -\lambda^2 Y$$

with general solution

$$Y = C\cos\lambda y + D\sin\lambda y$$

If the sign of K is negative $K = -\lambda^2$ the solutions will change to trigonometric in x and exponential in y.

These are the only two possibilities when we solve Laplace's equation using separation of variables and we must look at the boundary conditions of the problem to decide which is appropriate.
Here the boundary conditions are periodic in x (since $u(0,y) = u(\ell,y)$) and non-periodic in y which suggests we need a solution that is periodic in x and non-periodic in y.

Thus we choose $K = -\lambda^2$ to give

$$X(x) = (A\cos\lambda x + B\sin\lambda x)$$
$$Y(y) = (Ce^{\lambda y} + De^{-\lambda y})$$

(Note that had we chosen the incorrect sign for K at this stage we would later have found it impossible to satisfy all the given boundary conditions. You might like to verify this statement.)

The appropriate general solution of Laplace's equation for the given problem is

$$u(x,y) = (A\cos\lambda x + B\sin\lambda x)(Ce^{\lambda y} + De^{-\lambda y}).$$

(b) Inserting the boundary conditions produces the following consequences:

$$u(0,y) = 0 \qquad \text{gives} \qquad A = 0$$
$$u(\ell,y) = 0 \qquad \text{gives} \qquad \sin\lambda\ell = 0 \qquad \text{i.e. } \lambda = \frac{n\pi}{\ell}$$

where n is a positive integer $1, 2, 3, \ldots$. While $n = 0$ also satisfies the equation it leads to the trivial solution $u = 0$ only.)

$$u(x,0) = 0 \ \text{ gives } \ C + D = 0 \ \text{ i.e. } \ D = -C$$

At this point the solution can be written

$$u(x,y) = BC\sin\left(\frac{n\pi x}{\ell}\right)\left(e^{\frac{n\pi y}{\ell}} - e^{-\frac{n\pi y}{\ell}}\right)$$

This can be conveniently written as

$$u(x,y) = E\sin\left(\frac{n\pi x}{\ell}\right)\sinh\left(\frac{n\pi y}{\ell}\right) \qquad (1)$$

where $E = 2BC$.

At this stage we have just one final boundary condition to insert to obtain information about the constant E and the integer n. Our solution (1) gives

$$u(x,\ell) = E\sin\left(\frac{n\pi x}{\ell}\right)\sinh(n\pi)$$

and clearly this is not compatible, as it stands, with the given boundary condition

$$u(x,\ell) = U_0 = \text{constant}.$$

The way to proceed is again to **superpose** solutions of the form (1) for all positive integer values of n to give

$$u(x,y) = \sum_{n=1}^{\infty} E_n \sin\left(\frac{n\pi x}{\ell}\right)\sinh\left(\frac{n\pi y}{\ell}\right) \qquad (2)$$

from which the final boundary condition gives

$$\begin{aligned} U_0 &= \sum_{n=1}^{\infty} E_n \sin\left(\frac{n\pi x}{\ell}\right)\sinh(n\pi) \qquad 0 < x < \ell. \qquad (3) \\ &= \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{\ell}\right) \end{aligned}$$

What we have here is a Fourier (sine) series for the function

$$f(x) = U_0 \qquad 0 < x < \ell.$$

Recalling the work on half-range Fourier series (HELM 23.5) we must extend this definition to produce an odd function with period 2ℓ. Hence we define

$$f(x) = \begin{cases} U_0 & 0 < x < \ell \\ -U_0 & -\ell < x < 0 \end{cases}$$

$$f(x+\ell) = f(x)$$

illustrated in Figure 8.

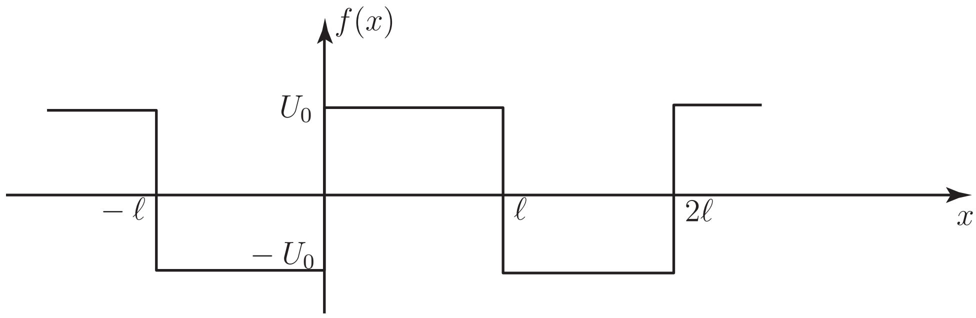


Figure 8

(c) We can now apply standard Fourier series theory to evaluate the Fourier coefficients b_n in (3).

We obtain

$$b_n = E_n \sinh n\pi = \frac{4U_0}{2\ell}\int_0^\ell \sin\left(\frac{n\pi x}{\ell}\right)\, dx$$

(Recall that, in general, $b_n = 2 \times$ the mean value of $f(x)\sin\left(\dfrac{n\pi x}{\ell}\right)$ over a period. Here, because $f(x)$ is odd, and hence $f(x)\sin\left(\dfrac{n\pi x}{\ell}\right)$ is even, we may take half the period for our averaging process.)

Carrying out the integration

$$E_n \sinh n\pi = \frac{2U_0}{n\pi}(1-\cos n\pi) \qquad \text{i.e.} \qquad E_n = \begin{cases} \dfrac{4U_0}{n\pi \sinh n\pi} & n = 1,3,5,\ldots \\ 0 & n = 2,4,6,\ldots \end{cases}$$

(Since $f(x)$ is a square wave with half-period symmetry we are not surprised that only odd harmonics arise in the Fourier series.)

Finally substituting these results for E_n into (2) we obtain the solution to the given problem as the infinite series:

$$u(x,y) = \frac{4U_0}{\pi}\sum_{\substack{n=1 \\ (n \text{ odd})}}^{\infty} \frac{\sin\left(\dfrac{n\pi x}{\ell}\right)\sinh\left(\dfrac{n\pi y}{\ell}\right)}{n \sinh n\pi}$$

Task

Solve Laplace's equation to determine the steady state temperature $u(x, y)$ in the semi-infinite plate $0 \leq x \leq 1, \quad y \geq 0$. Assume that the left and right sides are insulated and assume that the solution is bounded. The temperature along the bottom side is a known function $f(x)$.

First write this problem as a mathematical boundary value problem paying particular attention to the mathematical representation of the boundary conditions:

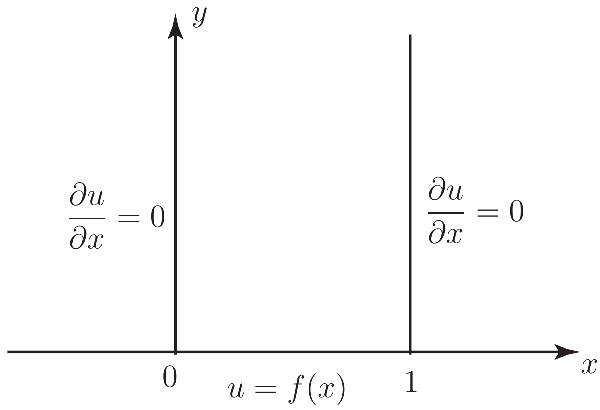
Your solution

Answer

Since the sides $x = 0$ and $x = 1$ are insulated, the temperature gradient across these sides is zero i.e. $\dfrac{\partial u}{\partial x} = 0$ for $x = 0$, $0 < y < \infty$ and $\dfrac{\partial u}{\partial x} = 0$ for $x = 1$, $0 < y < \infty$.

The third boundary condition is $u(x, 0) = f(x)$.

The fourth boundary condition is less obvious: since the solution should be bounded (ie not grow and grow) we must demand that $u(x, y) \to 0$ as $y \to \infty$. (See figure below.)



Now use the separation of variables method, putting $u(x, y) = X(x)Y(y)$, to find the differential equations satisfied by $X(x)$, $Y(y)$ and decide on the sign of the separation constant K:

Your solution

Answer
We have boundary conditions which, like the worked example above, are periodic in x. Hence the differential equations are, again,

$$X'' = -\lambda^2 X \qquad\qquad Y'' = +\lambda^2 Y$$

putting the separation constant K as $-\lambda^2$.

Write down the solutions for X, for Y and hence the product solution $u(x, y) = X(x)Y(y)$:

Your solution

Answer
$X = A\cos\lambda x + B\sin\lambda x \qquad\qquad Y = Ce^{\lambda y} + De^{-\lambda y}$

so

$$u = (A\cos\lambda x + B\sin\lambda x)(Ce^{\lambda y} + De^{-\lambda y}) \tag{4}$$

Impose the derivative boundary conditions on this solution:

Your solution

Answer

$$\frac{\partial u}{\partial x} = (-\lambda A \sin \lambda x + \lambda B \cos \lambda x)(Ce^{\lambda y} + De^{-\lambda y})$$

Hence $\dfrac{\partial u}{\partial x}(0, y) = 0$ gives $\lambda B(Ce^{\lambda y} + De^{-\lambda y}) = 0$ for all y.

The possibility $\lambda = 0$ can be excluded this would give a trivial constant solution in (4). Hence we must choose $B = 0$.

The condition $\dfrac{\partial u}{\partial x}(1, y) = 0$ gives

$$-\lambda A \sin \lambda(Ce^{\lambda y} + De^{-\lambda y}) = 0$$

Choosing $A = 0$ would make $u \equiv 0$ so we must force $\sin \lambda$ to be zero i.e. choose $\lambda = n\pi$ where n is a positive integer.

Thus, at this stage (4) becomes

$$\begin{aligned} u &= A \cos n\pi x(Ce^{n\pi y} + De^{-n\pi y}) \\ &= \cos n\pi x(Ee^{n\pi y} + Fe^{-n\pi y}) \end{aligned} \qquad (5)$$

Now impose the condition that this solution should be bounded:

Your solution

Answer

The region over which we are solving Laplace's equation is semi-infinite i.e. the y coordinate increases without limit. The solution for $u(x, y)$ in (5) will increase without limit as $y \to \infty$ due to the term $e^{n\pi y}$ (n being a positive integer.) This can be avoided i.e. the solution will be bounded if the constant E is chosen as zero.

Finally, use Fourier series techniques to deal with the final boundary condition $u(x, 0) = f(x)$:

Your solution

Your solution

Answer

Superposing solutions of the form (5) (with $E = 0$) gives

$$u(x, y) = \sum_{n=0}^{\infty} F_n \cos(n\pi x)\ e^{-n\pi y} \qquad (6)$$

so the boundary condition gives

$$f(x) = \sum_{n=0}^{\infty} F_n \cos n\pi x$$

We have here a half-range Fourier cosine series representation of a function $f(x)$ defined over $0 < x < 1$. Extending $f(x)$ as an even periodic function with period 2 and using standard Fourier series theory gives

$$F_n = 2\int_0^1 f(x) \cos n\pi x\ dx \qquad n = 1, 2, \ldots$$

with

$$\frac{F_0}{2} = \int_0^1 f(x)\ dx.$$

Hence (6) is the solution of this given boundary value problem, the integrals giving us in principle the Fourier coefficients F_n for a given function $f(x)$.