Contents 24

Fourier Transforms

Learning outcomes

In this Workbook you will learn about the Fourier transform which has many applications in science and engineering. You will learn how to find Fourier transforms of some standard functions and some of the properties of the Fourier transform. You will learn about the inverse Fourier transform and how to find inverse transforms directly and by using a table of transforms. Finally, you will learn about some special Fourier transform pairs.

The Fourier Transform 24.1

Introduction

Fourier transforms have for a long time been a basic tool of applied mathematics, particularly for solving differential equations (especially partial differential equations) and also in conjunction with integral equations.

There are really three Fourier transforms, the Fourier Sine and Fourier Cosine transforms and a complex form which is usually referred to as *the* Fourier transform.

The last of these transforms in particular has extensive applications in Science and Engineering, for example in physical optics, chemistry (e.g. in connection with Nuclear Magnetic Resonance and Crystallography), Electronic Communications Theory and more general Linear Systems Theory.

Prerequisites

Before starting this Section you should ...

- be familiar with basic Fourier series, particularly in the complex form

Learning Outcomes

On completion you should be able to ...

- calculate simple Fourier transforms from the definition
- state how the Fourier transform of a function (signal) depends on whether that function is even or odd or neither

1. The Fourier transform

Unlike Fourier series, which are mainly useful for periodic functions, the Fourier transform permits alternative representations of mostly non-periodic functions.

We shall firstly derive the Fourier transform from the complex exponential form of the Fourier series and then study its various properties.

2. Informal derivation of the Fourier transform

Recall that if $f(t)$ is a period T function, which we will temporarily re-write as $f_T(t)$ for emphasis, then we can expand it in a complex Fourier series,

$$f_T(t) = \sum_{n=-\infty}^{\infty} c_n e^{\mathrm{i}n\omega_0 t} \tag{1}$$

where $\omega_0 = \dfrac{2\pi}{T}$. In words, harmonics of frequency $n\omega_0 = n\dfrac{2\pi}{T} \quad n = 0, \pm 1, \pm 2, \ldots$ are present in the series and these frequencies are separated by

$$n\omega_0 - (n-1)\omega_0 = \omega_0 = \frac{2\pi}{T}.$$

Hence, as T increases the frequency separation becomes smaller and can be conveniently written as $\Delta\omega$. This suggests that as $T \to \infty$, corresponding to a non-periodic function, then $\Delta\omega \to 0$ and the frequency representation contains **all** frequency harmonics.

To see this in a little more detail, we recall (HELM 23: Fourier series) that the complex Fourier coefficients c_n are given by

$$c_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f_T(t) e^{-\mathrm{i}n\omega_0 t}\, dt. \tag{2}$$

Putting $\dfrac{1}{T}$ as $\dfrac{\omega_0}{2\pi}$ and then substituting (2) in (1) we get

$$f_T(t) = \sum_{n=-\infty}^{\infty} \left\{ \frac{\omega_0}{2\pi} \int_{-\frac{T}{2}}^{\frac{T}{2}} f_T(t) e^{-\mathrm{i}n\omega_0 t}\, dt \right\} e^{\mathrm{i}n\omega_0 t}.$$

In view of the discussion above, as $T \to \infty$ we can put ω_0 as $\Delta\omega$ and replace the sum over the discrete frequencies $n\omega_0$ by an integral over all frequencies. We replace $n\omega_0$ by a general frequency variable ω. We then obtain the double integral representation

$$f(t) = \int_{-\infty}^{\infty} \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{-\mathrm{i}\omega t}\, dt \right\} e^{\mathrm{i}\omega t}\, d\omega. \tag{3}$$

The inner integral (over all t) will give a function dependent only on ω which we write as $F(\omega)$. Then (3) can be written

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) e^{\mathrm{i}\omega t}\, d\omega \tag{4}$$

where

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-\mathrm{i}\omega t}\, dt. \qquad (5)$$

The representation (4) of $f(t)$ which involves all frequencies ω can be considered as the equivalent for a non-periodic function of the complex Fourier series representation (1) of a periodic function.

The expression (5) for $F(\omega)$ is analogous to the relation (2) for the Fourier coefficients c_n.

The function $F(\omega)$ is called the **Fourier transform** of the function $f(t)$. Symbolically we can write

$$F(\omega) = \mathcal{F}\{f(t)\}.$$

Equation (4) enables us, in principle, to write $f(t)$ in terms of $F(\omega)$. $f(t)$ is often called the **inverse Fourier transform** of $F(\omega)$ and we denote this by writing

$$f(t) = \mathcal{F}^{-1}\{F(\omega)\}.$$

Looking at the basic relation (3) it is clear that the position of the factor $\dfrac{1}{2\pi}$ is somewhat arbitrary in (4) and (5). If instead of (5) we define

$$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{-\mathrm{i}\omega t}\, dt.$$

then (4) must be written

$$f(t) = \int_{-\infty}^{\infty} F(\omega)e^{\mathrm{i}\omega t}\, d\omega.$$

A third, more symmetric, alternative is to write

$$F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-\mathrm{i}\omega t}\, dt$$

and, consequently:

$$f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\omega)e^{\mathrm{i}\omega t}\, d\omega.$$

We shall use (4) and (5) throughout this Section but you should be aware of these other possibilities which might be used in other texts.

Engineers often refer to $F(\omega)$ (whichever precise definition is used!) as the **frequency domain** representation of a function or signal and $f(t)$ as the **time domain** representation. In what follows we shall use this language where appropriate. However, (5) is really a mathematical transformation for obtaining one function from another and (4) is then the inverse transformation for recovering the initial function. In some applications of Fourier transforms (which we shall not study) the time/frequency interpretations are not relevant. However, in engineering applications, such as communications theory, the frequency representation is often used very literally.

As can be seen above, notationally we will use capital letters to denote Fourier transforms: thus a function $f(t)$ has a Fourier transform denoted by $F(\omega)$, $g(t)$ has a Fourier transform written $G(\omega)$ and so on. The notation $F(\mathrm{i}\omega)$, $G(\mathrm{i}\omega)$ is used in some texts because ω occurs in (5) only in the term $e^{-\mathrm{i}\omega t}$.

3. Existence of the Fourier transform

We will discuss this question in a little detail at a later stage when we will also consider briefly the relation between the Fourier transform and the Laplace Transform (HELM 20). For now we will use (5) to obtain the Fourier transforms of some important functions.

Example 1

Find the Fourier transform of the one-sided exponential function

$$f(t) = \begin{cases} 0 & t < 0 \\ e^{-\alpha t} & t > 0 \end{cases}$$

where α is a positive constant, shown below:

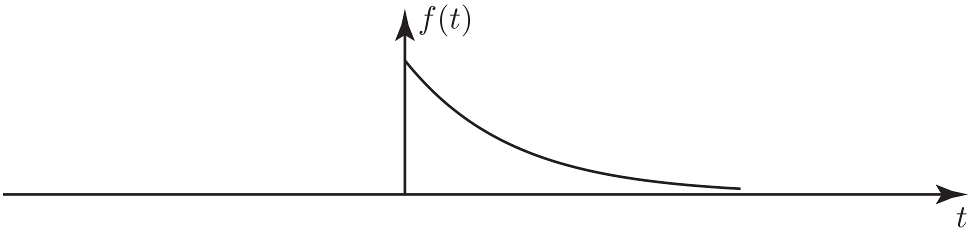


Figure 1

Solution

Using (5) then by straightforward integration

$$\begin{aligned} F(\omega) &= \int_0^\infty e^{-\alpha t} e^{-\mathrm{i}\omega t}\, dt \qquad \text{(since } f(t) = 0 \text{ for } t < 0\text{)} \\ &= \int_0^\infty e^{-(\alpha + \mathrm{i}\omega t)}\, dt \\ &= \left[\frac{e^{-(\alpha+\mathrm{i}\omega)t}}{-(\alpha + \mathrm{i}\omega)} \right]_0^\infty \\ &= \frac{1}{\alpha + \mathrm{i}\omega} \end{aligned}$$

since $e^{-\alpha t} \to 0$ as $t \to \infty$ for $\alpha > 0$.

This important Fourier transform is written in the following Key Point:

Key Point 1

$$\mathcal{F}\{e^{-\alpha t}u(t)\} = \frac{1}{\alpha + \mathrm{i}\omega}, \qquad \alpha > 0.$$

Note that this **real** function has a **complex** Fourier transform.

Note that if $u(t)$ is used to denote the **Heaviside unit step function**:

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$$

then we can write the function in Example 1 as: $f(t) = e^{-\alpha t}u(t)$. We shall frequently use this concise notation for one-sided functions.

Task

Write down the Fourier transforms of

(a) $e^{-t}u(t)$ (b) $e^{-3t}u(t)$ (c) $e^{-\frac{t}{2}}u(t)$

Use Key Point 1:

Your solution

(a)

(b)

(c)

Answer

(a) $\alpha = 1$ so $\mathcal{F}\{e^{-t}u(t)\} = \dfrac{1}{1 + \mathrm{i}\omega}$

(b) $\alpha = 3$ so $\mathcal{F}\{e^{-3t}u(t)\} = \dfrac{1}{3 + \mathrm{i}\omega}$

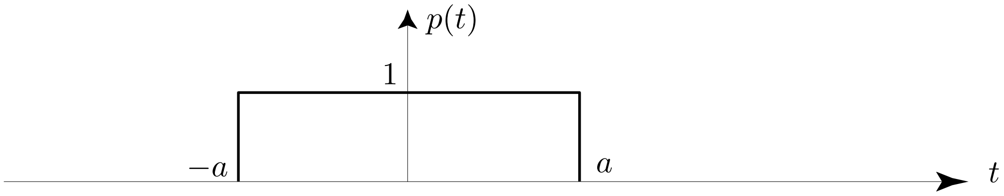
(c) $\alpha = \frac{1}{2}$ so $\mathcal{F}\{e^{-\frac{t}{2}}u(t)\} = \dfrac{1}{\frac{1}{2} + \mathrm{i}\omega}$

Task

Obtain, using the integral definition (5), the Fourier transform of the rectangular pulse

$$p(t) = \begin{cases} 1 & -a < t < a \\ 0 & \text{otherwise} \end{cases} .$$

Note that the pulse width is $2a$ as indicated in the diagram below.



First use (5) to write down the integral from which the transform will be calculated:

Your solution

Answer

$P(\omega) \equiv \mathcal{F}\{p(t)\} = \displaystyle\int_{-a}^{a} (1)e^{-\mathrm{i}\omega t}\,dt$ using the definition of $p(t)$

Now evaluate this integral and write down the final Fourier transform in trigonometric, rather than complex exponential form:

Your solution

Answer

$$\begin{aligned} P(\omega) &= \int_{-a}^{a} (1)e^{-\mathrm{i}\omega t}\,dt = \left[\frac{e^{-\mathrm{i}\omega t}}{(-\mathrm{i}\omega)}\right]_{-a}^{a} = \frac{e^{-\mathrm{i}\omega a} - e^{+\mathrm{i}\omega a}}{(-\mathrm{i}\omega)} \\ &= \frac{(\cos\omega a - \mathrm{i}\sin\omega a) - (\cos\omega a + \mathrm{i}\sin\omega a)}{(-\mathrm{i}\omega)} = \frac{2\mathrm{i}\sin\omega a}{\mathrm{i}\omega} \end{aligned}$$

i.e.

$$P(\omega) = \mathcal{F}\{p(t)\} = \frac{2\sin\omega a}{\omega} \qquad (6)$$

Note that in this case the Fourier transform is **wholly real**.

Engineers often call the function $\dfrac{\sin x}{x}$ the **sinc function**. Consequently if we write, the transform (6) of the rectangular pulse as

$$P(\omega) = 2a\frac{\sin \omega a}{\omega a},$$

we can say

$$P(\omega) = 2a\,\mathsf{sinc}(\omega a).$$

Using the result (6) in (4) we have the **Fourier integral representation** of the rectangular pulse.

$$p(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\frac{\sin \omega a}{\omega} e^{\mathrm{i}\omega t}\, d\omega.$$

As we have already mentioned, this corresponds to a Fourier series representation for a periodic function.

Key Point 2

The Fourier transform of a Rectangular Pulse

If $p_a(t) = \begin{cases} 1 & -a < t < a \\ 0 & \text{otherwise} \end{cases}$ then:

$$\mathcal{F}\{p_a(t)\} = 2a\frac{\sin \omega a}{\omega a} = 2a\,\mathsf{sinc}(\omega a)$$

Clearly, if the rectangular pulse has width 2, corresponding to $a = 1$ we have:

$$P_1(\omega) \equiv \mathcal{F}\{p_1(t)\} = 2\frac{\sin \omega}{\omega}.$$

As $\omega \to 0$, then $2\dfrac{\sin \omega}{\omega} \to 2$. Also, the function $2\dfrac{\sin \omega}{\omega}$ is an even function being the product of two odd functions $2\sin \omega$ and $\dfrac{1}{\omega}$. The graph of $P_1(\omega)$ is as follows:

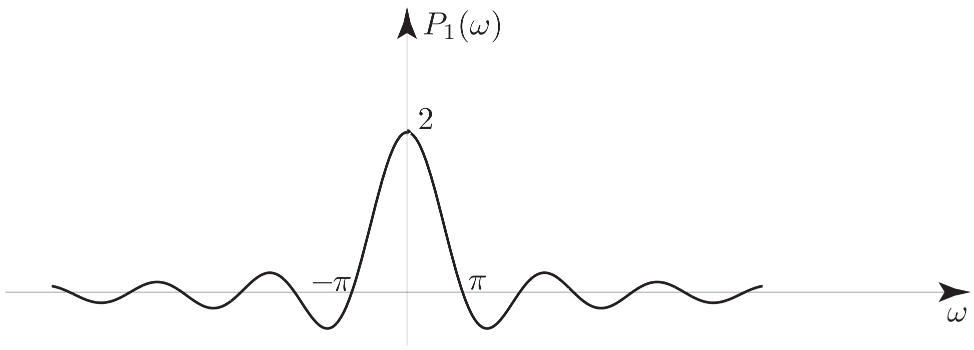


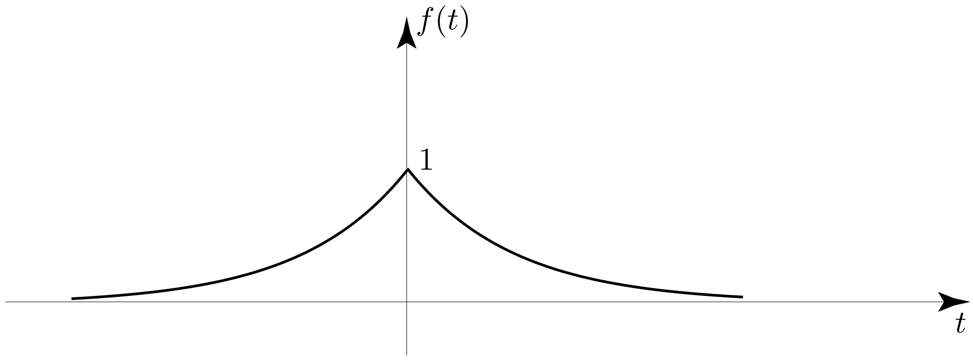
Figure 2

Task

Obtain the Fourier transform of the **two sided** exponential function

$$f(t) = \begin{cases} e^{\alpha t} & t < 0 \\ e^{-\alpha t} & t > 0 \end{cases}$$

where α is a positive constant.



Your solution

Answer

We must separate the range of the integrand into $[-\infty, 0]$ and $[0, \infty]$ since the function $f(t)$ is defined separately in these two regions: then

$$\begin{aligned} F(\omega) &= \int_{-\infty}^{0} e^{\alpha t} e^{-\mathrm{i}\omega t}\,dt + \int_{0}^{\infty} e^{-\alpha t} e^{-\mathrm{i}\omega t}\,dt = \int_{-\infty}^{0} e^{(\alpha-\mathrm{i}\omega)t}\,dt + \int_{0}^{\infty} e^{-(\alpha+\mathrm{i}\omega)t}\,dt \\ &= \left[\frac{e^{(\alpha-\mathrm{i}\omega)t}}{(\alpha-\mathrm{i}\omega)}\right]_{-\infty}^{0} + \left[\frac{e^{-(\alpha+\mathrm{i}\omega)t}}{-(\alpha+\mathrm{i}\omega)}\right]_{0}^{\infty} \\ &= \frac{1}{\alpha-\mathrm{i}\omega} + \frac{1}{\alpha+\mathrm{i}\omega} = \frac{2\alpha}{\alpha^2+\omega^2}. \end{aligned}$$

Note that, as in the case of the rectangular pulse, we have here a real even function of t giving a Fourier transform which is wholly real. Also, in both cases, the Fourier transform is an **even** (as well as real) function of ω.

Note also that it follows from the above calculation that

$$\mathcal{F}\{e^{-\alpha t}u(t)\} = \frac{1}{\alpha + \mathrm{i}\omega} \quad \text{(as we have already found)}$$

and

$$\mathcal{F}\{e^{\alpha t}u(-t)\} = \frac{1}{\alpha - \mathrm{i}\omega} \quad \text{where} \quad e^{\alpha t}u(-t) = \begin{cases} e^{\alpha t} & t < 0 \\ 0 & t > 0 \end{cases}.$$

4. Basic properties of the Fourier transform

Real and imaginary parts of a Fourier transform

Using the definition (5) we have,

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-\mathrm{i}\omega t}\,dt.$$

If we write $e^{-\mathrm{i}\omega t} = \cos\omega t - \mathrm{i}\sin\omega t$, then

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\cos\omega t\ dt - \mathrm{i}\int_{-\infty}^{\infty} f(t)\sin\omega t\ dt$$

where both integrals are real, assuming that $f(t)$ is real. Hence the real and imaginary parts of the Fourier transform are:

$$\text{Re}\ (F(\omega)) = \int_{-\infty}^{\infty} f(t)\cos\omega t\,dt \qquad\qquad \text{Im}\ (F(\omega)) = -\int_{-\infty}^{\infty} f(t)\sin\omega t\,dt.$$

Task

Recalling that if $h(t)$ is even and $g(t)$ is odd then $\displaystyle\int_{-a}^{a} h(t)\,dt = 2\int_0^a h(t)\,dt$ and $\displaystyle\int_{-a}^{a} g(t)\,dt = 0$, deduce Re$(F(\omega))$ and Im$(F(\omega))$ if

(a) $f(t)$ is a real even function

(b) $f(t)$ is a real odd function.

Your solution

(a)

Answer

If $f(t)$ is real and even

$$R(\omega) \equiv \text{Re } F(\omega) = 2\int_0^\infty f(t)\cos\omega t\,dt \quad \text{(because the integrand is even)}$$

$$I(\omega) \equiv \text{Im } F(\omega) = -\int_{-\infty}^\infty f(t)\sin\omega t\,dt = 0 \quad \text{(because the integrand is odd).}$$

Thus, any real even function $f(t)$ has a wholly real Fourier transform. Also since

$$\cos((-\omega)t) = \cos(-\omega t) = \cos\omega t$$

the Fourier transform in this case will be a real even function.

Your solution

(b)

Answer

Now

$$\text{Re } F(\omega) = \int_{-\infty}^\infty f(t)\cos\omega t\,dt = \int_{-\infty}^\infty \text{(odd)} \times \text{(even)}\,dt = \int_{-\infty}^\infty \text{(odd)}\,dt = 0$$

and

$$\text{Im } F(\omega) = -\int_{-\infty}^\infty f(t)\sin\omega t\,dt = -2\int_0^\infty f(t)\sin\omega t\,dt$$

(because the integrand is (odd)$\times$(odd)=(even)).

Also since $\sin((-\omega)t) = -\sin\omega t$, the Fourier transform in this case is an odd function of ω.

These results are summarised in the following Key Point:

Key Point 3

$f(t)$	$F(\omega) = \mathcal{F}\{f(t)\}$
real and even	real and even
real and odd	purely imaginary and odd
neither even nor odd	complex, $F(\omega) = R(\omega) + \mathrm{i}I(\omega)$

Polar form of a Fourier transform

Task

The one-sided exponential function $f(t) = e^{-\alpha t}u(t)$ has Fourier transform $F(\omega) = \dfrac{1}{\alpha + \mathrm{i}\omega}$. Find the real and imaginary parts of $F(\omega)$.

Your solution

Answer

$F(\omega) = \dfrac{1}{\alpha + \mathrm{i}\omega} = \dfrac{\alpha - \mathrm{i}\omega}{\alpha^2 + \omega^2}.$

Hence $R(\omega) = \text{Re } F(\omega) = \dfrac{\alpha}{\alpha^2 + \omega^2}$ $\qquad I(\omega) = \text{Im } F(\omega) = \dfrac{-\omega}{\alpha^2 + \omega^2}$

We can rewrite $F(\omega)$, like any other complex quantity, in **polar** form by calculating the magnitude and the argument (or phase). For the Fourier transform in the last Task

$$|F(\omega)| = \sqrt{R^2(\omega) + I^2(\omega)} = \sqrt{\frac{\alpha^2 + \omega^2}{(\alpha^2 + \omega^2)^2}} = \frac{1}{\sqrt{\alpha^2 + \omega^2}}$$

and $\qquad \arg F(\omega) = \tan^{-1}\dfrac{I(\omega)}{R(\omega)} = \tan^{-1}\left(\dfrac{-\omega}{\alpha}\right).$

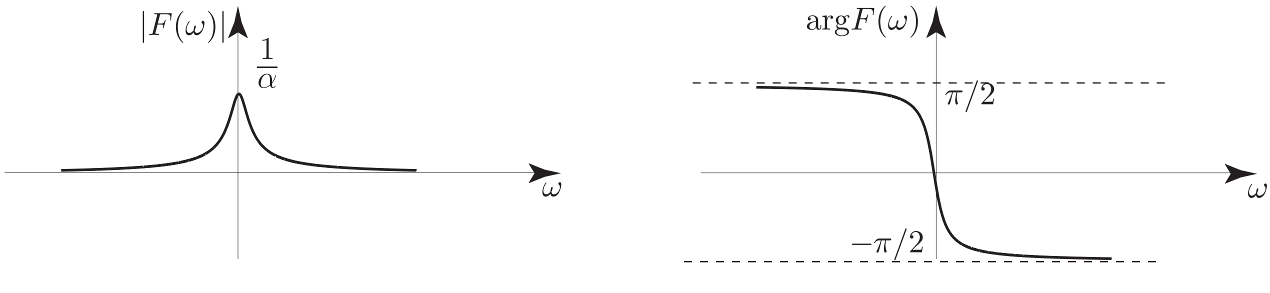


Figure 3

In general, a Fourier transform whose Cartesian form is $F(\omega) = R(\omega) + \mathrm{i}I(\omega)$ has a polar form $F(\omega) = |F(\omega)|e^{\mathrm{i}\phi(\omega)}$ where $\phi(\omega) \equiv \arg F(\omega)$.

Graphs, such as those shown in Figure 3, of $|F(\omega)|$ and arg $F(\omega)$ plotted against ω, are often referred to as **magnitude** spectra and **phase** spectra, respectively.

Exercises

1. Obtain the Fourier transform of the rectangular pulses

(a) $f(t) = \begin{cases} 1 & |t| \leq 1 \\ 0 & |t| > 1 \end{cases}$ (b) $f(t) = \begin{cases} \frac{1}{4} & |t| \leq 3 \\ 0 & |t| > 3 \end{cases}$

2. Find the Fourier transform of

$$f(t) = \begin{cases} 1 - \frac{t}{2} & 0 \leq t \leq 2 \\ 1 + \frac{t}{2} & -2 \leq t \leq 0 \\ 0 & |t| > 2 \end{cases}$$

Answers

1.(a) $F(\omega) = \frac{2}{\omega}\sin\omega$

(b) $F(\omega) = \frac{\sin 3\omega}{2\omega}$

2. $\frac{1 - \cos 2\omega}{\omega^2}$

Properties of the Fourier Transform

24.2

Introduction

In this Section we shall learn about some useful properties of the Fourier transform which enable us to calculate easily further transforms of functions and also in applications such as electronic communication theory.

Prerequisites

Before starting this Section you should . . .

- be aware of the basic definitions of the Fourier transform and inverse Fourier transform

Learning Outcomes

On completion you should be able to . . .

- state and use the linearity property and the time and frequency shift properties of Fourier transforms
- state various other properties of the Fourier transform

1. Linearity properties of the Fourier transform

(i) If $f(t)$, $g(t)$ are functions with transforms $F(\omega),\ G(\omega)$ respectively, then

- $\mathcal{F}\{f(t)+g(t)\} = F(\omega)+G(\omega)$

i.e. if we add 2 functions then the Fourier transform of the resulting function is simply the sum of the individual Fourier transforms.

(ii) If k is any constant,

- $\mathcal{F}\{kf(t)\} = kF(\omega)$

i.e. if we multiply a function by any constant then we must multiply the Fourier transform by the same constant. These properties follow from the definition of the Fourier transform and from the properties of integrals.

Examples

1.

$$\begin{aligned}
\mathcal{F}\{2e^{-t}u(t)+3e^{-2t}u(t)\} &= \mathcal{F}\{2e^{-t}u(t)\}+\mathcal{F}\{3e^{-2t}u(t)\} \\
&= 2\mathcal{F}\{e^{-t}u(t)\}+3\mathcal{F}\{e^{-2t}u(t)\} \\
&= \frac{2}{1+\mathrm{i}\omega}+\frac{3}{2+\mathrm{i}\omega}
\end{aligned}$$

2.

$$\begin{aligned}
\text{If} \qquad f(t) &= \begin{cases} 4 & -3\le t\le 3 \\ 0 & \text{otherwise}\end{cases} \\
\text{then} \qquad f(t) &= 4p_3(t) \\
\text{so} \qquad F(\omega) &= 4P_3(\omega) = \frac{8}{\omega}\sin 3\omega
\end{aligned}$$

using the standard result for $\mathcal{F}\{p_a(t)\}$.

Task

If $f(t) = \begin{cases} 6 & -2\le t\le 2 \\ 0 & \text{otherwise}\end{cases}$ write down $F(\omega)$.

Your solution

Answer

We have $f(t) = 6p_2(t)$ so $F(\omega) = \dfrac{12}{\omega}\sin 2\omega$.

2. Shift properties of the Fourier transform

There are two basic shift properties of the Fourier transform:

(i) Time shift property: • $\mathcal{F}\{f(t-t_0)\} = e^{-\mathrm{i}\omega t_0}F(\omega)$

(ii) Frequency shift property • $\mathcal{F}\{e^{\mathrm{i}\omega_0 t}f(t)\} = F(\omega-\omega_0)$.

Here t_0, ω_0 are constants.

In words, shifting (or translating) a function in one domain corresponds to a multiplication by a complex exponential function in the other domain.

We omit the proofs of these properties which follow from the definition of the Fourier transform.

Example 2

Use the time-shifting property to find the Fourier transform of the function

$$g(t) = \begin{cases} 1 & 3 \le t \le 5 \\ 0 & \text{otherwise} \end{cases}$$

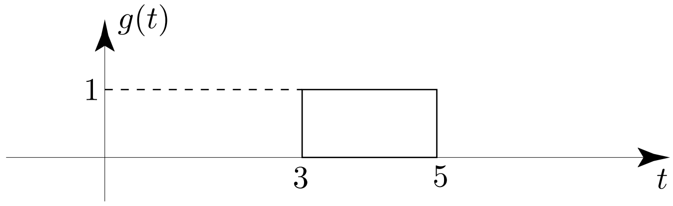


Figure 4

Solution

$g(t)$ is a pulse of width 2 and can be obtained by shifting the symmetrical rectangular pulse

$$p_1(t) = \begin{cases} 1 & -1 \le t \le 1 \\ 0 & \text{otherwise} \end{cases}$$

by 4 units to the right.

Hence by putting $t_0 = 4$ in the time shift theorem

$$G(\omega) = \mathcal{F}\{g(t)\} = e^{-4\mathrm{i}\omega}\frac{2}{\omega}\sin\omega.$$

Task

Verify the result of Example 2 by direct integration.

Your solution

Answer

$$G(\omega) = \int_3^5 1e^{-\mathrm{i}\omega t}dt \quad = \quad \left[\frac{e^{-\mathrm{i}\omega t}}{-\mathrm{i}\omega}\right]_3^5 = \frac{e^{-5\mathrm{i}\omega} - e^{-3\mathrm{i}\omega}}{-\mathrm{i}\omega} = e^{-4\mathrm{i}\omega}\left(\frac{e^{\mathrm{i}\omega} - e^{-\mathrm{i}\omega}}{\mathrm{i}\omega}\right) = e^{-4\mathrm{i}\omega}2\frac{\sin\omega}{\omega},$$

as obtained using the time-shift property.

Task

Use the frequency shift property to obtain the Fourier transform of the modulated wave

$$g(t) = f(t)\cos\omega_0 t$$

where $f(t)$ is an arbitrary signal whose Fourier transform is $F(\omega)$.

First rewrite $g(t)$ in terms of complex exponentials:

Your solution

Answer

$$g(t) = f(t)\left(\frac{e^{\mathrm{i}\omega_0 t} + e^{-\mathrm{i}\omega_0 t}}{2}\right) = \frac{1}{2}f(t)e^{\mathrm{i}\omega_0 t} + \frac{1}{2}f(t)e^{-\mathrm{i}\omega_0 t}$$

Now use the linearity property and the frequency shift property on each term to obtain $G(\omega)$:

Your solution

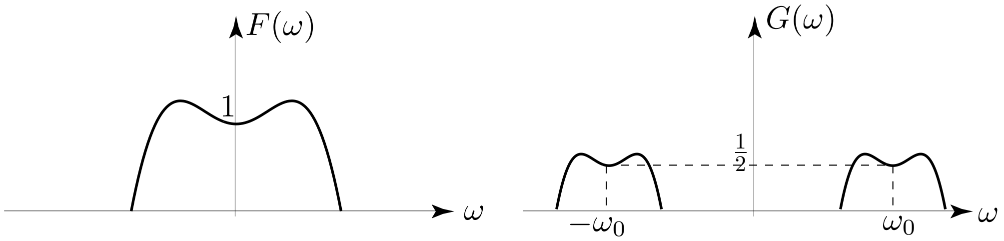
Answer

We have, by linearity:

$$\mathcal{F}\{g(t)\} = \frac{1}{2}\mathcal{F}\{f(t)e^{\mathrm{i}\omega_0 t}\} + \frac{1}{2}\mathcal{F}\{f(t)e^{-\mathrm{i}\omega_0 t}\}$$

and by the frequency shift property:

$$G(\omega) = \frac{1}{2}F(\omega - \omega_0) + \frac{1}{2}F(\omega + \omega_0).$$



3. Inversion of the Fourier transform

Formal inversion of the Fourier transform, i.e. finding $f(t)$ for a given $F(\omega)$, is sometimes possible using the inversion integral (4). However, in elementary cases, we can use a Table of standard Fourier transforms together, if necessary, with the appropriate properties of the Fourier transform.

The following Examples and Tasks involve such inversion.

Example 3

Find the inverse Fourier transform of $F(\omega) = 20\dfrac{\sin 5\omega}{5\omega}$.

Solution

The appearance of the sine function implies that $f(t)$ is a symmetric rectangular pulse.

We know the standard form $\mathcal{F}\{p_a(t)\} = 2a\dfrac{\sin \omega a}{\omega a}$ or $\mathcal{F}^{-1}\{2a\dfrac{\sin \omega a}{\omega a}\} = p_a(t)$.

Putting $a = 5$ $\mathcal{F}^{-1}\{10\dfrac{\sin 5\omega}{5\omega}\} = p_5(t)$. Thus, by the linearity property

$$f(t) = \mathcal{F}^{-1}\{20\frac{\sin 5\omega}{5\omega}\} = 2p_5(t)$$

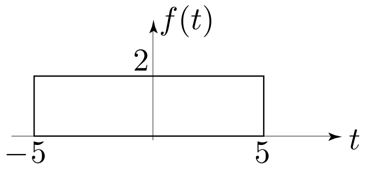


Figure 4

Example 4

Find the inverse Fourier transform of $G(\omega) = 20\dfrac{\sin 5\omega}{5\omega}\exp(-3\mathrm{i}\omega)$.

Solution

The occurrence of the complex exponential factor in the Fourier transform suggests the time-shift property with the time shift $t_0 = +3$ (i.e. a right shift).

From Example 3

$$\mathcal{F}^{-1}\{20\frac{\sin 5\omega}{5\omega}\} = 2p_5(t) \qquad \text{so} \qquad g(t) = \mathcal{F}^{-1}\{20\frac{\sin 5\omega}{5\omega}e^{-3\mathrm{i}\omega}\} = 2p_5(t-3)$$

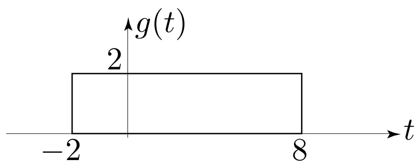


Figure 5

Task

Find the inverse Fourier transform of

$$H(\omega) = 6\frac{\sin 2\omega}{\omega}e^{-4\mathrm{i}\omega}.$$

Firstly ignore the exponential factor and find the inverse Fourier transform of the remaining terms:

Your solution

Answer

We use the result: $\mathcal{F}^{-1}\{2a\dfrac{\sin \omega a}{\omega a}\} = p_a(t)$

Putting $a = 2$ gives $\mathcal{F}^{-1}\{2\dfrac{\sin 2\omega}{\omega}\} = p_2(t)$ $\quad\therefore\quad$ $\mathcal{F}^{-1}\{6\dfrac{\sin 2\omega}{\omega}\} = 3p_2(t)$

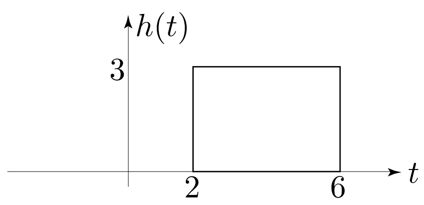
Now take account of the exponential factor:

Your solution

Answer

Using the time-shift theorem for $t_0 = 4$

$$h(t) = \mathcal{F}^{-1}\{6\frac{\sin 2\omega}{\omega}e^{-4\mathrm{i}\omega}\} = 3p_2(t-4)$$



Example 5

Find the inverse Fourier transform of

$$K(\omega) = \frac{2}{1 + 2(\omega - 1)\mathrm{i}}$$

Solution

The presence of the term $(\omega - 1)$ instead of ω suggests the frequency shift property.

Hence, we consider first

$$\hat{K}(\omega) = \frac{2}{1 + 2\mathrm{i}\omega}.$$

The relevant standard form is

$$\mathcal{F}\{e^{-\alpha t}u(t)\} = \frac{1}{\alpha + \mathrm{i}\omega} \qquad \text{or} \qquad \mathcal{F}^{-1}\{\frac{1}{\alpha + \mathrm{i}\omega}\} = e^{-\alpha t}u(t).$$

Hence, writing $\hat{K}(\omega) = \dfrac{1}{\frac{1}{2} + \mathrm{i}\omega}$ $\qquad \hat{k}(t) = e^{-\frac{1}{2}t}u(t).$

Then, by the frequency shift property with $\omega_0 = 1$ $\qquad k(t) = \mathcal{F}^{-1}\{\dfrac{2}{1 + 2(\omega - 1)\mathrm{i}}\} = e^{-\frac{1}{2}t}e^{\mathrm{i}t}u(t).$

Here $k(t)$ is a complex time-domain signal.

Task

Find the inverse Fourier transforms of

(a) $L(\omega) = 2\dfrac{\sin\{3(\omega - 2\pi)\}}{(\omega - 2\pi)}$ $\qquad$ (b) $M(\omega) = \dfrac{e^{\mathrm{i}\omega}}{1 + \mathrm{i}\omega}$

Your solution

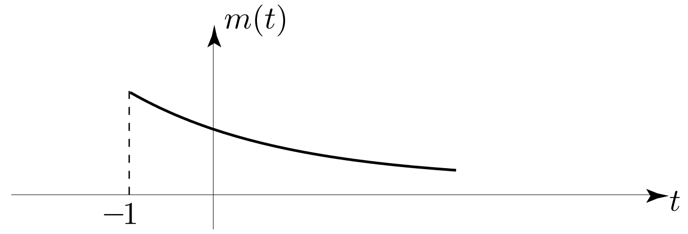
Answer

(a) Using the frequency shift property with $\omega_0 = 2\pi$

$$l(t) = \mathcal{F}^{-1}\{L(\omega)\} = p_3(t)e^{\mathrm{i}2\pi t}$$

(b) Using the time shift property with $t_0 = -1$

$$m(t) = e^{-(t+1)}u(t+1)$$



4. Further properties of the Fourier transform

We state these properties without proof. As usual $F(\omega)$ denotes the Fourier transform of $f(t)$.

(a) Time differentiation property:

$$\mathcal{F}\{f'(t)\} = \mathrm{i}\omega F(\omega)$$

(Differentiating a function is said to amplify the higher frequency components because of the additional multiplying factor ω.)

(b) Frequency differentiation property:

$$\mathcal{F}\{tf(t)\} = \mathrm{i}\frac{dF}{d\omega} \qquad \text{or} \qquad \mathcal{F}\{(-\mathrm{i}t)f(t)\} = \frac{dF}{d\omega}$$

Note the symmetry between properties (a) and (b).

(c) Duality property:

If $\mathcal{F}\{f(t)\} = F(\omega)$ then $\mathcal{F}\{F(t)\} = 2\pi f(-\omega)$.

Informally, the duality property states that we can, apart from the 2π factor, interchange the time and frequency domains provided we put $-\omega$ rather than ω in the second term, this corresponding to a reflection in the vertical axis. If $f(t)$ is even this latter is irrelevant.

For example, we know that if $f(t) = p_1(t) = \begin{cases} 1 & -1 < t < 1 \\ 0 & \text{otherwise} \end{cases}$, then $F(\omega) = 2\dfrac{\sin\omega}{\omega}$.

Then, by the duality property, since $p_1(\omega)$ is even, $\mathcal{F}\{2\dfrac{\sin t}{t}\} = 2\pi p_1(-\omega) = 2\pi p_1(\omega)$.

Graphically:

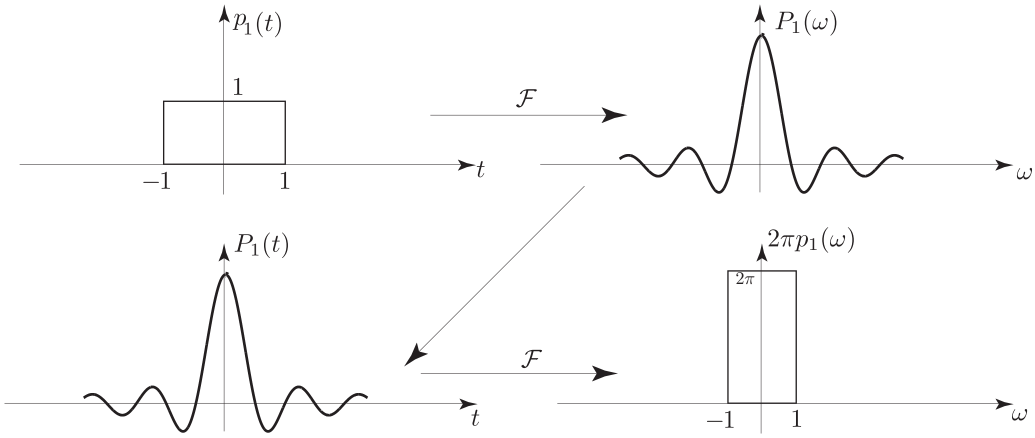


Figure 6

Recalling the Fourier transform pair

$$f(t) = \begin{cases} e^{-2t} & t > 0 \\ e^{2t} & t < 0 \end{cases} \qquad F(\omega) = \frac{4}{4+\omega^2},$$

obtain the Fourier transforms of

(a) $g(t) = \dfrac{1}{4+t^2}$ (b) $h(t) = \dfrac{1}{4+t^2}\cos 2t.$

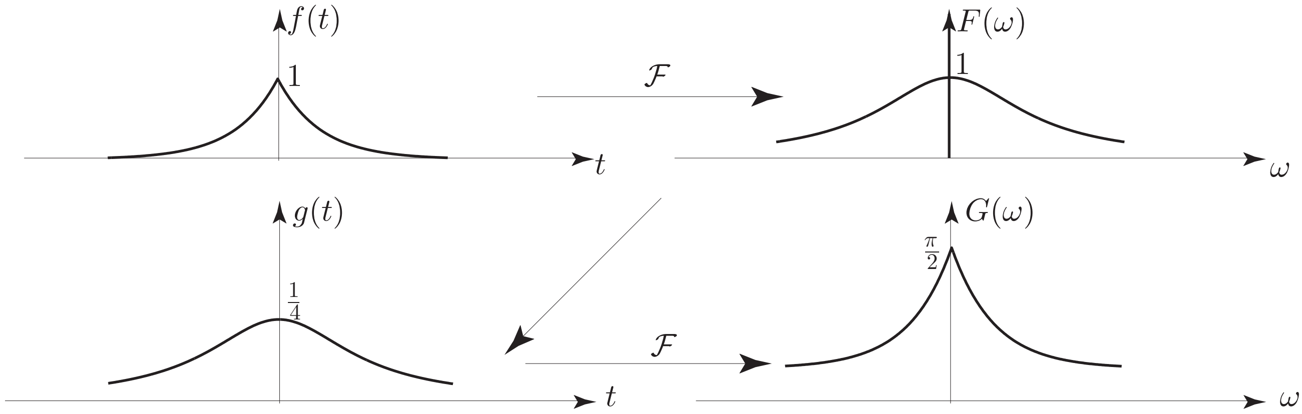
(a) Use the linearity and duality properties:

Your solution

Answer

We have $\mathcal{F}\{f(t)\} \equiv \mathcal{F}\{e^{-2|t|}\} = \dfrac{4}{4+\omega^2}$. $\quad\therefore\quad \mathcal{F}\{\dfrac{1}{4}e^{-2|t|}\} = \dfrac{1}{4+\omega^2}$ (by linearity)

$\therefore \quad \mathcal{F}\{\dfrac{1}{4+t^2}\} = 2\pi\dfrac{1}{4}e^{-2|-\omega|} = \dfrac{\pi}{2}e^{-2|\omega|} = G(\omega)$ (by duality).



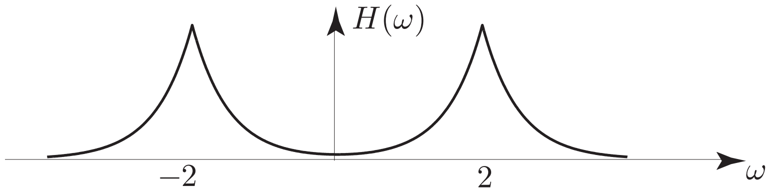
(b) Use the modulation property based on the frequency shift property:

Your solution

Answer

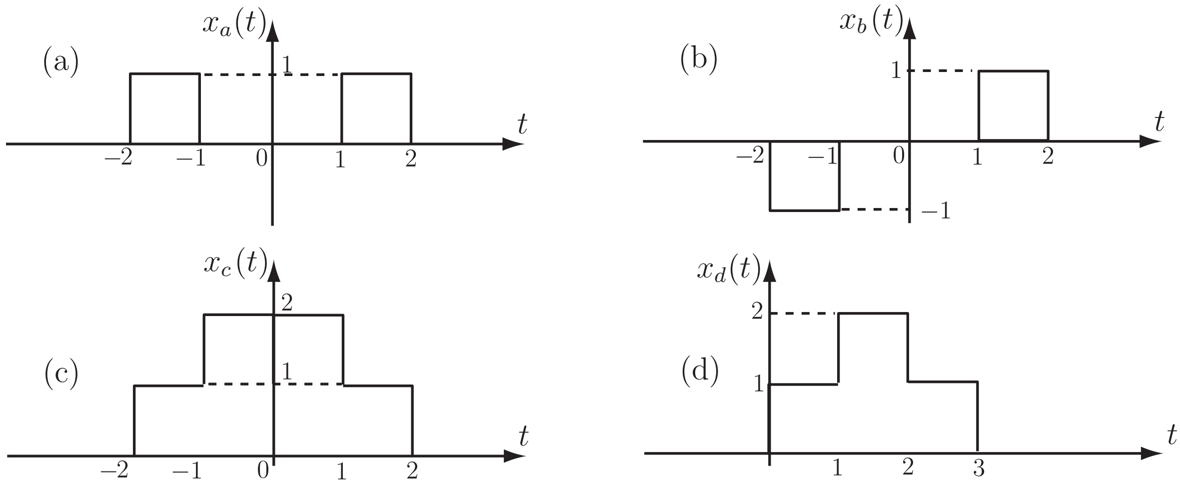
We have $h(t) = g(t)\cos 2t$. $\quad\therefore\quad \mathcal{F}\{g(t)\cos\omega_0 t\} = \frac{1}{2}(G(\omega-\omega_0) + G(\omega+\omega_0))$,

so with $\omega_0 = 2$ $\mathcal{F}\{h(t)\} = \dfrac{\pi}{4}\left\{e^{-2|\omega-2|} + e^{-2|\omega+2|}\right\} = H(\omega)$



Exercises

1. Using the superposition and time delay theorems and the known result for the transform of the rectangular pulse $p(t)$, obtain the Fourier transforms of each of the signals shown.



2. Obtain the Fourier transform of the signal

$$f(t) = \mathrm{e}^{-t}u(t) + \mathrm{e}^{-2t}u(t)$$

where $u(t)$ denotes the unit step function.

3. Use the time-shift property to obtain the Fourier transform of

$$f(t) = \begin{cases} 1 & 1 \leq t \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

Verify your result using the definition of the Fourier transform.

4. Find the inverse Fourier transforms of

   (a) $F(\omega) = 20\dfrac{\sin(5\omega)}{5\omega}\mathrm{e}^{-3\mathrm{i}\omega}$

   (b) $F(\omega) = \dfrac{8}{\omega}\sin 3\omega\ \mathrm{e}^{\mathrm{i}\omega}$

   (c) $F(\omega) = \dfrac{\mathrm{e}^{\mathrm{i}\omega}}{1 - \mathrm{i}\omega}$

5. If $f(t)$ is a signal with transform $F(\omega)$ obtain the Fourier transform of $f(t)\cos(\omega_0 t)\cos(\omega_0 t)$.

Answer

1.
$$X_a(\omega) = \frac{4}{\omega}\sin(\frac{\omega}{2})\cos(\frac{3\omega}{2})$$

$$X_b(\omega) = \frac{-4\mathrm{i}}{\omega}\sin(\frac{\omega}{2})\sin(\frac{3\omega}{2})$$

$$X_c(\omega) = \frac{2}{\omega}[\sin(2\omega) + \sin(\omega)]$$

$$X_d(\omega) = \frac{2}{\omega}\left(\sin(\frac{3\omega}{2}) + \sin(\frac{\omega}{2})\right)\mathrm{e}^{-3\mathrm{i}\omega/2}$$

2. $F(\omega) = \dfrac{3 + 2\mathrm{i}\omega}{2 - \omega^2 + 3\mathrm{i}\omega}$ (using the superposition property)

3. $F(\omega) = 2\dfrac{\sin\omega}{\omega}\mathrm{e}^{-2\mathrm{i}\omega}$

4. (a) $f(t) = \begin{cases} 2 & -2 < t < 8 \\ 0 & \text{otherwise} \end{cases}$

(b) $f(t) = \begin{cases} 4 & -4 < t < 2 \\ 0 & \text{otherwise} \end{cases}$

(c) $f(t) = \begin{cases} e^{t+1} & t < -1 \\ 0 & \text{otherwise} \end{cases}$

5. $\dfrac{1}{2}F(\omega) + \dfrac{1}{4}\left[F(\omega + 2\omega_0) + F(\omega - 2\omega_0)\right]$

Some Special Fourier Transform Pairs

24.3

Introduction

In this final Section on Fourier transforms we shall study briefly a number of topics such as Parseval's theorem and the relationship between Fourier transform and Laplace transforms. In particular we shall obtain, intuitively rather than rigorously, various Fourier transforms of functions such as the unit step function which actually violate the basic conditions which guarantee the existence of Fourier transforms!

Prerequisites

Before starting this Section you should ...

- be aware of the definitions and simple properties of the Fourier transform and inverse Fourier transform.

Learning Outcomes

On completion you should be able to ...

- use the unit impulse function (the Dirac delta function) to obtain various Fourier transforms

1. Parseval's theorem

Recall from HELM 23.2 on Fourier series that for a periodic signal $f_T(t)$ with complex Fourier coefficients $c_n (n = 0, \pm1, \pm2, \ldots)$ Parseval's theorem holds:

$$\frac{1}{T}\int_{-\frac{T}{2}}^{+\frac{T}{2}} f_T^2(t)dt = \sum_{n=-\infty}^{\infty} |c_n|^2,$$

where the left-hand side is the mean square value of the function (signal) over one period.

For a non-periodic real signal $f(t)$ with Fourier transform $F(\omega)$ the corresponding result is

$$\int_{-\infty}^{\infty} f^2(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2 d\omega.$$

This result is particularly significant in filter theory. For reasons that we do not have space to go into, the left-hand side integral is often referred to as the **total energy** of the signal. The integrand on the right-hand side

$$\frac{1}{2\pi}|F(\omega)|^2$$

is then referred to as the **energy density** (because it is the frequency domain quantity that has to be integrated to obtain the total energy).

Task

Verify Parseval's theorem using the one-sided exponential function

$$f(t) = e^{-t}u(t).$$

Firstly evaluate the integral on the left-hand side:

Your solution

Answer

$$\int_{-\infty}^{\infty} f^2(t)dt = \int_0^{\infty} e^{-2t}dt = \left[\frac{e^{-2t}}{-2}\right]_0^{\infty} = \frac{1}{2}.$$

Now obtain the Fourier transform $F(\omega)$ and evaluate the right-hand side integral:

Your solution

Answer

$$F(\omega) = \mathcal{F}\{e^{-t}u(t)\} = \frac{1}{1+\mathrm{i}\omega},$$

so

$$|F(\omega)|^2 = \frac{1}{(1+\mathrm{i}\omega)}.\frac{1}{(1-\mathrm{i}\omega)} = \frac{1}{1+\omega^2}.$$

Then

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{\infty}|F(\omega)|^2 d\omega &= \frac{1}{\pi}\int_0^{\infty}|F(\omega)|^2 d\omega \\ &= \frac{1}{\pi}\int_0^{\infty}\frac{1}{1+\omega^2}d\omega = \frac{1}{\pi}\left[\tan^{-1}\omega\right]_0^{\infty} = \frac{1}{\pi}\times\frac{\pi}{2} = \frac{1}{2}.\end{aligned}$$

Since both integrals give the same value, Parseval's theorem is verified for this case.

2. Existence of Fourier transforms

Formally, sufficient conditions for the Fourier transform of a function $f(t)$ to exist are

(a) $\int_{-\infty}^{\infty}|f(t)|^2 dt$ is finite

(b) $f(t)$ has a finite number of maxima and minima in any finite interval

(c) $f(t)$ has a finite number of discontinuities.

Like the equivalent conditions for the existence of Fourier series these conditions are known as **Dirichlet conditions**.

If the above conditions hold then $f(t)$ has a unique Fourier transform. However certain functions, such as the unit step function, which violate one or more of the Dirichlet conditions still have Fourier transforms in a more generalized sense as we shall see shortly.

3. Fourier transform and Laplace transforms

Suppose $f(t) = 0$ for $t < 0$. Then the Fourier transform of $f(t)$ becomes

$$\mathcal{F}\{f(t)\} = \int_0^{\infty} f(t)e^{-\mathrm{i}\omega t}dt. \qquad (1)$$

As you may recall from earlier units, the Laplace transform of $f(t)$ is

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} f(t)e^{-st}dt. \qquad (2)$$

Comparison of (1) and (2) suggests that for such one-sided functions, the Fourier transform of $f(t)$ can be obtained by simply replacing s by $\mathrm{i}\omega$ in the Laplace transform.
An obvious example where this can be done is the function

$$f(t) = e^{-\alpha t}u(t).$$

In this case $\mathcal{L}\{f(t)\} = \dfrac{1}{\alpha + s} = F(s)$ and, as we have seen earlier,

$$\mathcal{F}\{f(t)\} = \frac{1}{\alpha + \mathrm{i}\omega} = F(\mathrm{i}\omega).$$

However, care must be taken with such substitutions. We must be sure that the conditions for the existence of the Fourier transform are met. Thus, for the unit step function,

$$\mathcal{L}\{u(t)\} = \frac{1}{s},$$

whereas, $\mathcal{F}\{u(t)\} \neq \dfrac{1}{\mathrm{i}\omega}$. (We shall see that $\mathcal{F}\{u(t)\}$ does actually exist but is not equal to $\dfrac{1}{\mathrm{i}\omega}$.)

We should also point out that some of the properties we have discussed for Fourier transforms are similar to those of the Laplace transforms e.g. the time-shift properties:

Fourier: $\mathcal{F}\{f(t - t_0)\} = e^{-\mathrm{i}\omega t_0}F(\omega)$ Laplace: $\mathcal{L}\{f(t - t_0)\} = e^{-st_0}F(s)$.

4. Some special Fourier transform pairs

As mentioned in the previous subsection it is possible to obtain Fourier transforms for some important functions that violate the Dirichlet conditions. To discuss this situation we must introduce the **unit impulse function**, also known as the **Dirac delta function**. We shall study this topic in an inituitive, rather than rigorous, fashion.

Recall that a symmetrical rectangular pulse

$$p_a(t) = \begin{cases} 1 & -a < t < a \\ 0 & \text{otherwise} \end{cases}$$

has a Fourier transform

$$P_a(\omega) = \frac{2}{\omega}\sin\omega a.$$

If we consider a pulse whose height is $\dfrac{1}{2a}$ rather than 1 (so that the pulse encloses unit area), then we have, by the linearity property of Fourier transforms,

$$\mathcal{F}\left\{\frac{1}{2a}p_a(t)\right\} = \frac{\sin\omega a}{\omega a}.$$

As the value of a becomes smaller, the rectangular pulse becomes narrower and taller but still has unit area.

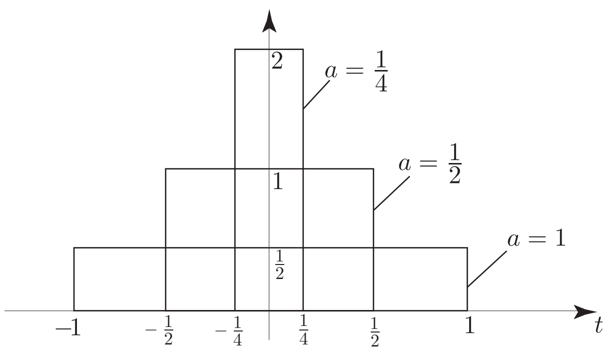


Figure 7

We define the unit impulse function $\delta(t)$ as

$$\delta(t) = \lim_{a \to 0} \frac{1}{2a} p_a(t)$$

and show it graphically as follows:

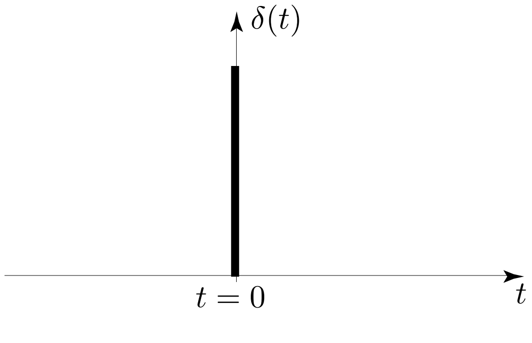


Figure 8

Then,

$$\begin{aligned} \mathcal{F}\{\delta(t)\} &= \mathcal{F}\left\{\lim_{a \to 0} \frac{1}{2a} p_a(t)\right\} = \lim_{a \to 0} \mathcal{F}\left\{\frac{1}{2a} p_a(t)\right\} \\ &= \lim_{a \to 0} \frac{\sin \omega a}{\omega a} \\ &= 1. \end{aligned}$$

Here we have assumed that interchanging the order of taking the Fourier transform with the limit operation is valid.

Now consider a shifted unit impulse $\delta(t - t_0)$:

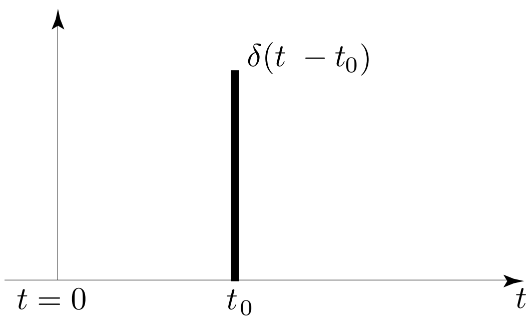


Figure 9

We have, by the time shift property

$$\mathcal{F}\{\delta(t - t_0)\} = e^{-\mathrm{i}\omega t_0}(1) = e^{-\mathrm{i}\omega t_0}.$$

These results are summarized in the following Key Point:

Key Point 4

The Fourier transform of a Unit Impulse

$$\mathcal{F}\{\delta(t-t_0)\} = e^{-\mathrm{i}\omega t_0}.$$

If $\quad t_0 = 0 \qquad$ then $\qquad \mathcal{F}\{\delta(t)\} = 1.$

Task

Apply the duality property to the result

$$\mathcal{F}\{\delta(t)\} = 1.$$

(From the way we have introduced the unit impluse function it must clearly be treated as an **even** function.)

Your solution

Answer

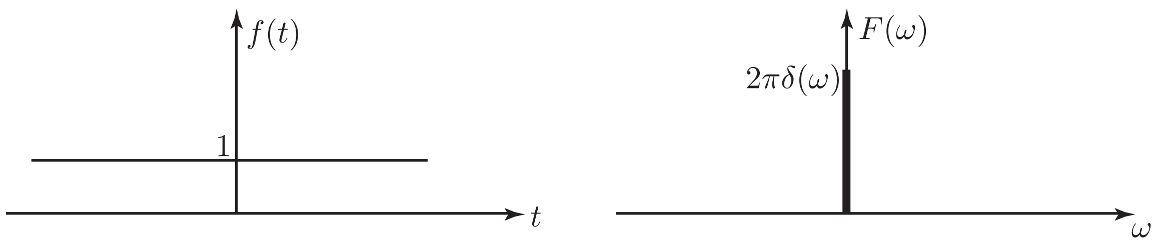
We have $\mathcal{F}\{\delta(t)\} = 1$. Therefore by the duality property

$$\mathcal{F}\{1\} = 2\pi\delta(-\omega) = 2\pi\delta(\omega).$$

We see that the signal

$$f(t) = 1, \qquad -\infty < t < \infty$$

which is infinitely wide, has Fourier transform $F(\omega) = 2\pi\delta(\omega)$ which is infinitesimally narrow. This reciprocal effect is characteristic of Fourier transforms.



This result is intuitively plausible since a constant signal would be expected to have a frequency representation which had only a component at zero frequency ($\omega = 0$).

Task

Use the result $\mathcal{F}\{1\} = 2\pi\delta(\omega)$ and the frequency shift property to obtain

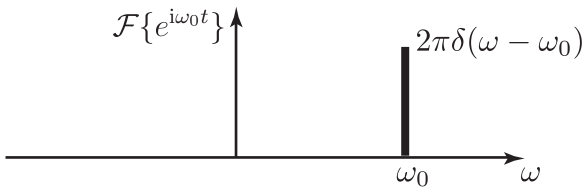
$\mathcal{F}\{e^{i\omega_0 t}\}$.

Your solution

Answer

$\mathcal{F}\{e^{i\omega_0 t}\} = \mathcal{F}\{e^{i\omega_0 t}f(t)\}$ where $f(t) = 1, \qquad -\infty < t < \infty.$

But $\mathcal{F}\{f(t)\} = 2\pi\delta(\omega)$, therefore, by the frequency shift property $\mathcal{F}\{e^{i\omega_0 t}\} = 2\pi\delta(\omega - \omega_0)$.



Task

Obtain the Fourier transform of a pure cosine wave

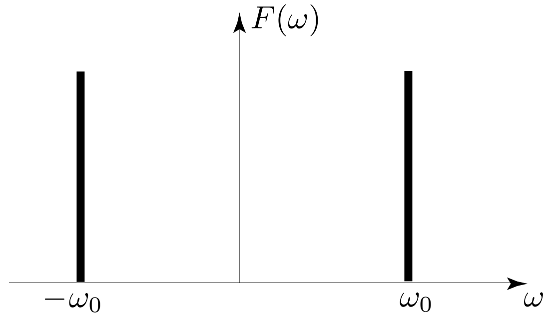
$f(t) = \cos\omega_0 t \qquad -\infty < t < \infty$

by writing $f(t)$ in terms of complex exponentials and using the result of the previous Task.

Your solution

Answer
We have $f(t) = \cos\omega_0 t = \frac{1}{2}\left\{e^{i\omega_0 t} + e^{-i\omega_0 t}\right\}$
so

$$\mathcal{F}\{\cos\omega_0 t\} = \frac{1}{2}\mathcal{F}\{e^{i\omega_0 t}\} + \frac{1}{2}\mathcal{F}\{e^{-i\omega_0 t}\} = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$



Note that because $\displaystyle\int_{-\infty}^{\infty} |\cos\omega_0 t|\, dt$ diverges, one of the Dirichlet conditions is violated. Nevertheless, as we can see via the use of the unit impulse functions, the Fourier transform of $\cos\omega_0 t$ exists.
By similar reasoning we can readily show

$$\mathcal{F}\{\sin\omega_0 t\} = \frac{\pi}{i}\delta(\omega - \omega_0) - \frac{\pi}{i}\delta(\omega + \omega_0).$$

Note that the usual results for Fourier transforms of even and odd functions still hold.

5. Fourier transform of the unit step function

We have already pointed out that although

$$\mathcal{L}\{u(t)\} = \frac{1}{s}$$

we cannot simply replace s by $i\omega$ to obtain the Fourier transform of the unit step.
We proceed via the Fourier transform of the **signum function** sgn(t) which is defined as

$$\text{sgn } t = \begin{cases} 1 & t > 0 \\ -1 & t < 0 \end{cases}$$

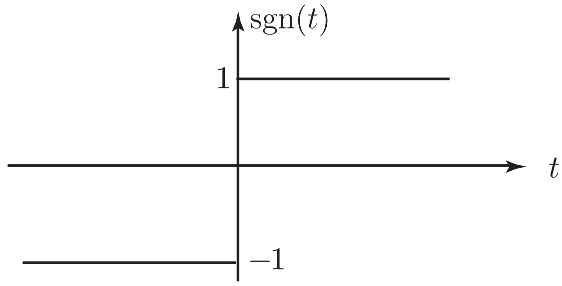


Figure 10

We obtain $\mathcal{F}\{\text{sgn}(t)\}$ as follows.

Consider the odd two-sided exponential function $f_\alpha(t)$ defined as

$$f_\alpha(t) = \begin{cases} e^{-\alpha t} & t > 0 \\ -e^{\alpha t} & t < 0 \end{cases},$$

where $\alpha > 0$:

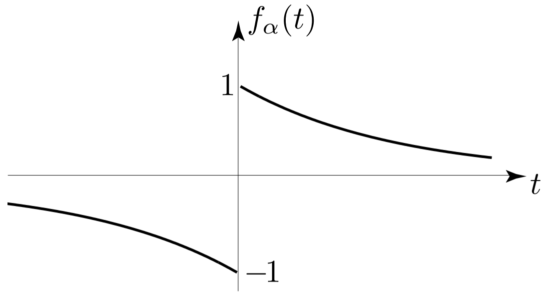


Figure 11

By slightly adapting our earlier calculation for the even two-sided exponential function we find

$$\begin{aligned}
\mathcal{F}\{f_\alpha(t)\} &= -\frac{1}{(\alpha - \mathrm{i}\omega)} + \frac{1}{(\alpha + \mathrm{i}\omega)} \\
&= \frac{-(\alpha + \mathrm{i}\omega) + (\alpha - \mathrm{i}\omega)}{\alpha^2 + \omega^2} \\
&= -\frac{2\mathrm{i}\omega}{\alpha^2 + \omega^2}.
\end{aligned}$$

The parameter α controls how rapidly the exponential function varies:

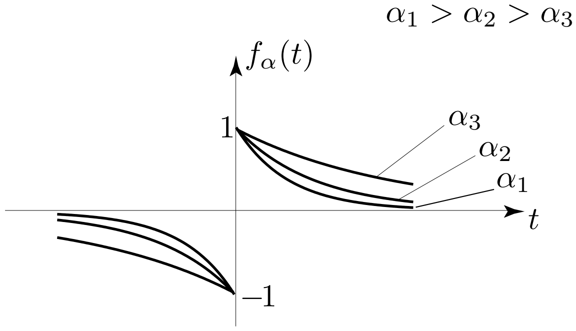


Figure 12

As we let $\alpha \to 0$ the exponential function resembles more and more closely the signum function. This suggests that

$$\begin{aligned}
\mathcal{F}\{\mathrm{sgn}(t)\} &= \lim_{\alpha \to 0} \mathcal{F}\{f_\alpha(t)\} \\
&= \lim_{\alpha \to 0} \left(-\frac{2\mathrm{i}\omega}{\alpha^2 + \omega^2} \right) = -\frac{2\mathrm{i}}{\omega} = \frac{2}{\mathrm{i}\omega}.
\end{aligned}$$

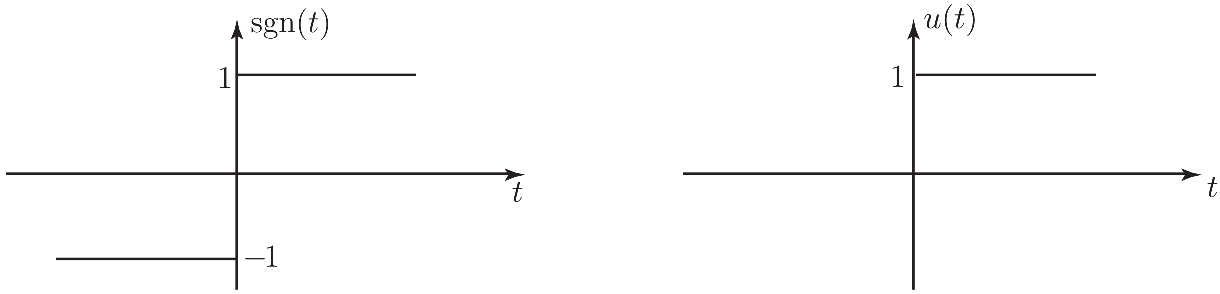
Task

Write the unit step function in terms of the signum function and hence obtain $\mathcal{F}\{u(t)\}$.

First express $u(t)$ in terms of sgn(t):

Your solution

Answer
From the graphs



the step function can be obtained by adding 1 to the signum function for all t and then dividing the resulting function by 2 i.e.

$$u(t) = \frac{1}{2}(1 + \text{sgn}(t)).$$

Now, using the linearity property of Fourier transforms and previously obtained Fourier transforms, find $\mathcal{F}\{u(t)\}$:

Your solution

Answer

We have, using linearity,

$$\mathcal{F}\{u(t)\} \;=\; \frac{1}{2}\mathcal{F}\{1\} + \frac{1}{2}\mathcal{F}\{\mathsf{sgn}(t)\} = \frac{1}{2}2\pi\delta(\omega) + \frac{1}{2}\frac{2}{\mathrm{i}\omega} = \pi\delta(\omega) + \frac{1}{\mathrm{i}\omega}$$

Thus, the Fourier transform of the unit step function contains the additional impulse term $\pi\delta(\omega)$ as well as the odd term $\dfrac{1}{\mathrm{i}\omega}$.

Exercises

1. Use Parseval's theorem and the Fourier transform of a 'two-sided' exponential function to show that

$$\int_{-\infty}^{\infty} \frac{d\omega}{(a^2+\omega^2)^2} = \frac{\pi}{2|a|^3}$$

2. Using $\mathcal{F}\{\mathsf{sgn}(t)\} = \dfrac{2}{\mathrm{i}\omega}$ find the Fourier transforms of (a) $f_1(t) = \dfrac{1}{t}$ (b) $f_2(t) = |t|$

   Hence obtain the transforms of (c) $f_3(t) = -\dfrac{1}{t^2}$ (d) $f_4(t) = \dfrac{2}{t^3}$

3. Show that

$$\mathcal{F}\{\sin\omega_0 t\} = \mathrm{i}\pi[\delta(\omega+\omega_0) - \delta(\omega-\omega_0)]$$

   Verify your result using inverse Fourier transform properties.

Answers

2 (a) $\mathcal{F}\{\dfrac{1}{t}\} = -\pi\mathrm{i}\ \mathsf{sgn}(\omega)$ (by the duality property)

(b) $\mathcal{F}\{|t|\} = -\dfrac{2}{\omega^2}$

(c) $\mathcal{F}\{-\dfrac{1}{t^2}\} = \pi\omega\ \mathsf{sgn}(\omega) = \begin{cases} \pi\omega, & \omega > 0 \\ -\pi\omega, & \omega < 0 \end{cases}$

(d) $\mathcal{F}\{\dfrac{1}{t^3}\} = \dfrac{\mathrm{i}\pi\omega^2}{2}\mathsf{sgn}(\omega)$

(Using time differentiation property in (b), (c) and (d).)