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Learning outcomes

In this Workbook you will learn what a causal function is, what the Laplace transform is, and how to obtain the Laplace transform of many commonly occurring causal functions. You will learn how the inverse Laplace transform can be obtained by using a look-up table and by using the so-called shift theorems. You will understand how to apply the Laplace transform to solve single and systems of ordinary differential equations. Finally you will gain some appreciation of transfer functions and some of their applications in solving linear systems.

Causal Functions

20.1

Introduction

The Laplace transformation is a technique employed primarily to solve constant coefficient ordinary differential equations. It is also used in modelling engineering systems. In this section we look at those functions to which the Laplace transformation is normally applied; so-called **causal** or **one-sided functions**. These are functions $f(t)$ of a single variable t such that $f(t) = 0$ if $t < 0$. In particular we consider the simplest causal function: the unit step function (often called the Heaviside function) $u(t)$:

$$u(t) = \begin{cases} 1 & \text{if} \quad t \geq 0 \\ 0 & \text{if} \quad t < 0 \end{cases}$$

We then use this function to show how signals (functions of time t) may be 'switched on' and 'switched off'.

Prerequisites

Before starting this Section you should . . .

- understand what a function is
- be able to integrate simple functions

Learning Outcomes

On completion you should be able to . . .

- explain what a causal function is
- be able to apply the step function to 'switch on' and 'switch off' signals

1. Transforms and causal functions

Without perhaps realising it, we are used to employing transformations in mathematics. For example, we often transform problems in algebra to an equivalent problem in geometry in which our natural intuition and experience can be brought to bear. Thus, for example, if we ask:
q What are those values of x for which $x(x-1)(x+2) > 0$' then perhaps the simplest way to solve this problem is to sketch the curve $y = x(x-1)(x+2)$ and then, by inspection, find for what values of x it is positive. We obtain the following figure.

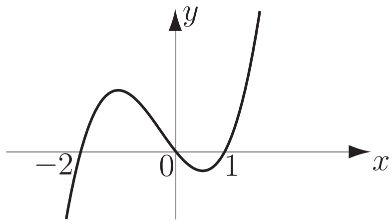


Figure 1

We have transformed a problem in algebra into an equivalent geometrical problem.
Clearly, by inspection of the curve, this inequality is satisfied if

$$-2 < x < 0 \quad \text{or if} \quad x > 1$$

and we have transformed back again to algebraic form.

The Laplace transform is a more complicated transformation than the simple geometric transformation considered above. What is done is to transform a function $f(t)$ of a single variable t into another function $F(s)$ of a single variable s through the relation:

$$F(s) = \int_0^\infty \mathrm{e}^{-st} f(t)\,dt.$$

The procedure is to produce, for each $f(t)$ of interest, the corresponding expression $F(s)$. As a simple example, if $f(t) = \mathrm{e}^{-2t}$ then

$$\begin{aligned} F(s) &= \int_0^\infty \mathrm{e}^{-st}\mathrm{e}^{-2t}\,dt \\ &= \int_0^\infty \mathrm{e}^{-(s+2)t}\,dt \\ &= \left[\frac{\mathrm{e}^{-(s+2)t}}{-(s+2)}\right]_0^\infty \\ &= 0 - \frac{\mathrm{e}^0}{-(s+2)} = \frac{1}{s+2} \end{aligned}$$

(We remind the reader that $\mathrm{e}^{-kt} \to 0$ as $t \to \infty$ if $k > 0$.)

Task

Find $F(s)$ if $f(t) = t$ using $F(s) = \int_0^\infty \mathrm{e}^{-st} t\, dt$

Your solution

Answer

You should obtain $F(s) = 1/s^2$. You do this by integrating by parts:

$$F(s) = \int_0^\infty \mathrm{e}^{-st} t\, dt = \left[t \frac{\mathrm{e}^{-st}}{(-s)} \right]_0^\infty - \int_0^\infty \frac{\mathrm{e}^{-st}}{(-s)}\, dt \;=\; 0 + \int_0^\infty \frac{\mathrm{e}^{-st}}{s}\, dt$$

$$= \left[-\frac{\mathrm{e}^{-st}}{s^2} \right]_0^\infty = \frac{1}{s^2}$$

The integral $\int_0^\infty \mathrm{e}^{-st} f(t)\, dt$ is called the Laplace transform of $f(t)$ and is denoted by $\mathcal{L}\{f(t)\}$.

Key Point 1

The Laplace Transform

$$\mathcal{L}\{f(t)\} = \int_0^\infty \mathrm{e}^{-st} f(t)\, dt = F(s)$$

Causal functions

As we have seen above, the Laplace transform involves an integral with limits $t = 0$ and $t = \infty$. Because of this, the nature of the function being transformed, $f(t)$, when t is negative is of no importance. In order to emphasize this we shall only consider so-called **causal functions** all of which take the value 0 when $t < 0$.

The simplest causal function is the Heaviside or step function denoted by $u(t)$ and defined by:

$$u(t) = \begin{cases} 1 & \text{if} \quad t \geq 0 \\ 0 & \text{if} \quad t < 0 \end{cases}$$

with graph as in Figure 2.

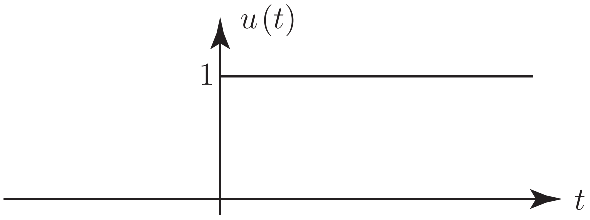


Figure 2

Similarly we can consider other 'step-functions'. For example, from the above definition we deduce

$$u(t-3) = \begin{cases} 1 & \text{if} \quad t-3 \geq 0 \\ 0 & \text{if} \quad t-3 < 0 \end{cases} \quad \text{or, rearranging the inequalities:} \quad u(t-3) = \begin{cases} 1 & \text{if} \quad t \geq 3 \\ 0 & \text{if} \quad t < 3 \end{cases}$$

with graph as in Figure 3:

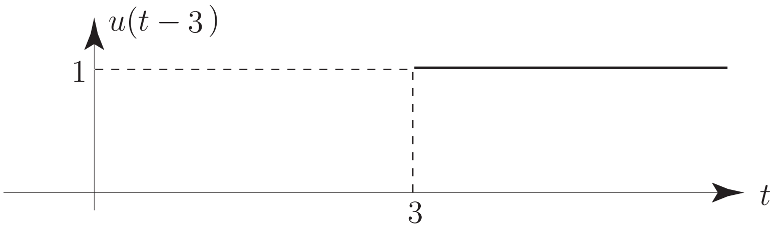


Figure 3

The step function has a useful property: multiplying an ordinary function $f(t)$ by the step function $u(t)$ changes it into a causal function; e.g. if $f(t) = \sin t$ then $\sin t.u(t)$ is causal. This is illustrated in the change from Figure 4 to Figure 5:

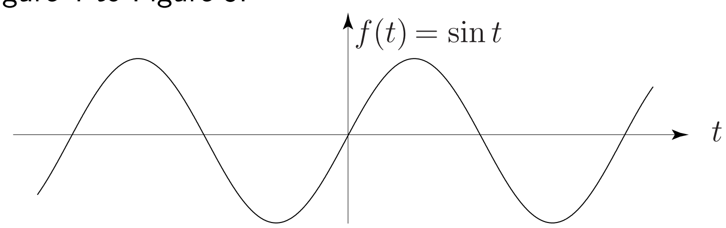


Figure 4

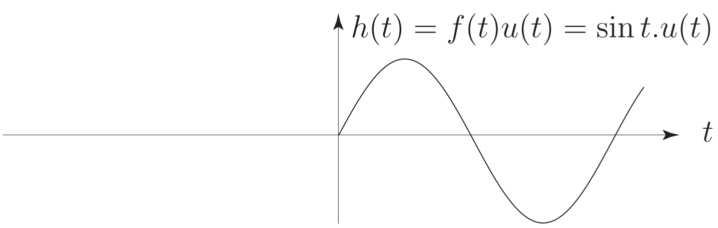


Figure 5

Key Point 2

Causal Functions

If $u(t)$ is the unit step function and $f(t)$ is any function then

$$f(t)u(t) \text{ is a causal function}$$

The step function can be used to 'switch on' functions at other values of t (which we will normally interpret as time). For example $u(t-1)$ has the value 1 if $t \geq 1$ and 0 otherwise so that $\sin t.u(t-1)$ is described by the (solid) curve in Figure 6:

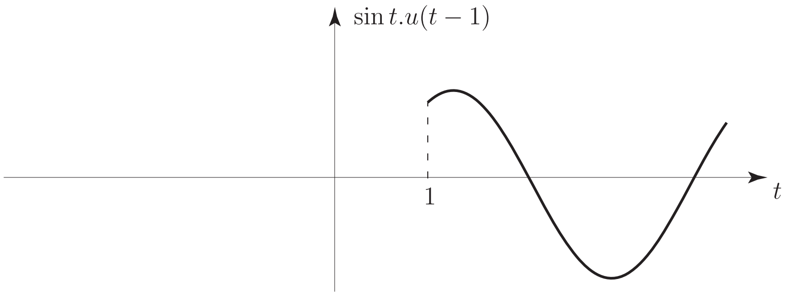


Figure 6

The step function can also be used to 'switch-off' signals. For example, the step function $u(t-1) - u(t-3)$ in Figure 7 has the effect on $f(t)$ such that $f(t)\,[u(t-1) - u(t-3)]$ (described by the solid curve in Figure 8) switches on at $t = 1$ (because then $u(t-1) - u(t-3)$ takes the value 1), remains 'on' for $1 \leq t \leq 3$, and then switches 'off' when $t > 3$ (because then $u(t-1) - u(t-3) = 1 - 1 = 0$).

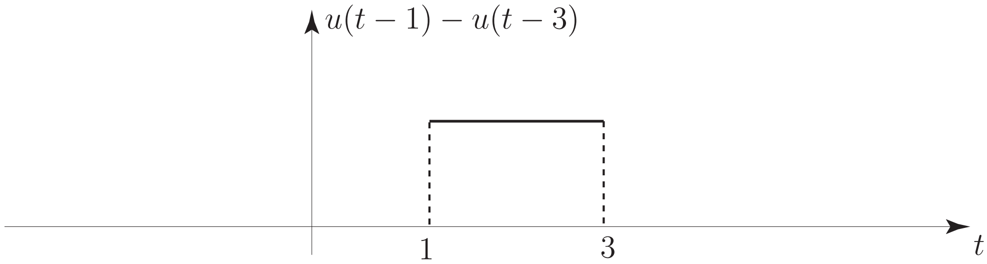


Figure 7

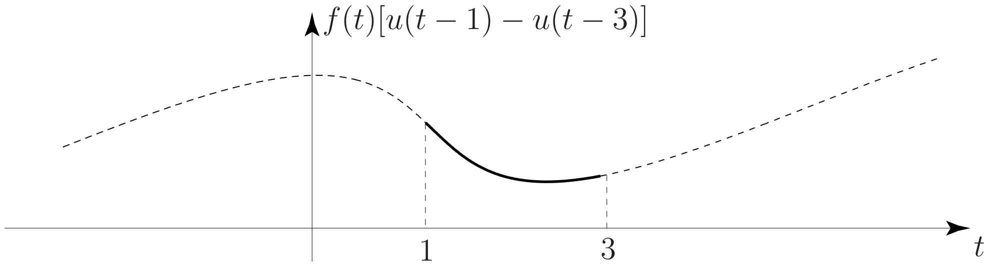


Figure 8

If we have an expression $f(t-a)u(t-a)$ then this is the function $f(t)$ translated along the t-axis through a time a. For example $\sin(t-2).u(t-2)$ is simply the causal sine curve $\sin t.u(t)$ shifted to the right by two units as described in the following Figure 9.

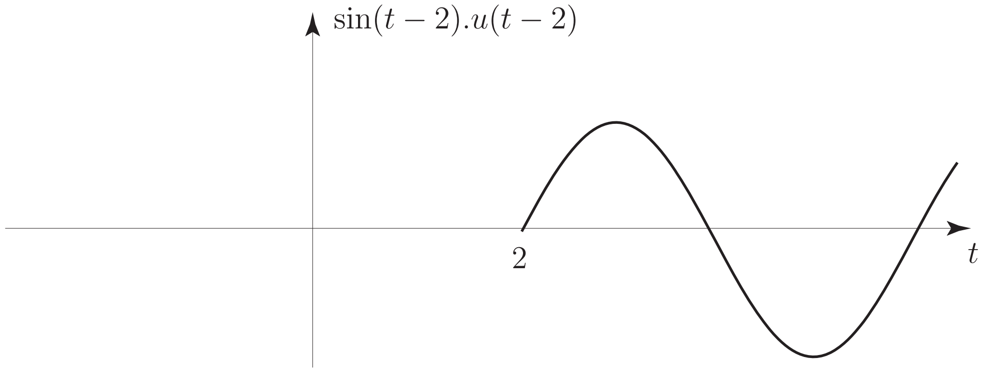


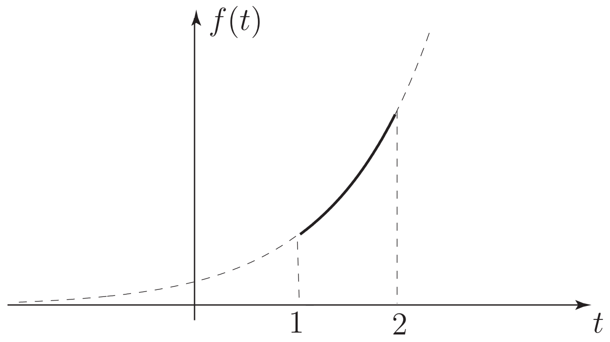
Figure 9

Task

Sketch the curve $f(t) = \mathrm{e}^t(u(t-1) - u(t-2))$.

Your solution

Answer
You should obtain



This is obtained since, if $t < 1$ then $t - 1 < 0$ and $t - 2 < 0$ and so

$$u(t-1) = 0, \qquad u(t-2) = 0 \qquad \text{leading to} \quad f(t) = 0$$

Also if $1 < t < 2$ then $t - 1 > 0$ and $t - 2 < 0$ so

$$u(t-1) = 1 \qquad \text{and} \qquad u(t-2) = 0 \qquad \text{implying} \quad f(t) = \mathrm{e}^t \quad \text{for this range of } t\text{-values.}$$

Finally if $t > 2$ then $t - 1 > 0$ and $t - 2 > 0$ and so

$$u(t-1) = 1, \qquad u(t-2) = 1 \qquad \text{giving} \quad f(t) = \mathrm{e}^t(1-1) = 0.$$

2. Properties of causal functions

Even though a function $f(t)$ may be causal we shall still often use the step function $u(t)$ to emphasize its causality and write $f(t)\ u(t)$. The following properties are easily verified.

(a) The sum of casusal functions is causal:

$$f(t)u(t) + g(t)u(t) = [f(t) + g(t)]\, u(t)$$

(b) The product of causal functions is causal:

$$\{f(t)u(t)\}\ \{g(t)u(t)\} = f(t)g(t).u(t)$$

(c) The derivative of a causal function is causal:

$$\frac{d}{dt}\{f(t)u(t)\} = \frac{df}{dt}.u(t)$$

(d) The definite integral of a causal function is a constant.

Calculating the definite integral of a causal function needs care.

Consider $\displaystyle\int_a^b f(t)u(t)\,dt$ where $a < b$. There are 3 cases to consider (i) $b < 0$ (ii) $a < 0, \ \ b > 0$ and (iii) $a > 0$ which are described in Figure 10:

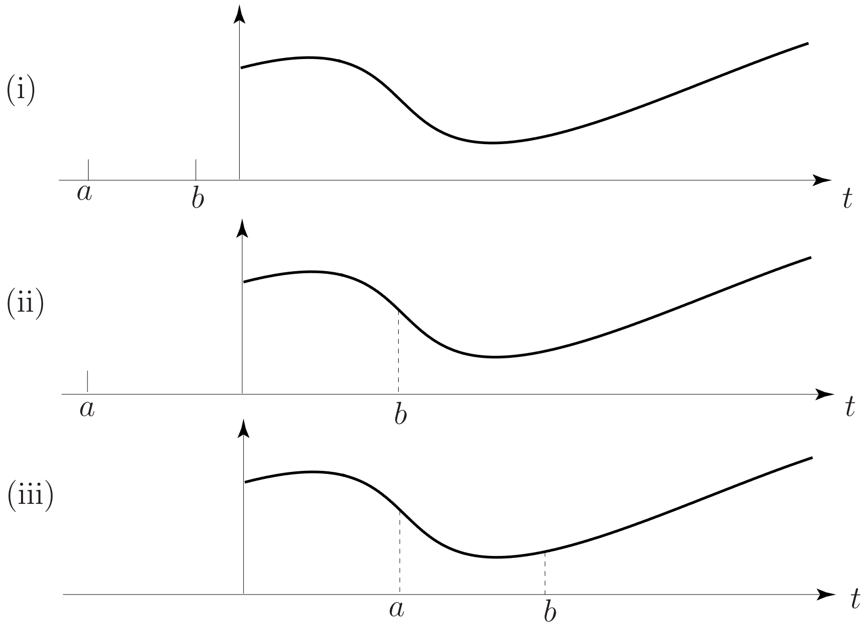


Figure 10

(i) If $b < 0$ then $t < 0$ and so $u(t) = 0 \quad \therefore \qquad \displaystyle\int_a^b f(t)u(t)\,dt = 0$

(ii) If $a < 0,\ b > 0$ then

$$F(t) = \int_a^b f(t)u(t)\,dt = \int_a^0 f(t)u(t)\,dt + \int_0^b f(t)u(t)\,dt = 0 + \int_0^b f(t)u(t)\,dt = \int_0^b f(t)\,dt$$

since, in the first integral $t < 0$ and so $u(t) = 0$ whereas, in the second integral $t > 0$ and so $u(t) = 1$.

(iii) If $a > 0$ then $\displaystyle\int_a^b f(t)u(t)\,dt = \int_a^b f(t)\,dt$ since $t > 0$ and so $u(t) = 1$.

Task

If $f(t) = (\mathrm{e}^{-t} + t)u(t)$ then find $\dfrac{df}{dt}$ and $\displaystyle\int_{-3}^{4} f(t)\,dt$

Find the derivative first:

Your solution

Answer

$$\frac{df}{dt} = (-\mathrm{e}^{-t} + 1)u(t)$$

Now obtain another integral representing $\int_{-3}^{4} f(t)\,dt$:

Your solution

Answer

You should obtain $\int_0^4 (\mathrm{e}^{-t} + t)\,dt$ since

$$\int_{-3}^{4} f(t)\,dt = \int_{-3}^{4} (\mathrm{e}^{-t} + t)u(t)\,dt = \int_0^4 (\mathrm{e}^{-t} + t)\,dt$$

This follows because in the range $t = -3$ to $t = 0$ the step function $u(t) = 0$ and so that part of the integral is zero. In the other part of the integral $u(t) = 1$.

Now complete the integration:

Your solution

Answer

You should obtain 8.9817 (to 4 d.p.) since

$$\int_0^4 (\mathrm{e}^{-t} + t)\,dt = \left[-\mathrm{e}^{-t} + \frac{t^2}{2}\right]_0^4 = (-\mathrm{e}^{-4} + 8) - (-1) = -\mathrm{e}^{-4} + 9 \approx 8.9817$$

Exercises

1. Find the derivative with respect to t of $(t^3 + \sin t)\ u(t)$.
2. Find the area under the curve $(t^3 + \sin t)u(t)$ between $t = -3$ and $t = 1$.
3. Find the area under the curve $\dfrac{1}{(t+3)}\ [u(t-1) - u(t-3)]$ between $t = -2$ and $t = 2.5$.

Answers

1. $(3t^2 + \cos t)u(t)$
2. 0.7097
3. 0.3185

The Transform and its Inverse

20.2

Introduction

In this Section we formally introduce the Laplace transform. The transform is only applied to causal functions which were introduced in Section 20.1. We find the Laplace transform of many commonly occurring 'signals'and produce a table of standard Laplace transforms.
We also consider the inverse Laplace transform. To begin with, the inverse Laplace transform is obtained 'by inspection' using a table of transforms. This approach is developed by employing techniques such as partial fractions and completing the square introduced in HELM 3.6.

Prerequisites

Before starting this Section you should ...

- understand what a causal function is
- be able to find and use partial fractions
- be able to perform integration by parts
- be able to use the technique of completing the square

Learning Outcomes

On completion you should be able to ...

- find the Laplace transform of many commonly occurrring causal functions
- obtain the inverse Laplace transform using techniques involving
  (i) a table of transforms
  (ii) partial fractions
  (iii) completing the square
  (iv) the first shift theorem

1. The Laplace transform

If $f(t)$ is a **causal function** then the Laplace transform of $f(t)$ is written $\mathcal{L}\{f(t)\}$ and defined by:

$$\mathcal{L}\{f(t)\} = \int_0^\infty \mathrm{e}^{-st} f(t)\, dt.$$

Clearly, once the integral is performed and the limits substituted the resulting expression will involve the s parameter alone since the dependence upon t is removed in the integration process. This resulting expression in s is denoted by $F(s)$; its precise form is dependent upon the form taken by $f(t)$. We now refine Key Point 1 (page 4).

Key Point 3

The Laplace Transform of a Causal Function

$$\mathcal{L}\{f(t)u(t)\} \equiv \int_0^\infty \mathrm{e}^{-st} f(t)u(t)\, dt \equiv F(s)$$

To begin, we determine the Laplace transform of some simple causal functions. For example, if we consider the **ramp function** $f(t) = t.u(t)$ with graph

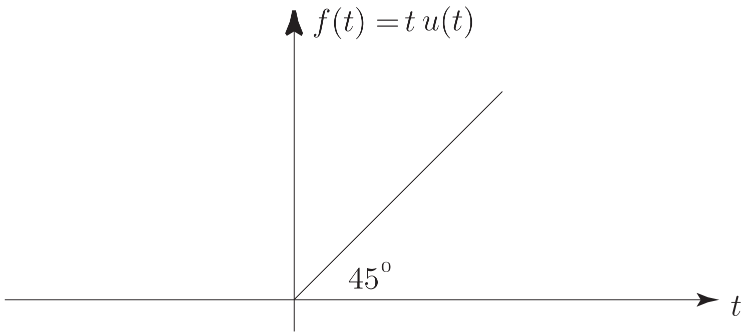


Figure 11

we find:

$$\begin{aligned}
\mathcal{L}\{t\ u(t)\} &= \int_0^\infty \mathrm{e}^{-st} t\ u(t)\, dt \\
&= \int_0^\infty \mathrm{e}^{-st} t\ \ dt \qquad \text{since in the range of the integral } u(t) = 1 \\
&= \left[\frac{t\ \mathrm{e}^{-st}}{(-s)}\right]_0^\infty - \int_0^\infty \frac{\mathrm{e}^{-st}}{(-s)}\ dt \qquad \text{using integration by parts} \\
&= \left[\frac{t\mathrm{e}^{-st}}{(-s)}\right]_0^\infty - \left[\frac{\mathrm{e}^{-st}}{(-s)^2}\right]_0^\infty
\end{aligned}$$

Now we have the difficulty of substituting in the limits of integration. The only problem arises with the upper limit ($t = \infty$). We shall always assume that the parameter s is so chosen that no contribution ever arises from the upper limit ($t = \infty$). In this particular case we need only demand that s is real and positive. Using this 'rule of thumb':

$$\begin{aligned}\mathcal{L}\{t\ u(t)\} &= [0-0] - \left[0 - \left(\frac{1}{(-s)^2}\right)\right] \\ &= \frac{1}{s^2}\end{aligned}$$

Thus, if $f(t) = t\ u(t)$ then $F(s) = 1/s^2$.
A similar, but more tedious, calculation yields the result that if $f(t) = t^n u(t)$ in which n is a positive integer then:

$$\mathcal{L}\{t^n u(t)\} = \frac{n!}{s^{n+1}}$$

[We remember $\quad n! \equiv n(n-1)(n-2)\ldots(3)(2)(1)$.]

Task

Find the Laplace transform of the step function $u(t)$.

Begin by obtaining the Laplace integral:

Your solution

Answer

You should obtain $\displaystyle\int_0^\infty \mathrm{e}^{-st}\,dt$ since in the range of integration, $t > 0$ and so $u(t) = 1$ leading to

$$\mathcal{L}\{u(t)\} = \int_0^\infty \mathrm{e}^{-st}u(t)\ dt = \int_0^\infty \mathrm{e}^{-st}\,dt$$

Your solution

Now complete the integration:

Answer

You should have obtained:

$$\begin{aligned}\mathcal{L}\{u(t)\} &= \int_0^\infty \mathrm{e}^{-st}\,dt \\ &= \left[\frac{\mathrm{e}^{-st}}{(-s)}\right]_0^\infty = 0 - \left[\frac{1}{(-s)}\right] = \frac{1}{s}\end{aligned}$$

where, again, we have assumed the contribution from the upper limit is zero.

As a second example, we consider the decaying exponential $f(t) = \mathrm{e}^{-at}u(t)$ where a is a positive constant. This function has graph:

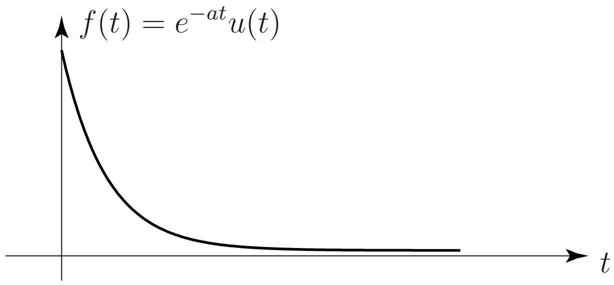


Figure 12

In this case,

$$\begin{aligned}
\mathcal{L}\{\mathrm{e}^{-at}u(t)\} &= \int_0^\infty \mathrm{e}^{-st}\mathrm{e}^{-at}\,dt \\
&= \int_0^\infty \mathrm{e}^{-(s+a)t}\,dt \\
&= \left[\frac{\mathrm{e}^{-(s+a)t}}{-(s+a)}\right]_0^\infty = \frac{1}{s+a} \qquad \text{(zero contribution from the upper limit)}
\end{aligned}$$

Therefore, if $f(t) = \mathrm{e}^{-at}u(t)$ then $F(s) = \dfrac{1}{s+a}$.

Following this approach we can develop a table of Laplace transforms which records, for each causal function $f(t)$ listed, its corresponding transform function $F(s)$. Table 1 gives a limited table of transforms.

The linearity property of the Laplace transformation

If $f(t)$ and $g(t)$ are causal functions and c_1, c_2 are constants then

$$\begin{aligned}
\mathcal{L}\{c_1f(t) + c_2g(t)\} &= \int_0^\infty \mathrm{e}^{-st}[c_1f(t) + c_2g(t)]\,dt \\
&= c_1\int_0^\infty \mathrm{e}^{-st}f(t)\,dt + c_2\int_0^\infty \mathrm{e}^{-st}g(t)\,dt \\
&= c_1\mathcal{L}\{f(t)\} + c_2\mathcal{L}\{g(t)\}
\end{aligned}$$

Key Point 4

Linearity Property of the Laplace Transform

$$\mathcal{L}\{c_1f(t) + c_2g(t)\} = c_1\mathcal{L}\{f(t)\} + c_2\mathcal{L}\{g(t)\}$$

Table 1. Table of Laplace Transforms

Rule	Causal function	Laplace transform
1	$f(t)$	$F(s)$
2	$u(t)$	$\dfrac{1}{s}$
3	$t^n u(t)$	$\dfrac{n!}{s^{n+1}}$
4	$\mathrm{e}^{-at}u(t)$	$\dfrac{1}{s+a}$
5	$\sin at\,.\,u(t)$	$\dfrac{a}{s^2+a^2}$
6	$\cos at\,.\,u(t)$	$\dfrac{s}{s^2+a^2}$
7	$\mathrm{e}^{-at}\sin bt\,.\,u(t)$	$\dfrac{b}{(s+a)^2+b^2}$
8	$\mathrm{e}^{-at}\cos bt\, u(t)$	$\dfrac{s+a}{(s+a)^2+b^2}$

Note: For convenience, this table is repeated at the end of the Workbook.

That is, the Laplace transform of a linear sum of causal functions is a linear sum of Laplace transforms. For example,

$$\begin{aligned}\mathcal{L}\{2\cos t\,.\,u(t) - 3t^2u(t)\} &= 2\mathcal{L}\{\cos t\,.\,u(t)\} - 3\mathcal{L}\{t^2u(t)\} \\ &= 2\left(\frac{s}{s^2+1}\right) - 3\left(\frac{2}{s^3}\right)\end{aligned}$$

Task

Obtain the Laplace transform of the hyperbolic function $\sinh at$.

Begin by expressing $\sinh at$ in terms of exponential functions:

Your solution

Answer

$\sinh at = \frac{1}{2}(\mathrm{e}^{at} - \mathrm{e}^{-at})$

Now use the linearity property (Key Point 4) to obtain the Laplace transform of the causal function $\sinh at.u(t)$:

Your solution

Answer

You should obtain $a/(s^2 - a^2)$ since

$$\begin{aligned}
\mathcal{L}\{\sinh at.u(t)\} = \mathcal{L}\left\{\frac{\text{e}^{at} - \text{e}^{-at}}{2}.u(t)\right\} &= \frac{1}{2}\mathcal{L}\{\text{e}^{at}.u(t)\} - \frac{1}{2}\mathcal{L}\{\text{e}^{-at}.u(t)\} \\
&= \frac{1}{2}\left[\frac{1}{s-a}\right] - \frac{1}{2}\left[\frac{1}{s+a}\right] \qquad \text{(Table 1, Rule 4)} \\
&= \frac{1}{2}\left[\frac{2a}{(s-a)(s+a)}\right] = \frac{a}{s^2 - a^2}
\end{aligned}$$

Task

Obtain the Laplace transform of the hyperbolic function $\cosh at$.

Your solution

Answer

You should obtain $\dfrac{s}{s^2 - a^2}$ since

$$\begin{aligned}
\mathcal{L}\{\cosh at.u(t)\} = \mathcal{L}\left\{\frac{\text{e}^{at} + \text{e}^{-at}}{2}.u(t)\right\} &= \frac{1}{2}\mathcal{L}\{\text{e}^{at}.u(t)\} + \frac{1}{2}\mathcal{L}\{\text{e}^{-at}.u(t)\} \\
&= \frac{1}{2}\left[\frac{1}{s-a}\right] + \frac{1}{2}\left[\frac{1}{s+a}\right] \qquad \text{(Table 1, Rule 4)} \\
&= \frac{1}{2}\left[\frac{2s}{(s-a)(s+a)}\right] = \frac{s}{s^2 - a^2}
\end{aligned}$$

Task

Find the Laplace transform of the **delayed step-function** $u(t-a)$, $a > 0$.

Write the delayed step-function here in terms of an integral:

Your solution

Answer

You should obtain $\mathcal{L}\{u(t-a)\} = \int_a^{\infty} \text{e}^{-st}\,dt$ (note the lower limit is a) since:

$$\mathcal{L}\{u(t-a)\} = \int_0^{\infty} \text{e}^{-st}u(t-a)\,dt = \int_0^{a} \text{e}^{-st}u(t-a)\,dt + \int_a^{\infty} \text{e}^{-st}u(t-a)\,dt$$

In the first integral $0 < t < a$ and so $(t-a) < 0$, therefore $u(t-a) = 0$.

In the second integral $a < t < \infty$ and so $(t-a) > 0$, therefore $u(t-a) = 1$. Hence

$$\mathcal{L}\{u(t-a)\} = 0 + \int_a^{\infty} \text{e}^{-st}\,dt.$$

Now complete the integration:

Your solution

Answer

$$\mathcal{L}\{u(t-a)\} = \int_a^{\infty} \text{e}^{-st}\,dt = \left[\frac{\text{e}^{-st}}{(-s)}\right]_a^{\infty} = \frac{\text{e}^{-sa}}{s}$$

Exercise

Determine the Laplace transform of the following functions.

(a) $\text{e}^{-3t}u(t)$ (b) $u(t-3)$ (c) $\text{e}^{-t}\sin 3t.u(t)$ (d) $(5\cos 3t - 6t^3).u(t)$

Answer (a) $\dfrac{1}{s+3}$ (b) $\dfrac{\text{e}^{-3s}}{s}$ (c) $\dfrac{3}{(s+1)^2+9}$ (d) $\dfrac{5s}{s^2+9} - \dfrac{36}{s^4}$

2. The inverse Laplace transform

The Laplace transform takes a causal function $f(t)$ and transforms it into a function of s, $F(s)$:

$$\mathcal{L}\{f(t)\} \equiv F(s)$$

The inverse Laplace transform operator is denoted by $\mathcal{L}^{-1}$ and involves recovering the original causal function $f(t)$. That is,

Key Point 5

Inverse Laplace Transform

$$\mathcal{L}^{-1}\{F(s)\} = f(t) \quad \text{where} \quad \mathcal{L}\{f(t)\} = F(s)$$

For example, using standard transforms from Table 1:

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} = \cos 2t\,.\,u(t) \text{ since } \mathcal{L}\{\cos 2t\,.\,u(t)\} = \frac{s}{s^2+4}. \qquad \text{(Table 1, Rule 6)}$$

Also

$$\mathcal{L}^{-1}\left\{\frac{3}{s^2}\right\} = 3t\,u(t) \text{ since } \mathcal{L}\{3t\,u(t)\} = \frac{3}{s^2}. \qquad \text{(Table 1, Rule 3)}$$

Because the Laplace transform is a linear operator it follows that the inverse Laplace transform is also linear, so if $c_1,\ c_2$ are constants:

Key Point 6

Linearity Property of Inverse Laplace Transforms

$$\mathcal{L}^{-1}\{c_1F(s) + c_2G(s)\} = c_1\mathcal{L}^{-1}\{F(s)\} + c_2\mathcal{L}^{-1}\{G(s)\}$$

For example, to find the inverse Laplace transform of $\dfrac{2}{s^4} - \dfrac{6}{s^2+4}$ we have

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{2}{s^4} - \frac{6}{s^2+4}\right\} &= \frac{2}{6}\mathcal{L}^{-1}\left\{\frac{6}{s^4}\right\} - 3\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} \\ &= \frac{1}{3}t^3u(t) - 3\sin 2t\,.\,u(t) \qquad \text{(from Table 1)}\end{aligned}$$

Note that the fractions have had to be manipulated slightly in order that the expressions match precisely with the expressions in Table 1.

Although the inverse Laplace transform can be examined at a deeper mathematical level we shall be content with this simple-minded approach to finding inverse Laplace transforms by using the table of Laplace transforms. However, even this approach is not always straightforward and considerable algebraic manipulation is often required before an inverse Laplace transform can be found. Next we consider two standard rearrangements which often occur.

Inverting through the use of partial fractions

The function

$$F(s) = \frac{1}{(s-1)(s+2)}$$

does not appear in our table of transforms and so we cannot, by inspection, write down the inverse Laplace transform. However, by using partial fractions we see that

$$F(s) = \frac{1}{(s-1)(s+2)} = \frac{\frac{1}{3}}{s-1} - \frac{\frac{1}{3}}{s+2}$$

and so, using the linearity property:

$$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{1}{(s-1)(s+2)}\right\} &= \mathcal{L}^{-1}\left\{\frac{\frac{1}{3}}{s-1}\right\} - \mathcal{L}^{-1}\left\{\frac{\frac{1}{3}}{s+2}\right\} \\ &= \tfrac{1}{3}\mathrm{e}^{t} - \tfrac{1}{3}\mathrm{e}^{-2t} \qquad \text{(Table 1, Rule 4)} \end{aligned}$$

Task

Find the inverse Laplace transform of $\dfrac{3}{(s-1)(s^2+1)}$.

Begin by using partial fractions to write the given expression in a more suitable form:

Your solution

Answer

$$\frac{3}{(s-1)(s^2+1)} = \frac{\frac{3}{2}}{s-1} - \frac{\frac{3}{2}s + \frac{3}{2}}{s^2+1}$$

Now continue to obtain the inverse:

Your solution

Answer

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{3}{(s-1)(s^2+1)}\right\} &= \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} - \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} \\ &= \frac{3}{2}\left[\mathrm{e}^t - \cos t - \sin t\right]u(t) \qquad \text{(Table 1, Rules 4, 6, 5)}\end{aligned}$$

3. The first shift theorem

The first and second shift theorems enable an even wider range of Laplace transforms to be easily obtained than the transforms we have already found. They also enable a significantly wider range of inverse transforms to be found. Here we introduce the first shift theorem. If $f(t)$ is a causal function with Laplace transform $F(s)$, i.e. $\mathcal{L}\{f(t)\} = F(s)$, then as we shall see, the Laplace transform of $\mathrm{e}^{-at}f(t)$, where a is a given constant, can easily be found in terms of $F(s)$.
Using the definition of the Laplace transform:

$$\begin{aligned}\mathcal{L}\{\mathrm{e}^{-at}f(t)\} &= \int_0^\infty \mathrm{e}^{-st}\left[\mathrm{e}^{-at}f(t)\right]dt \\ &= \int_0^\infty \mathrm{e}^{-(s+a)t}f(t)\,dt\end{aligned}$$

But if

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty \mathrm{e}^{-st}f(t)\,dt$$

then simply replacing 's' by '$s+a$' on both sides gives:

$$F(s+a) = \int_0^\infty \mathrm{e}^{-(s+a)t}f(t)\,dt$$

That is, the parameter s is shifted to the value $s+a$.

We have then the statement of the **first shift theorem**:

Key Point 7

First Shift Theorem

If $\quad \mathcal{L}\{f(t)\} = F(s) \quad$ then $\quad \mathcal{L}\{\mathrm{e}^{-at}f(t)\} = F(s+a).$

For example, we already know (from Table 1) that

$$\mathcal{L}\{t^3u(t)\} = \frac{6}{s^4}$$

and so, by the first shift theorem:

$$\mathcal{L}\{\mathrm{e}^{-2t}t^3u(t)\} = \frac{6}{(s+2)^4}$$

Task

Use the first shift theorem to determine $\mathcal{L}\{\mathrm{e}^{2t}\cos 3t.u(t)\}$.

Your solution

Answer

You should obtain $\dfrac{s-2}{(s-2)^2+9}$ since $\mathcal{L}\{\cos 3t.u(t)\} = \dfrac{s}{s^2+9}$ (Table 1, Rule 6)

and so by the first shift theorem (with $a = -2$)

$$\mathcal{L}\{\mathrm{e}^{2t}\cos 3t.u(t)\} = \frac{s-2}{(s-2)^2+9}$$

obtained by simply replacing 's' by '$s-2$'.

We can also employ the first shift theorem to determine some inverse Laplace transforms.

Task

Find the inverse Laplace transform of $F(s) = \dfrac{3}{s^2-2s-8}$.

Begin by completing the square in the denominator:

Your solution

Answer

$$\frac{3}{s^2-2s-8} = \frac{3}{(s-1)^2-9}$$

Recalling that $\mathcal{L}\{\sinh 3t\ u(t)\} = \dfrac{3}{s^2-9}$ (from the Task on page 15) complete the inversion using the first shift theorem:

Your solution

Answer
You should obtain

$$\mathcal{L}^{-1}\left\{\frac{3}{(s-1)^2-9}\right\} = \mathrm{e}^t \sinh 3t\ u(t)$$

Here, in the notation of the shift theorem:

$$f(t) = \sinh 3t\ u(t) \qquad F(s) = \frac{3}{s^2-9} \qquad \text{and} \quad a = -1$$

Inverting using completion of the square

The function:

$$F(s) = \frac{4s}{s^2+2s+5}$$

does not appear in the table of transforms and, again, needs amending before we can find its inverse transform. In this case, because s^2+2s+5 does not have nice factors, we complete the square in the denominator:

$$s^2+2s+5 \equiv (s+1)^2+4$$

and so

$$F(s) = \frac{4s}{s^2+2s+5} = \frac{4s}{(s+1)^2+4}$$

Now the numerator needs amending slightly to enable us to use the appropriate rule in the table of transforms (Table 1, Rule 8):

$$\begin{aligned} F(s) = \frac{4s}{(s+1)^2+4} &= 4\left\{\frac{s+1-1}{(s+1)^2+4}\right\} \\ &= 4\left\{\frac{s+1}{(s+1)^2+4} - \frac{1}{(s+1)^2+4}\right\} \\ &= \frac{4(s+1)}{(s+1)^2+4} - 2\left[\frac{2}{(s+1)^2+4}\right] \end{aligned}$$

Hence

$$\begin{aligned} \mathcal{L}^{-1}\{F(s)\} &= 4\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+4}\right\} - 2\mathcal{L}^{-1}\left\{\frac{2}{(s+1)^2+4}\right\} \\ &= 4\mathrm{e}^{-t}\cos 2t\,.\,u(t) - 2\mathrm{e}^{-t}\sin 2t\,.\,u(t) \\ &= \mathrm{e}^{-t}[4\cos 2t - 2\sin 2t]u(t) \end{aligned}$$

Task Find the inverse Laplace transform of $\dfrac{3}{s^2 - 4s + 6}$.

Begin by completing the square in the denominator of this expression:

Your solution

Answer

$$\frac{3}{s^2 - 4s + 6} = \frac{3}{(s-2)^2 + 2}$$

Now obtain the inverse:

Your solution

Answer

You should obtain:

$$\mathcal{L}^{-1}\left\{\frac{3}{(s-2)^2+2}\right\} = \mathcal{L}^{-1}\left\{\frac{3}{\sqrt{2}}\left[\frac{\sqrt{2}}{(s-2)^2+2}\right]\right\} = \frac{3}{\sqrt{2}}\mathrm{e}^{2t}\sin\sqrt{2}t.u(t) \qquad \text{(Table 1, Rule 7)}$$

Exercise

Determine the inverse Laplace transforms of the following functions.

(a) $\dfrac{10}{s^4}$ (b) $\dfrac{s-1}{s^2+8s+17}$ (c) $\dfrac{3s-7}{s^2+9}$ (d) $\dfrac{3s+3}{(s-1)(s+2)}$ (e) $\dfrac{s+3}{s^2+4s}$

(f) $\dfrac{2}{(s+1)(s^2+1)}$

Answer

(a) $\frac{10}{6}t^3$ (b) $\mathrm{e}^{-4t}\cos t - 5\mathrm{e}^{-4t}\sin t$ (c) $3\cos 3t - \frac{7}{3}\sin 3t$ (d) $2\mathrm{e}^{t} + \mathrm{e}^{-2t}$

(e) $\frac{3}{4}u(t) + \frac{1}{4}\mathrm{e}^{-4t}u(t)$ (f) $(\mathrm{e}^{-t} - \cos t + \sin t)u(t)$

Further Laplace Transforms

20.3

Introduction

In this Section we introduce the second shift theorem which simplifies the determination of Laplace and inverse Laplace transforms in some complicated cases.

Then we obtain the Laplace transform of derivatives of causal functions. This will allow us, in the next Section, to apply the Laplace transform in the solution of ordinary differential equations.

Finally, we introduce the delta function and obtain its Laplace transform. The delta function is often needed to model the effect on a system of a forcing function which acts for a very short time.

Prerequisites

Before starting this Section you should ...

- be able to find Laplace transforms and inverse Laplace transforms of simple causal functions
- be familiar with integration by parts
- understand what an initial-value problem is
- have experience of the first shift theorem

Learning Outcomes

On completion you should be able to ...

- use the second shift theorem to obtain Laplace transforms and inverse Laplace transforms
- find the Laplace transform of the derivative of a causal function

1. The second shift theorem

The second shift theorem is similar to the first except that, in this case, it is the time-variable that is shifted not the s-variable. Consider a causal function $f(t)u(t)$ which is shifted to the right by amount a, that is, the function $f(t-a)u(t-a)$ where $a > 0$. Figure 13 illustrates the two causal functions.

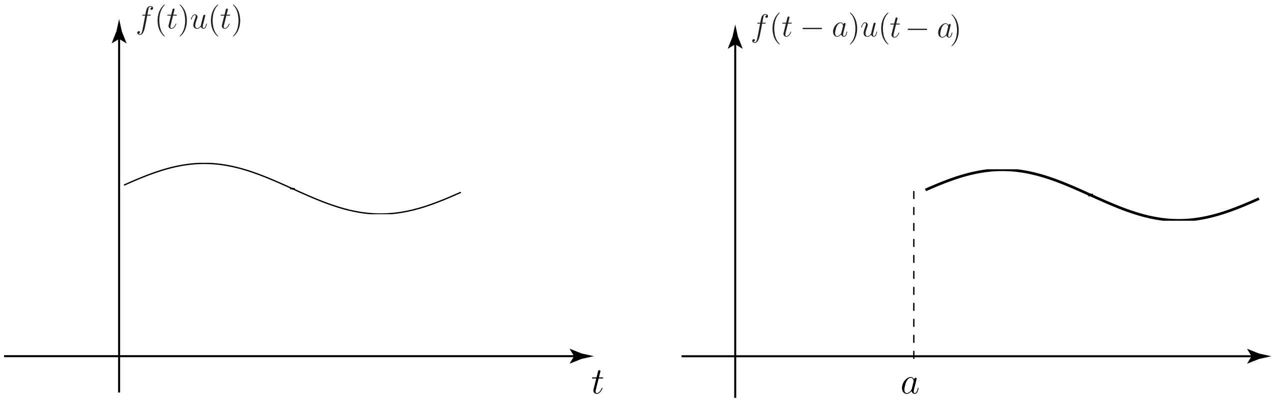


Figure 13

The Laplace transform of the shifted function is easily obtained:

$$\begin{aligned}\mathcal{L}\{f(t-a)u(t-a)\} &= \int_0^\infty \mathrm{e}^{-st} f(t-a)u(t-a)\,dt \\ &= \int_a^\infty \mathrm{e}^{-st} f(t-a)\,dt\end{aligned}$$

(Note the change in the lower limit from 0 to a resulting from the step function switching on at $t = a$). We can re-organise this integral by making the substitution $x = t - a$. Then $dt = dx$ and when $t = a, x = 0$ and when $t = \infty$ then $x = \infty$.

Therefore

$$\begin{aligned}\int_a^\infty \mathrm{e}^{-st} f(t-a)\,dt &= \int_0^\infty \mathrm{e}^{-s(x+a)} f(x)\,dx \\ &= \mathrm{e}^{-sa}\int_0^\infty \mathrm{e}^{-sx} f(x)\,dx\end{aligned}$$

The final integral is simply the Laplace transform of $f(x)$, which we know is $F(s)$ and so, finally, we have the statement of the second shift theorem:

Key Point 8

Second Shift Theorem

If $\quad \mathcal{L}\{f(t)\} = F(s) \quad$ then $\quad \mathcal{L}\{f(t-a)u(t-a)\} = \mathrm{e}^{-sa}F(s)$

Obviously, this theorem has its uses in finding the Laplace transform of time-shifted causal functions but it is also of considerable use in finding inverse Laplace transforms since, using the inverse formulation of the theorem of Key Point 8 we get:

Key Point 9

Inverse Second Shift Theorem

If $\mathcal{L}^{-1}\{F(s)\} = f(t)$ then $\mathcal{L}^{-1}\{\mathrm{e}^{-sa}F(s)\} = f(t-a)u(t-a)$

Task

Find the inverse Laplace transform of $\dfrac{\mathrm{e}^{-3s}}{s^2}$.

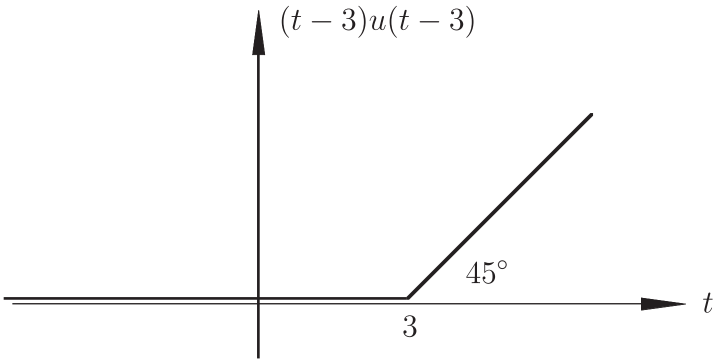
Your solution

Answer

You should obtain $(t-3)u(t-3)$ for the following reasons. We know that the inverse Laplace transform of $1/s^2$ is $t.u(t)$ (Table 1, Rule 3) and so, using the second shift theorem (with $a = 3$), we have

$$\mathcal{L}^{-1}\left\{\mathrm{e}^{-3s}\frac{1}{s^2}\right\} = (t-3)u(t-3)$$

This function is graphed in the following figure:



Task Find the inverse Laplace transform of $\dfrac{s}{s^2 - 2s + 2}$

Your solution

Answer
You should obtain $\mathrm{e}^t(\cos t + \sin t)$.

To obtain this, complete the square in the denominator: $s^2 - 2s + 2 = (s-1)^2 + 1$ and so

$$\frac{s}{s^2 - 2s + 2} = \frac{s}{(s-1)^2 + 1} = \frac{(s-1) + 1}{(s-1)^2 + 1} = \frac{s-1}{(s-1)^2 + 1} + \frac{1}{(s-1)^2 + 1}$$

Now, using the first shift theorem

$$\mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2 + 1}\right\} = \mathrm{e}^t \cos t . u(t) \quad \text{since} \quad \mathcal{L}^{-1}\left\{\frac{s}{s^2 + 1}\right\} = \cos t . u(t) \qquad \text{(Table 1, Rule 6)}$$

and

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2 + 1}\right\} = \mathrm{e}^t \sin t . u(t) \quad \text{since} \quad \mathcal{L}^{-1}\left\{\frac{1}{s^2 + 1}\right\} = \sin t . u(t) \qquad \text{(Table 1. Rule 5)}$$

Thus

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2 - 2s + 2}\right\} = \mathrm{e}^t(\cos t + \sin t)u(t)$$

2. The Laplace transform of a derivative

Here we consider not a causal function $f(t)$ directly but its derivatives $\dfrac{df}{dt}, \dfrac{d^2f}{dt^2}, \ldots$ (which are also causal.) The Laplace transform of derivatives will be invaluable when we apply the Laplace transform to the solution of constant coefficient ordinary differential equations.

If $\mathcal{L}\{f(t)\}$ is $F(s)$ then we shall seek an expression for $\mathcal{L}\{\dfrac{df}{dt}\}$ in terms of the function $F(s)$.

Now, by the definition of the Laplace transform

$$\mathcal{L}\left\{\frac{df}{dt}\right\} = \int_0^\infty \mathrm{e}^{-st} \frac{df}{dt}\, dt$$

This integral can be simplified using integration by parts:

$$\begin{aligned}\int_0^\infty \mathrm{e}^{-st}\frac{df}{dt}\,dt &= \left[\mathrm{e}^{-st}f(t)\right]_0^\infty - \int_0^\infty (-s)\mathrm{e}^{-st}f(t)\,dt \\ &= -f(0) + s\int_0^\infty \mathrm{e}^{-st}f(t)\,dt\end{aligned}$$

(As usual, we assume that contributions arising from the upper limit, $t = \infty$, are zero.) The integral on the right-hand side is precisely the Laplace transform of $f(t)$ which we naturally replace by $F(s)$. Thus

$$\mathcal{L}\left\{\frac{df}{dt}\right\} = -f(0) + sF(s)$$

As an example, we know that if $f(t) = \sin t\; u(t)$ then

$$\mathcal{L}\{f(t)\} = \frac{1}{s^2+1} = F(s) \qquad \text{(Table 1, Rule 5)}$$

and so, according to the result just obtained,

$$\begin{aligned}\mathcal{L}\left\{\frac{df}{dt}\right\} = \mathcal{L}\{\cos t\; u(t)\} &= -f(0) + sF(s) \\ &= 0 + s\left(\frac{1}{s^2+1}\right) \\ &= \frac{s}{s^2+1}\end{aligned}$$

a result we know to be true.
We can find the Laplace transform of the second derivative in a similar way to find:

$$\mathcal{L}\left\{\frac{d^2f}{dt^2}\right\} = -f'(0) - sf(0) + s^2F(s)$$

(The reader might wish to derive this result.) Here $f'(0)$ is the derivative of $f(t)$ evaluated at $t = 0$.

Key Point 10

Laplace Transforms of Derivatives

If $\quad\mathcal{L}\{f(t)\} = F(s)\quad$ then

$$\begin{aligned}\mathcal{L}\left\{\frac{df}{dt}\right\} &= -f(0) + sF(s) \\ \mathcal{L}\left\{\frac{d^2f}{dt^2}\right\} &= -f'(0) - sf(0) + s^2F(s)\end{aligned}$$

Task

If $\mathcal{L}\{f(t)\} = F(s)$ and $\dfrac{d^2f}{dt^2} - \dfrac{df}{dt} = 3t$ with initial conditions $f(0) = 1, \quad f'(0) = 0$, find the explicit expression for $F(s)$.

Begin by finding $\mathcal{L}\left\{\dfrac{d^2f}{dt^2}\right\}, \ \mathcal{L}\left\{\dfrac{df}{dt}\right\}$ and $\mathcal{L}\{3t\}$:

Your solution

Answer

$$\begin{aligned}
\mathcal{L}\{3t\} &= 3/s^2 \\
\mathcal{L}\left\{\frac{df}{dt}\right\} &= -f(0) + sF(s) = -1 + sF(s) \\
\mathcal{L}\left\{\frac{d^2f}{dt^2}\right\} &= -f'(0) - sf(0) + s^2F(s) = -s + s^2F(s)
\end{aligned}$$

Now complete the calculation to find $F(s)$:

Your solution

Answer

You should find $\quad F(s) = \dfrac{s^3 - s^2 + 3}{s^3(s-1)} \quad$ since, using the transforms we have found:

$$\begin{aligned}
-s + s^2F(s) - (-1 + sF(s)) &= \frac{3}{s^2} \\
\text{so} \qquad F(s)[s^2 - s] &= \frac{3}{s^2} + s - 1 = \frac{s^3 - s^2 + 3}{s^2}
\end{aligned}$$

leading to $\quad F(s) = \dfrac{s^3 - s^2 + 3}{s^3(s-1)}$

Exercises

1. Find the Laplace transforms of
   (a) $t^3 e^{-2t}u(t)$ (b) $e^t \sinh 3t.u(t)$ (c) $\sin(t-3).u(t-3)$

2. If $F(s) = \mathcal{L}\{f(t)\}$ find expressions for $F(s)$ if
   (a) $\dfrac{d^2y}{dt^2} - 3\dfrac{dy}{dt} + 4y = \sin t \qquad y(0) = 1, \quad y'(0) = 0$
   (b) $7\dfrac{dy}{dt} - 6y = 3u(t) \qquad y(0) = 0,$

3. Find the inverse Laplace transforms of
   (a) $\dfrac{6}{(s+3)^4}$ (b) $\dfrac{15}{s^2-2s+10}$ (c) $\dfrac{3s^2+11s+14}{s^3+2s^2-11s-52}$ (d) $\dfrac{e^{-3s}}{s^4}$ (e) $\dfrac{e^{-2s-2}(s+1)}{s^2+2s+5}$

Answers

1. (a) $\dfrac{6}{(s+2)^4}$ (b) $\dfrac{3}{(s-1)^2-9}$ (c) $\dfrac{e^{-3s}}{s^2+1}$

2. (a) $\dfrac{s^3-3s^2+s-2}{(s^2+1)(s^2-3s+4)}$ (b) $\dfrac{3}{s(7s-6)}$

3. (a) $e^{-3t}t^3u(t)$ (b) $5e^t \sin 3t.u(t)$ (c) $(2e^{4t} + e^{-3t}\cos 2t)u(t)$ (d) $\frac{1}{6}(t-3)^3u(t-3)$
   (e) $e^{-t}\cos 2(t-2).u(t-2)$

3. The delta function (or impulse function)

There is often a need for considering the effect on a system (modelled by a differential equation) by a forcing function which acts for a very short time interval. For example, how does the current in a circuit behave if the voltage is switched on and then very shortly afterwards switched off? How does a cantilevered beam vibrate if it is hit with a hammer (providing a force which acts over a very short time interval)? Both of these engineering 'systems' can be modelled by a differential equation. There are many ways the 'kick' or 'impulse' to the system can be modelled. The function we have in mind could have the graphical representation (when a is small) shown in Figure 14.

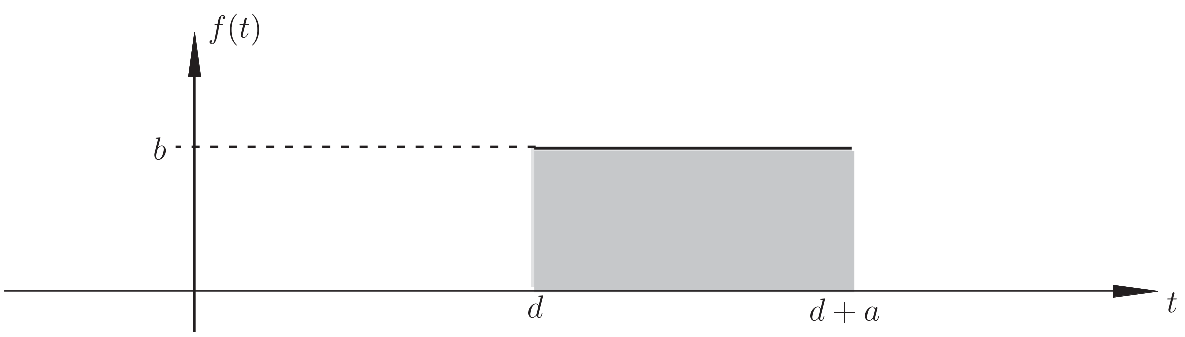


Figure 14

This can be represented formally using step functions; it switches on at $t = d$ and switches off at $t = d + a$ and has amplitude b:

$$f(t) = b[u(t-d) - u(t - \{d+a\})]$$

The effect on the system is related to the area under the curve rather than just the amplitude b. Our aim is to reduce the time interval over which the forcing function acts (i.e. reduce a) whilst at the same time keeping the total effect (i.e. the area under the curve) a constant. To do this we shall take $b = 1/a$ so that the area is always equal to 1. Reducing the value of a then gives the sequence of inputs shown in Figure 15.

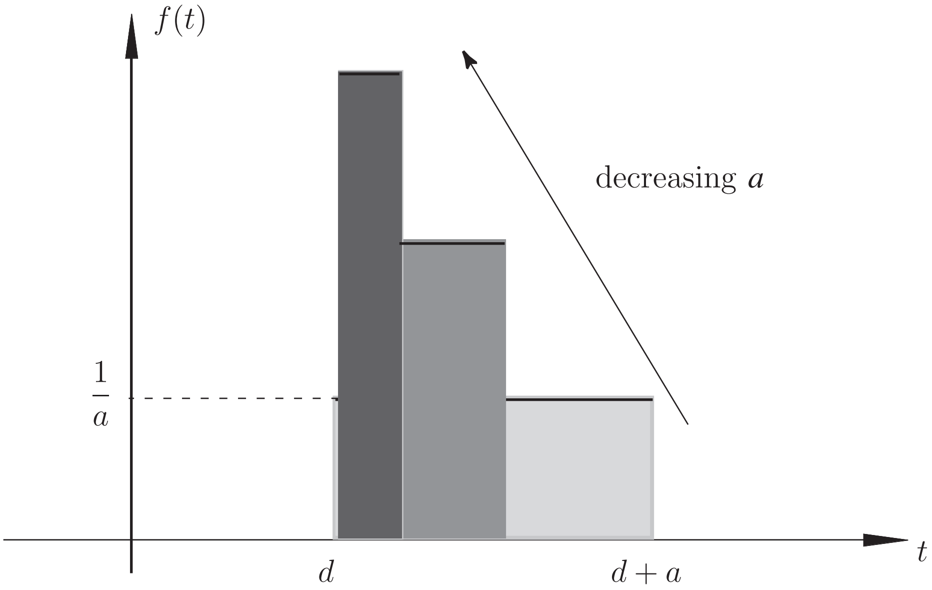


Figure 15

As the value of a decreases the height of the rectangle increases (to ensure the value of the area under the curve is fixed at value 1) until, in the limit as $a \to 0$, the 'function' becomes a 'spike' at $t = d$. The resulting function is called a **delta function** (or **impulse function**) and denoted by $\delta(t-d)$. This notation is used because, in a very obvious sense, the delta function described here is 'located' at $t = d$. Thus the delta function $\delta(t-1)$ is 'located' at $t = 1$ whilst the delta function $\delta(t)$ is 'located' at $t = 0$.

If we were defining an ordinary function we would write

$$\delta(t-d) = \lim_{a\to 0} \frac{1}{a}[u(t-d) - u(t - \{d+a\})]$$

However, this limit does not exist. The important property of the delta function relates to its integral:

$$\int_{-\infty}^{\infty} \delta(t-d)\,dt = \lim_{a\to 0}\int_{-\infty}^{\infty} \frac{1}{a}[u(t-d) - u(t-\{d+a\})]\,dt = \lim_{a\to 0}\int_{d}^{d+a} \frac{1}{a}\,dt$$
$$= \lim_{a\to 0}\left[\frac{d+a}{a} - \frac{d}{a}\right] = 1$$

which is what we expect since the area under each of the limiting curves is equal to 1.

A more technical discussion obtains the more general result:

Key Point 11

Sifting Property of the Delta Function

$$\int_{-\infty}^{\infty} f(t)\delta(t-d)\,dt = f(d)$$

This is called the **sifting property** of the delta function as it sifts out the value $f(d)$ from the function $f(t)$. Although the integral here ranges from $t = -\infty$ to $t = +\infty$ in fact the same result is obtained for any range if the range of the integral includes the point $t = d$. That is, if $\alpha \leq d \leq \beta$ then

$$\int_{\alpha}^{\beta} f(t)\delta(t-d)\,dt = f(d)$$

Thus, as long as the delta function is 'located' within the range of the integral the sifting property holds. For example,

$$\int_1^2 \sin t\ \delta(t-1.1)\,dt = \sin 1.1 = 0.8112 \qquad \int_0^{\infty} \mathrm{e}^{-t}\delta(t-1)\,dt = \mathrm{e}^{-1} = 0.3679$$

Task

Write expressions for delta functions located at $t = -1.7$ and at $t = 2.3$

Your solution

Answer

$\delta(t+1.7)$ and $\delta(t-2.3)$

Task

Evaluate the integral $\displaystyle\int_{-1}^{3} (\sin t\ \delta(t+2) - \cos t\ \delta(t))\,dt$

Your solution

Answer
You should obtain the value -1 since the first delta function, $\delta(t+2)$, is located outside the range of integration and thus

$$\int_{-1}^{3} (\sin t\ \delta(t+2) - \cos t\ \delta(t))\, dt = \int_{-1}^{3} -\cos t\ \delta(t)\, dt = -\cos 0 = -1$$

The Laplace transform of the delta function

Here we consider $\mathcal{L}\{\delta(t-d)\}$. From the definition of the Laplace transform:

$$\mathcal{L}\{\delta(t-d)\} = \int_0^\infty \mathrm{e}^{-st}\delta(t-d)\, dt = \mathrm{e}^{-sd}$$

by the sifting property of the delta function. Thus

Key Point 12

Laplace Transform of the Sifting Function

$\mathcal{L}\{\delta(t-d)\} = \mathrm{e}^{-sd}$ and, putting $d = 0$, $\mathcal{L}\{\delta(t)\} = \mathrm{e}^0 = 1$

Exercise

Find the Laplace transforms of $3\delta(t-3)$.

Answer

$3e^{-3s}$

Solving Differential Equations

20.4

Introduction

In this Section we employ the Laplace transform to solve constant coefficient ordinary differential equations. In particular we shall consider initial value problems. We shall find that the initial conditions are automatically included as part of the solution process. The idea is simple; the Laplace transform of each term in the differential equation is taken. If the unknown function is $y(t)$ then, on taking the transform, an algebraic equation involving $Y(s) = \mathcal{L}\{y(t)\}$ is obtained. This equation is solved for $Y(s)$ which is then inverted to produce the required solution $y(t) = \mathcal{L}^{-1}\{Y(s)\}$.

Prerequisites

Before starting this Section you should ...

- understand how to find Laplace transforms of simple functions and of their derivatives
- be able to find inverse Laplace transforms using a variety of techniques
- know what an initial-value problem is

Learning Outcomes

On completion you should be able to ...

- solve initial-value problems using the Laplace transform method

1. Solving ODEs using Laplace transforms

We begin with a straightforward initial value problem involving a first order constant coefficient differential equation. Let us find the solution of

$$\frac{dy}{dt} + 2y = 12e^{3t} \qquad y(0) = 3$$

using the Laplace transform approach.

Although it is not stated explicitly we shall assume that $y(t)$ is a causal function (we have no interest in the value of $y(t)$ if $t < 0$.) Similarly, the function on the right-hand side of the differential equation ($12e^{3t}$), the 'forcing function', will be assumed to be causal. (Strictly, we should write $12e^{3t}u(t)$ but the step function $u(t)$ will often be omitted.) Let us write $\mathcal{L}\{y(t)\} = Y(s)$. Then, taking the Laplace transform of every term in the differential equation gives:

$$\mathcal{L}\{\frac{dy}{dt}\} + \mathcal{L}\{2y\} = \mathcal{L}\{12e^{3t}\}$$

Now

$$\begin{aligned} \mathcal{L}\{\frac{dy}{dt}\} &= -y(0) + sY(s) = -3 + sY(s) \\ \mathcal{L}\{2y\} &= 2Y(s) \qquad \text{and} \qquad \mathcal{L}\{12e^{3t}\} = \frac{12}{s-3} \end{aligned}$$

Substituting these expressions into the transformed version of the differential equation gives:

$$[-3 + sY(s)] + 2Y(s) = \frac{12}{s-3}$$

Solving for $Y(s)$ we have

$$(s+2)Y(s) = \frac{12}{s-3} + 3 = \frac{3+3s}{s-3}$$

Therefore

$$Y(s) = \frac{3(s+1)}{(s+2)(s-3)}$$

Now, using partial fractions, this last expression can be written in a more convenient form:

$$Y(s) = \frac{3/5}{(s+2)} + \frac{12/5}{(s-3)}$$

and then, inverting:

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \tfrac{3}{5}\mathcal{L}^{-1}\{\frac{1}{s+2}\} + \tfrac{12}{5}\mathcal{L}^{-1}\{\frac{1}{s-3}\}$$

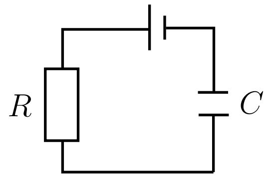
thus

$$y(t) = \tfrac{3}{5}e^{-2t}u(t) + \tfrac{12}{5}e^{3t}u(t)$$

This is the solution to the given initial value problem.

Task

The equation governing the build up of charge, $q(t)$, on the capacitor of an RC circuit is $\quad R\dfrac{dq}{dt} + \dfrac{1}{C}q = v_0$



where v_0 is the constant d.c. voltage. Initially, the circuit is relaxed and the circuit is then 'closed' at $t = 0$ and so $q(0) = 0$ is the initial condition for the charge. Use the Laplace transform method to solve the differential equation for $q(t)$.

Assume the forcing term v_0 is causal.

Begin by finding an expression for $Q(s) = \mathcal{L}\{q(t)\}$:

Your solution

Answer

$Q(s) = \dfrac{v_0 C}{s(RCs+1)}$ since, taking the Laplace transform of each term in the differential equation:

$$\begin{aligned} R\mathcal{L}\{\frac{dq}{dt}\} + \frac{1}{C}\mathcal{L}\{q\} &= \mathcal{L}\{v_0\} \\ \text{i.e.} \qquad R[-q(0) + sQ(s)] + \frac{1}{C}Q(s) &= \frac{v_0}{s} \end{aligned}$$

where, we emphasize, the Laplace transform of the constant term v_0 is $\dfrac{v_0}{s}$.

Inserting $q(0) = 0$ we have, after some rearrangement,

$$Q(s) = \frac{v_0 C}{s(RCs+1)}$$

Now expand the expression using partial fractions:

Your solution

Answer

You should obtain $Q(s) = v_0 C \left[\frac{1}{s} - \frac{RC}{RCs+1}\right]$

Now obtain $q(t)$ by taking inverse Laplace transforms:

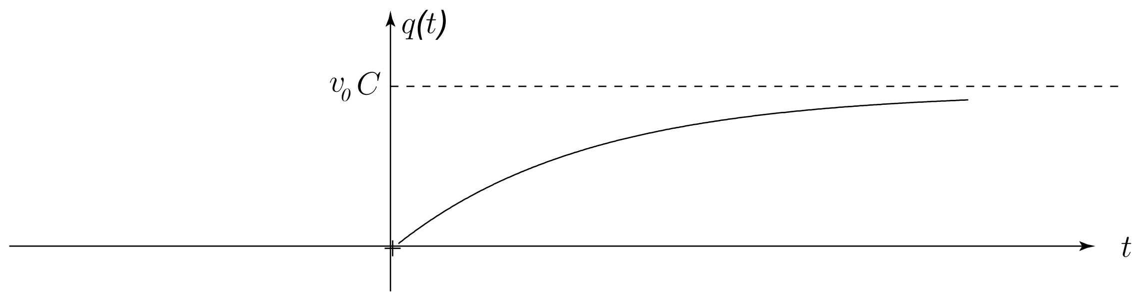
Your solution

Answer

$q(t) = v_0 C(1 - \text{e}^{-t/RC})u(t)$ since

$$\mathcal{L}^{-1}\{\frac{1}{s}\} = 1 \qquad \text{and} \qquad \mathcal{L}^{-1}\{\frac{RC}{RCs+1}\} = \mathcal{L}^{-1}\{\frac{1}{s+(1/RC)}\} = \text{e}^{-t/RC}$$

The solution to this problem is illustrated in the following diagram.



The Laplace transform method is also applied to higher-order differential equations in a similar way.

Example 1

Solve the second-order initial-value problem:

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 2y = \mathrm{e}^{-t} \qquad y(0) = 0, \quad y'(0) = 0$$

using the Laplace transform method.

Solution

As usual we shall assume the forcing function is causal (i.e. is really $\mathrm{e}^{-t}u(t)$.0 Taking the Laplace transform of each term:

$$\mathcal{L}\{\frac{d^2y}{dt^2}\} + 2\mathcal{L}\{\frac{dy}{dt}\} + 2\mathcal{L}\{y\} = \mathcal{L}\{\mathrm{e}^{-t}\}$$

that is,

$$[-y'(0) - sy(0) + s^2Y(s)] + 2[-y(0) + sY(s)] + 2Y(s) = \frac{1}{s+1}$$

Inserting the initial conditions and rearranging:

$$Y(s)[s^2 + 2s + 2] = \frac{1}{s+1} \qquad \text{i.e.} \qquad Y(s) = \frac{1}{(s+1)(s^2+2s+2)}$$

Then, using partial fractions:

$$\frac{1}{(s+1)(s^2+2s+2)} \equiv \frac{1}{s+1} - \frac{(s+1)}{s^2+2s+2} \equiv \frac{1}{s+1} - \frac{(s+1)}{(s+1)^2+1}$$

where we have completed the square in the second term of the right-hand side. We can now take the inverse Laplace transform:

$$\begin{aligned} y(t) = \mathcal{L}^{-1}\{Y(s)\} &= \mathcal{L}^{-1}\{\frac{1}{s+1}\} - \mathcal{L}^{-1}\{\frac{s+1}{(s+1)^2+1}\} \\ &= (\mathrm{e}^{-t} - \mathrm{e}^{-t}\cos t)u(t) \end{aligned}$$

which is the solution to the initial value problem.

Exercises

Use Laplace transforms to solve:

1. $\dfrac{dx}{dt} + x = 9\mathrm{e}^{2t} \qquad x(0) = 3$

2. $\dfrac{d^2x}{dt^2} + x = 2t \qquad x(0) = 0 \quad x'(0) = 5$

Answers 1. $x(t) = 3\mathrm{e}^{2t}$ 2. $x(t) = 3\sin t + 2t$

Example 2

A damped spring, constrained to move in one direction, such as might be found in a railway buffer, is subjected to an impulse of duration 5 seconds. The spring constant divided by the mass causing the impulse is 10 m^{-2} s^{-2} and the frictional force divided by this mass is 2 m^{-2}s^{-2}.

(a) Write down the equation governing the motion in terms of the displacement x m and time t seconds including the impulse $u(t)$.

(b) Write down the initial conditions on the displacement (x) and velocity.

(c) Solve the equation for displacement as a function of time.

(d) Draw a graph of the oscillations for $t = 0$ to 10 s.

Solution

(a) Since the system involves a restoring force and friction, after dividing through by the mass, the equation of motion may be written:

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 10x = u(t) - u(t-5)$$

where the right-hand side represents the impulse being switched on at $t = 0$ s and switched off at $t = 5$ s.

(b) Since the system starts from rest $x(0) = x'(0) = 0$.

(c) Taking the Laplace Transform of each term of the differential equation gives

$$\mathcal{L}\left[\frac{d^2x}{dt^2}\right] + 2\mathcal{L}\left[\frac{dx}{dt}\right] + 10\mathcal{L}\left[x\right] = \mathcal{L}\left[u(t)\right] - \mathcal{L}\left[u(t-5)\right]$$

i.e. $$s^2X(s) - x(0) - s\,x'(0) + 2(s\,X(s) - x(0)) + 10X(s) = \frac{1}{s} - \frac{1}{s}\mathrm{e}^{-5s}$$

but as $x(0) = x'(0) = 0$, this simplifies to $s^2X(s) + 2\,s\,X(s) + 10X(s) = \frac{1}{s}\left[1 - e^{-5s}\right]$

i.e.
$$\begin{aligned} X(s) &= \frac{1}{s(s^2+2s+10)}\left[1-\mathrm{e}^{-5s}\right] \\ &= \left[\frac{1}{10}\cdot\frac{1}{s} - \frac{1}{10}\cdot\frac{s+2}{s^2+2s+10}\right]\left[1-\mathrm{e}^{-5s}\right] \quad \text{(using partial fractions)} \\ &= \left[\frac{1}{10}\cdot\frac{1}{s} - \frac{1}{10}\cdot\frac{s+1}{(s+1)^2+3^2} - \frac{1}{30}\cdot\frac{3}{(s+1)^2+3^2}\right]\left[1-\mathrm{e}^{-5s}\right] \\ &= \frac{1}{10}\cdot\frac{1}{s} - \frac{1}{10}\cdot\frac{s+1}{(s+1)^2+3^2} - \frac{1}{30}\cdot\frac{3}{(s+1)^2+3^2} \\ &\quad -\frac{1}{10}\cdot\frac{1}{s}\mathrm{e}^{-5s} + \frac{1}{10}\cdot\frac{s+1}{(s+1)^2+3^2}\mathrm{e}^{-5s} + \frac{1}{30}\cdot\frac{3}{(s+1)^2+3^2}\mathrm{e}^{-5s} \end{aligned}$$

Solution (contd.)

so, on taking inverse Laplace Transforms,

$$\begin{aligned} x(t) &= \frac{1}{10} - \frac{1}{10}\mathrm{e}^{-t}\cos 3t - \frac{1}{30}\mathrm{e}^{-t}\sin 3t \\ &\quad -\frac{1}{10}u(t-5) + \frac{1}{10}\mathrm{e}^{-(t-5)}\cos 3(t-5)u(t-5) + \frac{1}{30}\mathrm{e}^{-(t-5)}\sin 3(t-5)u(t-5) \end{aligned}$$

(d)

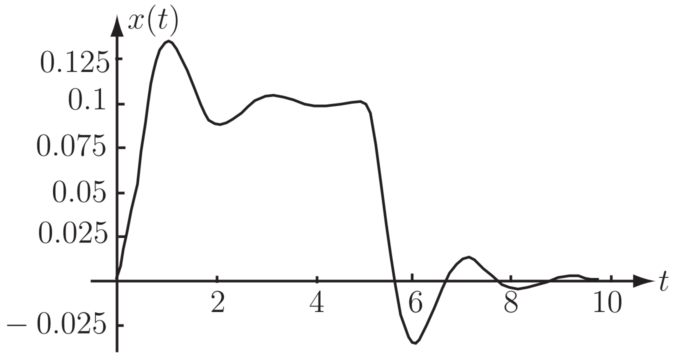


Figure 16

According to the graph the damped spring has a damped oscillation about a displacement of 0.1 m after the start of the impulse and a damped oscillation about a displacement of zero after the impulse has finished.

2. Solving systems of differential equations

The Laplace transform method is also well suited to solving systems of differential equations. A simple example will illustrate the technique.

Let $x(t),\ y(t)$ be two independent functions which satisfy the coupled differential equations

$$\begin{aligned} \frac{dx}{dt} + y &= \mathrm{e}^{-t} \\ \frac{dy}{dt} - x &= 3\mathrm{e}^{-t} \\ x(0) &= 0, \quad y(0) = 1 \end{aligned}$$

Now, using a traditional approach, we could try to eliminate one of the unknown functions from this system: for example, from the first:

$$\frac{dy}{dt} = -\mathrm{e}^{-t} - \frac{d^2x}{dt^2} \qquad \text{(taking the derivative and rearranging)}$$

This can then be substituted in the second equation: $\dfrac{dy}{dt} - x = 3\mathrm{e}^{-t}$, to give:

$$-\frac{d^2x}{dt^2} - x = 4\mathrm{e}^{-t}$$

which can then be solved in the normal way (either using the complementary function/particular integral approach or else the Laplace transform approach.) However, this approach is not workable if we have large numbers of first order differential equations to deal with. Let us instead use the Laplace transform directly.
If we use the notation that

$$\mathcal{L}\{x(t)\} = X(s) \qquad \text{and} \qquad \mathcal{L}\{y(t)\} = Y(s)$$

then, by taking the Laplace transform of every term in the given differential equations, we obtain:

$$\begin{aligned} -x(0) + sX(s) + Y(s) &= \frac{1}{s+1} \\ -y(0) + sY(s) - X(s) &= \frac{3}{s+1} \end{aligned}$$

which, using the initial conditions and rearranging gives

$$\begin{aligned} sX(s) + Y(s) &= \frac{1}{s+1} \\ -X(s) + sY(s) &= \frac{s+4}{s+1} \end{aligned}$$

Key Point 13

Taking the Laplace transform converts a system of differential equations into a system of algebraic simultaneous equations.

We can solve these algebraic equations (in $X(s)$ and $Y(s)$) using a variety of techniques (inverse matrix; Cramer's determinant method etc.) Here we will use Cramer's method.

$$X(s) = \frac{\begin{vmatrix} \frac{1}{s+1} & 1 \\ \frac{s+4}{s+1} & s \end{vmatrix}}{\begin{vmatrix} s & 1 \\ -1 & s \end{vmatrix}} = \frac{\dfrac{s}{s+1} - \dfrac{s+4}{s+1}}{s^2+1}$$

$$= \frac{-4}{(s^2+1)(s+1)} = \frac{2(s-1)}{s^2+1} - \frac{2}{s+1}$$

and

$$Y(s) = \frac{\begin{vmatrix} s & \frac{1}{s+1} \\ -1 & \frac{s+4}{s+1} \end{vmatrix}}{\begin{vmatrix} s & 1 \\ -1 & s \end{vmatrix}} = \frac{\dfrac{s(s+4)}{s+1} + \dfrac{1}{s+1}}{s^2+1}$$

$$= \frac{s^2+4s+1}{(s^2+1)(s+1)} = -\frac{1}{s+1} + \frac{2(s+1)}{s^2+1}$$

The last lines in each case having been obtained using partial fractions. We can now invert $X(s),\ Y(s)$ to find $x(t),\ y(t)$:

$$\begin{aligned} x(t) = \mathcal{L}^{-1}\{X(s)\} &= 2\mathcal{L}^{-1}\{\frac{s}{s^2+1}\} - 2\mathcal{L}^{-1}\{\frac{1}{s^2+1}\} - 2\mathcal{L}^{-1}\{\frac{1}{s+1}\} \\ &= (2\cos t - 2\sin t - 2\mathrm{e}^{-t})u(t) \\ y(t) = \mathcal{L}^{-1}\{Y(s)\} &= -\mathcal{L}^{-1}\{\frac{1}{s+1}\} + 2\mathcal{L}^{-1}\{\frac{s}{s^2+1}\} + 2\mathcal{L}^{-1}\{\frac{1}{s^2+1}\} \\ &= (-\mathrm{e}^{-t} + 2\cos t + 2\sin t)u(t) \end{aligned}$$

(Note that once the solution for $x(t)$ is found the solution for $y(t)$ may be easier to obtain by substituting in the differential equation: $y = \mathrm{e}^{-t} - \dfrac{dx}{dt}$ rather than using Laplace transforms.)

Use the Laplace transform to solve the coupled differential equations:

$$\frac{dy}{dt} - x = 0, \qquad \frac{dx}{dt} + y = 1, \qquad x(0) = -1, \quad y(0) = 1$$

Begin by obtaining a system of algebraic equations for $X(s)$ and $Y(s)$:

Your solution

Answer

Writing $\mathcal{L}\{x(t)\} = X(s)$ and $\mathcal{L}\{y(t)\} = Y(s)$ you should obtain the set of transformed equations

$$-1 + sY(s) - X(s) = 0$$

$$1 + sX(s) + Y(s) = \frac{1}{s}$$

which, when re-arranged, are

$$-X(s) + sY(s) = 1$$

$$sX(s) + Y(s) = \frac{1-s}{s}$$

Now solve these equations for $X(s)$ and $Y(s)$:

Your solution

Answer

$$X(s) = -\frac{s}{1+s^2} \qquad Y(s) = \frac{1}{s} - \frac{1}{1+s^2}$$

Now find the required solution by obtaining the inverse Laplace transforms:

Your solution

Answer

You should obtain $x(t) = -\cos t.u(t)$ and $y(t) = (1 - \sin t).u(t)$. This follows since

$$\mathcal{L}^{-1}\{-\frac{s}{1+s^2}\} = -\cos t.u(t) \qquad \mathcal{L}^{-1}\{\frac{1}{s}\} = u(t) \qquad \mathcal{L}^{-1}\{-\frac{1}{1+s^2}\} = -\sin t.u(t)$$

Exercises

1. Solve the given system of differential equations for the initial conditions specified.

   (a) $\dfrac{dx}{dt} = y \qquad \dfrac{dy}{dt} = x \qquad x(0) = 1 \quad y(0) = 0$

   (b) $\dfrac{dx}{dt} = 4x - 2y \qquad \dfrac{dy}{dt} = 5x + 2y \qquad x(0) = 2 \quad y(0) = -2$

2. The Laplace transform can also be used to solve a pair of coupled second order differential equations.

   Solve, for the given initial conditions,

$$\frac{d^2x}{dt^2} = y + \sin t \qquad x(0) = 1 \quad x'(0) = 0$$

$$\frac{d^2y}{dt^2} = -\frac{dx}{dt} + \cos t \qquad y(0) = -1 \quad y'(0) = -1$$

   (Note that the initial conditions on each of $x(t)$ and $y(t)$ are needed in the second order situation.)

Answer

1. (a) $x = \cosh t, \quad y = \sinh t$ (b) $x = \mathrm{e}^{3t}(2\cos 3t + 2\sin 3t), \quad y = \mathrm{e}^{3t}(-2\cos 3t + 4\sin 3t)$
2. $x = \cos t, \quad y = -\cos t - \sin t$

3. Applications of systems of differential equations

Coupled electrical circuits and mechanical vibrating systems involving several masses in springs offer examples of engineering systems modelled by systems of differential equations.

Electrical circuits

Consider the RL (resistance/inductance) circuit with a voltage $v(t)$ applied as shown in Figure 17.

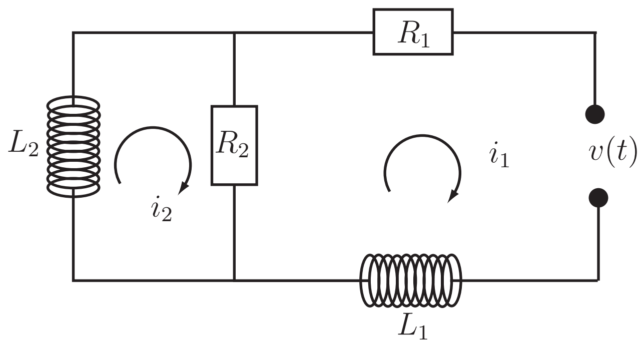


Figure 17

If i_1 and i_2 denote the currents in each loop we obtain, using Kirchhoff's voltage law:

(i) in the right loop: $L_1\dfrac{di_1}{dt} + R_2(i_1 - i_2) + R_1 i_1 = v(t)$

(ii) in the left loop: $L_2\dfrac{di_2}{dt} + R_2(i_2 - i_1) = 0$

Suppose, in the above circuit, that

$L_1 = 0.8$ henry, $L_2 = 1$ henry, $R_1 = 1.4\ \Omega$ $R_2 = 1\ \Omega$.

Assume zero initial conditions: $i_1(0) = i_2(0) = 0$.

Suppose that the applied voltage is constant: $v(t) = 100$ volts $t \geq 0$.

Solve the problem by Laplace transforms.

Begin by obtaining $V(s)$, the Laplace transform of $v(t)$:

Your solution

Answer

We have, from the definition of the Laplace transform:

$$V(s) = \int_0^\infty 100\mathrm{e}^{-st}dt = 100\left[\frac{\mathrm{e}^{-st}}{-s}\right]_0^\infty = \frac{100}{s}$$

This is simply the Laplace transform of the step function of height 100.

Now insert the parameter values into the differential equations and obtain the Laplace transform of each equation. Denote by $I_1(s), I_2(s)$ the Laplace transforms of the unknown currents. (These are equivalent to $X(s)$ and $Y(s)$ of the theory.):

Your solution

Answer

$$0.8\frac{di_1}{dt} + i_1 - i_2 + 1.4i_1 = v(t)$$

$$\frac{di_2}{dt} + i_2 - i_1 = 0$$

Rearranging and dividing the first equation by 0.8:

$$\frac{di_1}{dt} + 3i_1 - 1.25i_2 = 1.25v(t)$$

$$\frac{di_2}{dt} - i_1 + i_2 = 0$$

Taking Laplace transforms and inserting the initial conditions $i_1(0) = 0,\ i_2(0) = 0$:

$$(s+3)I_1(s) - 1.25I_2(s) = \frac{125}{s}$$

$$-I_1(s) + (s+1)I_2(s) = 0$$

Now solve these equations for $I_1(s)$ and $I_2(s)$. Put each expression into partial fractions and finally take the inverse Laplace transform to obtain $i_1(t)$ and $i_2(t)$:

Your solution

Answer

We find

$$I_1(s) = \frac{125(s+1)}{s(s+1/2)(s+7/2)} = \frac{500}{7s} - \frac{125}{3(s+1/2)} - \frac{625}{21(s+7/2)}$$

in partial fractions.

Hence $\quad i_1(t) = \frac{500}{7} - \frac{125}{3}\mathrm{e}^{-t/2} - \frac{625}{21}\mathrm{e}^{-7t/2}$

Similarly

$$I_2(s) = \frac{125}{s(s+1/2)(s+7/2)} = \frac{500}{7s} - \frac{250}{3(s+1/2)} + \frac{250}{21(s+7/2)}$$

which has inverse Laplace transform:

$$i_2(t) = \frac{500}{7} - \frac{250}{3}\mathrm{e}^{-t/2} + \frac{250}{21}\mathrm{e}^{-7t/2}$$

Notice in both cases that $i_1(t)$ and $i_2(t)$ tend to the steady state value $\frac{500}{7}$ as t increases.

Two masses on springs

Consider the vibrating system shown:

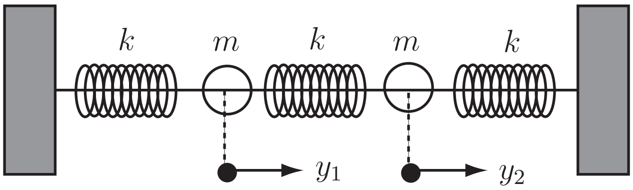


Figure 18

As you can see, the system consists of two equal masses, both m, and 3 springs of the same stiffness k. The governing differential equations can be obtained by applying Newton's second law ('force equals mass times acceleration'): (recall that a single spring of stiffness k will experience a force $-ky$ if it is displaced a distance y from its equilibrium.)

In our system therefore

$$m\frac{d^2y_1}{dt^2} = -ky_1 + k(y_2 - y_1)$$

$$m\frac{d^2y_2}{dt^2} = -k(y_2 - y_1) - ky_2$$

which is a **pair** of second order differential equations.

Task

For the above system, if $m = 1$, $k = 2$ and the initial conditions are

$$y_1(0) = 1 \quad y_1'(0) = \sqrt{6} \quad y_2(0) = 1 \quad y_2'(0) = -\sqrt{6}$$

use Laplace transforms to solve the system of differential equations to find $y_1(t)$ and $y_2(t)$.

Begin by letting $Y_1(s), Y_2(s)$ be the Laplace transforms of $y_1(t), y_2(t)$ respectively and take the transforms of the differential equations, inserting the initial conditions:

Your solution

Answer

$$(s^2 + 4)Y_1 - 2Y_2 = s + \sqrt{6}$$

$$-2Y_1 + (s^2 + 4)Y_2 = s - \sqrt{6}$$

Solve these equations (e.g. by Cramer's rule or by Gauss elimination) then use partial fractions and finally take inverse Laplace transforms:

Your solution

(Perform the calculation on separate paper and summarise the results here.)

Answer

$$Y_1(s) = \frac{(s+\sqrt{6})(s^2+4)+2(s-\sqrt{6})}{(s^2+4)^2-4} = \frac{s}{s^2+2} + \frac{\sqrt{6}}{s^2+6}$$

from which $\quad y_1(t) = \cos\sqrt{2}t + \sin\sqrt{6}t$

A similar calculation gives $\quad y_2(t) = \cos\sqrt{2}t - \sin\sqrt{6}t$

We see that the motion of each mass is composed of two harmonic oscillations; the system model was undamped so, on this model, the vibration continues indefinitely.

Engineering Example 1

Charge on a capacitor

In the circuit shown in Figure 19, the switch S is closed at $t = 0$ with a capacitor charge $q(0) = q_0 =$ constant and $dq/dt(0) = 0$.

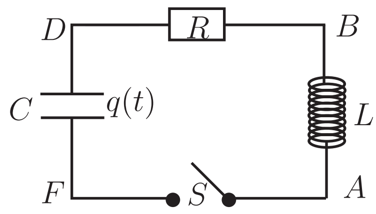


Figure 19

Show that $\quad q(t) = q_0(t)\mathrm{e}^{-\alpha t}\left[\cos\omega t + \dfrac{\alpha}{w}\sin\omega t\right] \quad$ where $\alpha = \dfrac{R}{2L}$ and $\omega^2 = \dfrac{1}{LC} - \alpha^2$

Laplace transform properties required

The following properties are needed to solve this problem.

$$F(s+a) = \mathcal{L}\{e^{-at}f(t)\} \qquad \text{(P1)}$$

$$\mathcal{L}\left\{\frac{df(t)}{dt}\right\} = s\{f(t)\} - f(0) \qquad \text{(P2)}$$

$$\mathcal{L}\left\{\frac{d^2f(t)}{dt^2}\right\} = s^2\mathcal{L}\{f(t)\} - \frac{df}{dt}(0) - s\ f(0) \qquad \text{(P3)}$$

$$\mathcal{L}\{\sin kt\} = \frac{k}{s^2+k^2} \text{ with } s > 0 \qquad \text{(P4)}$$

$$\mathcal{L}\{\cos kt\} = \frac{s}{s^2+k^2} \text{ with } s > 0 \qquad \text{(P5)}$$

$$\mathcal{L}^{-1}\{\mathcal{L}\{f(t)\}\} = f(t) \qquad \text{(P6)}$$

STEP 1 Establish the differential equation for $q(t)$ using, for example, Kirchhoff's law.

Solution

When the switch S is closed, the inductance L, capacitance C and resistance R give rise to a.c. voltages related by

$$V_A - V_B = L\frac{di}{dt},\ V_B - V_D = R\,i,\ V_D - V_F = q/C \ \text{ respectively.}$$

So since $V_A - V_F = (V_A - V_B) + (V_B - V_D) + (V_D - V_F) = 0$ and $i = \dfrac{dq}{dt}$ we have

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = 0 \qquad (1)$$

STEP 2 Write the Laplace transform of the differential equation substituting for the initial conditions:

Solution

Since the Laplace transform is linear, the transform of differential Equation (1) is

$$\mathcal{L}\left\{L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{q}{C}\right\}=L\mathcal{L}\left\{\frac{d^2q}{dt^2}\right\}+R\mathcal{L}\left\{\frac{dq}{dt}\right\}+\mathcal{L}\{\frac{q}{C}\}=0. \qquad (2)$$

We deal with each derivative term in turn: Using property (P3),

$$\mathcal{L}\left\{\frac{d^2q}{dt^2}\right\}=s^2\mathcal{L}\{q(t)\}-\frac{dq}{dt}(0)-s\ q(0).$$

So, using the initial conditions $q(0)=q_0$ and $\frac{dq}{dt}(0)=0$

$$\mathcal{L}\left\{\frac{d^2q}{dt^2}\right\}=s^2\mathcal{L}\{q(t)\}-s\ q_0. \qquad (3)$$

By means of property (2)

$$\mathcal{L}\left\{\frac{dq}{dt}\right\}=s\mathcal{L}\{q(t)\}-q_0 \qquad (4)$$

STEP 3 Solve for the function $\mathcal{L}\{q(t)\}$ by substituting from (3) and (4) into Equation (2):

Solution

$$L[s^2\mathcal{L}\{q(t)\}-sq_0]+R[s\mathcal{L}\{q(t)\}-q_0]+\frac{1}{C}\mathcal{L}\{q(t)\}=0$$

$$\Rightarrow \qquad \mathcal{L}\{q(t)\}[Ls^2+Rs+\frac{1}{C}]=Lsq_0+Rq_0$$

$$\Rightarrow \qquad \mathcal{L}\{q(t)\}=\frac{(Ls+R)}{(Ls^2+Rs+\frac{1}{C})}q_0 \qquad (5)$$

Using the definitions $\alpha=\frac{R}{2L}$ and $\omega^2=\frac{1}{LC}-\alpha^2$ enables the denominator in Equation (5) to be expressed as the sum of two squares,

$$L\ s^2+R\ s+\frac{1}{C}=L[s^2+\frac{Rs}{L}+\frac{1}{LC}]=L[s^2+2\alpha s+\frac{1}{LC}]$$
$$=L[s^2+2\alpha s+\alpha^2+\omega^2]=L[\{s+\alpha\}^2+\omega^2].$$

Consequently, with the new expression for the denominator, Equation (5) becomes

$$\mathcal{L}\{q(t)\}=q_0\left[\frac{s}{(s+\alpha)^2+\omega^2}+\frac{R}{L}\ \frac{1}{(s+\alpha)^2+\omega^2}\right]. \qquad (6)$$

STEP 4 Use the inverse Laplace transform to obtain $q(t)$:

Solution

The inverse Laplace transform is used to find $q(t)$.

Taking the inverse Laplace transform of Equation (6) and using the linearity properties

$$\mathcal{L}^{-1}\{\mathcal{L}\{q(t)\}\} = q_0\mathcal{L}^{-1}\left\{\frac{s}{(s+\alpha)^2+\omega^2} + \frac{R}{L}\,\frac{1}{(s+\alpha)^2+\omega^2}\right\}.$$

Using property (P6) this can be written as

$$q(t) = q_0\mathcal{L}^{-1}\left\{\frac{s+\alpha}{(s+\alpha)^2+\omega^2} + \frac{-\alpha}{(s+\alpha)^2+\omega^2} + \frac{R}{L\omega}\,\frac{\omega}{(s+\alpha)^2+\omega^2}\right\}.$$

Using the linearity of the Laplace transform again

$$q(t) = q_0\mathcal{L}^{-1}\left\{\frac{s+\alpha}{(s+\alpha)^2+\omega^2}\right\} + \mathcal{L}^{-1}\left\{\frac{-\alpha}{(s+\alpha)^2+\omega^2}\right\} + \mathcal{L}^{-1}\left\{\frac{R}{L\omega}\,\frac{\omega}{(s+\alpha)^2+\omega^2}\right\}. \qquad (7)$$

Using properties (P1) and (P5)

$$\mathcal{L}^{-1}\left\{\frac{s+\alpha}{(s+\alpha)^2+\omega^2}\right\} = \mathsf{e}^{-\alpha t}\cos\omega t. \qquad (8)$$

Similarly,

$$\mathcal{L}^{-1}\left\{\frac{-\alpha}{(s+\alpha)^2+\omega^2}\right\} = -(\frac{\alpha}{\omega})\{\mathsf{e}^{-\alpha t}\sin\omega t\} \qquad (9)$$

and

$$\mathcal{L}^{-1}\left\{\frac{R}{L\omega}\,\frac{\omega}{(s+a)^2+\omega^2}\right\} = (\frac{R}{L\omega})\mathsf{e}^{-\alpha t}\sin\omega t. \qquad (10)$$

Substituting (8), (9) and (10) in (7) gives

$$q(t) = q_0\mathsf{e}^{-\alpha t}\left[\cos\omega t + \left\{-\frac{\alpha}{\omega} + \frac{R}{L\omega}\right\}\mathsf{e}^{-\alpha t}\sin\omega t\right]. \qquad (11)$$

STEP 5 Finally, show that for $t > 0$ the solution is

$$q(t) = q_0\mathsf{e}^{-\alpha t}[\cos\omega t + (\frac{\alpha}{\omega})\sin\omega t] \text{ where } \alpha = \frac{R}{2L} \text{ and } \omega^2 = \frac{1}{LC} - \alpha^2.$$

Solution

Substituting $\alpha = \dfrac{R}{2L}$ in (11) gives

$$q(t) = q_0\mathsf{e}^{-\alpha t}\left[\cos\omega t + [-\frac{\alpha}{\omega} + \frac{2\alpha}{\omega}]\mathsf{e}^{-\alpha t}\sin\omega t\right]$$

$$= q_0\mathsf{e}^{-\alpha t}[\cos\omega t + \frac{\alpha}{\omega}\sin\omega t\ ]$$

Engineering Example 2

Deflection of a uniformly loaded beam

Introduction

A uniformly loaded beam of length L is supported at both ends. The deflection $y(x)$ is a function of horizontal position x and obeys the differential equation

$$\frac{d^4y}{dx^4}(x) = \frac{1}{EI}q(x) \qquad (1)$$

where E is Young's modulus, I is the moment of inertia and $q(x)$ is the load per unit length at point x. We assume in this problem that $q(x) = q$ (a constant). The boundary conditions are (i) no deflection at $x = 0$ and $x = L$ (ii) no curvature of the beam at $x = 0$ and $x = L$.

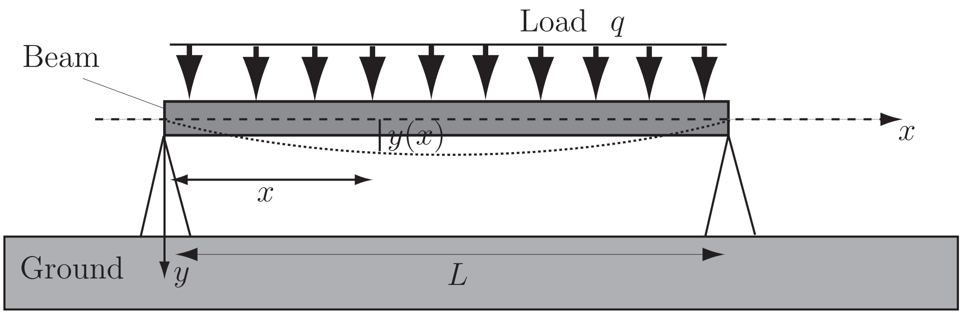


Figure 20

Problem in words

In addition to being subject to a uniformly distributed load, a beam is supported so that there is no deflection and no curvature of the beam at its ends. Applying a Laplace Transform to the differential equation (1), find the deflection of the beam as function of horizontal position along the beam.

Mathematical formulation of the problem

Find the equation of the curve $y(x)$ assumed by the bending beam that solves (1). Use the coordinate system shown in Figure 1 where the origin is at the left extremity of the beam. In this coordinate system, the mathematical formulations of the boundary conditions which require that there is no deflection at $x = 0$ and $x = L$, and that there is no curvature of the beam at $x = 0$ and $x = L$, are

(a) $y(0) = 0$

(b) $y(L) = 0$

(c) $\frac{d^2y}{dx^2}\Big|_{x=0} = 0$

(d) $\frac{d^2y}{dx^2}\Big|_{x=L} = 0$

Note that $\frac{dy(x)}{dx}$ and $\frac{d^2y(x)}{dx^2}$ are respectively the slope and the radius of curvature of the curve at point (x, y).

Mathematical analysis

The following Laplace transform properties are needed:

$$\mathcal{L}\left\{\frac{d^n f(t)}{dt^n}\right\} = s^n F(s) - \sum_{k=1}^{n} s^{k-1} \left.\frac{d^{n-k} f}{dx^{n-k}}\right|_{x=0} \quad \text{(P1)}$$

$$\mathcal{L}\{1\} = 1/s \quad \text{(P2)}$$

$$\mathcal{L}\{t^n\} = n!/s^{n+1} \quad \text{(P3)}$$

$$\mathcal{L}^{-1}\{\mathcal{L}\{f(t)\}\} = f(t) \quad \text{(P4)}$$

To solve a differential equation involving the unknown function $f(t)$ using Laplace transforms

(a) Write the Laplace transform of the differential equation using property (P1)

(b) Solve for the function $\mathcal{L}\{f(t)\}$ using properties (P2) and (P3)

(c) Use the inverse Laplace transform to obtain $f(t)$ using property (P4)

Using the linearity properties of the Laplace transform, (1) becomes

$$\mathcal{L}\left\{\frac{d^4 y}{dx^4}(x)\right\} - \mathcal{L}\{\frac{q}{EI}\} = 0.$$

Using (P1) and (P2)

$$s^4\mathcal{L}\{y(x)\} - \sum_{k=1}^{4} s^{k-1} \left.\frac{d^{4-k}y}{dx^{4-k}}\right|_{x=0} - \frac{q}{EI}\,\frac{1}{s} = 0. \quad (2)$$

The four terms of the sum are

$$\sum_{k=1}^{4} s^{k-1}\frac{d^{4-k}y}{dx^{4-k}} = \left.\frac{d^3y}{dx^3}\right|_{x=0} + d\left.\frac{d^2y}{dx^2}\right|_{x=0} + s^2\left.\frac{dy}{dx}\right|_{x=0} + s^3 y(0).$$

The boundary conditions give $y(0) = 0$ and $\dfrac{d^2y}{dx^2} = 0$. So (2) becomes

$$s^4 L\{y(x)\} - \left.\frac{d^3y}{dx^3}\right|_{x=0} - s^2\left.\frac{dy}{dx}\right|_{x=0} - \frac{q}{EI}\,\frac{1}{s} = 0. \quad (3)$$

Here $\left.\dfrac{d^3y}{dx^3}\right|_{x=0}$ and $\left.\dfrac{dy}{dx}\right|_{x=0}$ are unknown *constants*, but they can be determined by using the remaining two boundary conditions $y(L) = 0$ and $\left.\dfrac{d^2y}{dx^2}\right|_{x=L} = 0.$

Solving for $\mathcal{L}\{y(x)\}$, (3) leads to

$$\mathcal{L}\{y(x)\} = \frac{1}{s^4}\left.\frac{d^3y}{dx^3}\right|_{x=0} + \frac{1}{s^2}\left.\frac{dy}{dx}\right|_{x=0} + \frac{q}{EI}\,\frac{1}{s^5}.$$

Using the linearity of the Laplace transform, the inverse Laplace transform of this equation gives

$$\mathcal{L}^{-1}\{\mathcal{L}\{y(x)\}\} = \left.\frac{d^3y}{dx^3}\right|_{x=0} \times \mathcal{L}^{-1}\left\{\frac{1}{s^4}\right\} + \left.\frac{dy}{dx}\right|_{x=0} \times \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + \frac{q}{EI}\,\mathcal{L}^{-1}\left\{\frac{1}{s^5}\right\}.$$

Hence

$$y(x) = \left.\frac{d^3y}{dx^3}\right|_{x=0} \times \mathcal{L}^{-1}\left\{3!\frac{1}{s^4}\right\}/3! + \left.\frac{dy}{dx}\right|_{x=0} \times \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + \frac{q}{EI}\mathcal{L}^{-1}\left\{4!\,\frac{1}{s^5}\right\}/4!$$

So using (P3)

$$y(x) = \left.\frac{d^3y}{dx^3}\right|_{x=0} \times \mathcal{L}^{-1}\{\mathcal{L}\{x^3\}\}/6 + \left.\frac{dy}{dx}\right|_{x=0} \times \mathcal{L}^{-1}\{\mathcal{L}\{x^1\}\} + \frac{q}{EI}\mathcal{L}^{-1}\{\mathcal{L}\{x^4\}\}/24.$$

Simplifying by means of (P4)

$$y(x) = \left.\frac{d^3y}{dx^3}\right|_{x=0} \times x^3/6 + \left.\frac{dy}{dx}\right|_{x=0} \times x + \frac{q}{EI}x^4/24. \qquad (4)$$

To use the boundary condition $\left.\frac{d^2y}{dx^2}\right|_{x=L} = 0$, take the second derivative of (4), to obtain

$$\frac{d^2y}{dx^2}(x) = \left.\frac{d^3y}{dx^3}\right|_{x=0} \times x + \frac{q}{2EI}x^2.$$

The boundary condition $\left.\frac{d^2y}{dx^2}\right|_{x=L} = 0$ implies

$$\left.\frac{d^3y}{dx^3}\right|_{x=0} = -\frac{q}{2EI}L. \qquad (5)$$

Using the last boundary condition $y(L) = 0$ with (5) in (4)

$$\left.\frac{dy}{dx}\right|_{x=0} = \frac{qL^3}{24EI} \qquad (6)$$

Finally substituting (5) and (6) in (4) gives

$$y(x) = \frac{q}{24EI}x^4 - \frac{qL}{12EI}x^3 + \frac{qL^3}{24EI}x.$$

Interpretation

The predicted deflection is zero at both ends as required.

Note This problem was solved by an entirely different means (integrating the ODE) in HELM 19.4, page 65.

The Convolution Theorem

20.5

Introduction

In this Section we introduce the convolution of two functions $f(t),\ g(t)$ which we denote by $(f*g)(t)$. The convolution is an important construct because of the convolution theorem which allows us to find the inverse Laplace transform of a product of two transformed functions:

$$\mathcal{L}^{-1}\{F(s)G(s)\} = (f*g)(t)$$

Prerequisites

Before starting this Section you should ...

- be able to find Laplace transforms and inverse Laplace transforms of simple functions
- be able to integrate by parts
- understand how to use step functions in integration

Learning Outcomes

On completion you should be able to ...

- calculate the convolution of simple functions
- apply the convolution theorem to obtain inverse Laplace transforms

1. Convolution

Let $f(t)$ and $g(t)$ be two functions of t. The **convolution** of $f(t)$ and $g(t)$ is also a function of t, denoted by $(f*g)(t)$ and is defined by the relation

$$(f*g)(t) = \int_{-\infty}^{\infty} f(t-x)g(x)\,dx$$

However if f and g are both **causal** functions then (strictly) $f(t), g(t)$ are written $f(t)u(t)$ and $g(t)u(t)$ respectively, so that

$$(f*g)(t) = \int_{-\infty}^{\infty} f(t-x)u(t-x)g(x)u(x)\,dx = \int_0^t f(t-x)g(x)\,dx$$

because of the properties of the step functions: $u(t-x) = 0$ if $x > t$ and $u(x) = 0$ if $x < 0$.

Key Point 14

Convolution

If $f(t)$ and $g(t)$ are causal functions then their convolution is defined by:

$$(f*g)(t) = \int_0^t f(t-x)g(x)\,dx$$

This is an odd looking definition but it turns out to have considerable use both in Laplace transform theory and in the modelling of linear engineering systems. The reader should note that the variable of integration is x. As far as the integration process is concerned the t-variable is (temporarily) regarded as a constant.

Example 3

Find the convolution of f and g if $f(t) = tu(t)$ and $g(t) = t^2u(t)$.

Solution

$$f(t-x) = (t-x)u(t-x) \quad \text{and} \quad g(x) = x^2u(x)$$

Therefore

$$\begin{aligned}(f*g)(t) = \int_0^t (t-x)x^2\,dx &= \left[\tfrac{1}{3}x^3t - \tfrac{1}{4}x^4\right]_0^t \\ &= \tfrac{1}{3}t^4 - \tfrac{1}{4}t^4 = \tfrac{1}{12}t^4\end{aligned}$$

Example 4

Find the convolution of $f(t) = t.u(t)$ and $g(t) = \sin t.u(t)$.

Solution

Here $f(t-x) = (t-x)u(t-x)$ and $g(x) = \sin x.u(x)$ and so

$$(f * g)(t) = \int_0^t (t-x)\sin x\, dx$$

We need to integrate by parts. We find, remembering again that t is a constant in the integration process,

$$\begin{aligned}\int_0^t (t-x)\sin x\, dx &= \Big[-(t-x)\cos x\Big]_0^t - \int_0^t (-1)(-\cos x)\, dx \\ &= [0+t] - \int_0^t \cos x\, dx \\ &= t - \Big[\sin x\Big]_0^t = t - \sin t\end{aligned}$$

so that

$(f * g)(t) = t - \sin t$ or, equivalently, in this case $(t * \sin t)(t) = t - \sin t$

Task

In Example 4 we found the convolution of $f(t) = t.u(t)$ and $g(t) = \sin t.u(t)$. In this Task you are asked to find the convolution $(g * f)(t)$ that is, to reverse the order of f and g.

Begin by writing $(g * f)(t)$ as an appropriate integral:

Your solution

Answer

$g(t-x) = \sin(t-x).u(t-x)$ and $f(x) = xu(x)$, so $(g * f)(t) = \displaystyle\int_0^t \sin(t-x).x\, dx$

Now evaluate the convolution integral:

Your solution

Answer

$$
\begin{aligned}
(g * f)(t) &= \int_0^t \sin(t-x).x\,dx \\
&= \Big[x\cos(t-x)\Big]_0^t - \int_0^t \cos(t-x)\,dx \\
&= [t-0] + \Big[\sin(t-x)\Big]_0^t = t - \sin t
\end{aligned}
$$

This Task illustrates the general result in the following Key Point:

Key Point 15

Commutativity Property of Convolution

$$(f * g)(t) = (g * f)(t)$$

In words: the convolution of $f(t)$ with $g(t)$ is the same as the convolution of $g(t)$ with $f(t)$.

Task

Obtain the Laplace transforms of $f(t) = t.u(t)$ and $g(t) = \sin t.u(t)$ and $(f*g)(t)$.

Begin by finding $\mathcal{L}\{f(t)\}$, $\mathcal{L}\{g(t)\}$:

Your solution

Answer

$$\mathcal{L}\{f(t)\} = \frac{1}{s^2} \qquad \mathcal{L}\{g(t)\} = \frac{1}{s^2+1} \qquad \text{(from Table 1)}$$

Now find $\mathcal{L}\{(f*g)(t)\}$:

Your solution

Answer

From Example 4 $(f*g)(t) = t - \sin t$ and so $\mathcal{L}\{(f*g)(t)\} = \mathcal{L}\{t - \sin t\} = \dfrac{1}{s^2} - \dfrac{1}{s^2+1}$

Now compare $\mathcal{L}\{f(t)\} \times \mathcal{L}\{g(t)\}$ with $\mathcal{L}\{f*g(t)\}$. What do you observe?

Your solution

Answer

$$\mathcal{L}\{(f*g)(t)\} = \frac{1}{s^2} - \frac{1}{s^2+1} = \frac{1}{s^2}\left(\frac{1}{s^2+1}\right) = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\} = F(s)G(s)$$

We see that the Laplace transform of the convolution of $f(t)$ and $g(t)$ is the product of their separate Laplace transforms. This, in fact, is a general result which is expressed in the statement of the **convolution theorem** which we discuss in the next subsection.

2. The convolution theorem

Let $f(t)$ and $g(t)$ be causal functions with Laplace transforms $F(s)$ and $G(s)$ respectively, i.e. $\mathcal{L}\{f(t)\} = F(s)$ and $\mathcal{L}\{g(t)\} = G(s)$. Then it can be shown that

Key Point 16

The Convolution Theorem

$\mathcal{L}^{-1}\{F(s)G(s)\} = (f*g)(t)$ or equivalently $\mathcal{L}\{(f*g)(t)\} = F(s)G(s)$

Example 5

Find the inverse transform of $\dfrac{6}{s(s^2+9)}$.

(a) Using partial fractions (b) Using the convolution theorem.

Solution

(a) $\dfrac{6}{s(s^2+9)} = \dfrac{(2/3)}{s} - \dfrac{(2/3)s}{s^2+9}$ and so

$$\mathcal{L}^{-1}\left\{\frac{6}{s(s^2+9)}\right\} = \tfrac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \tfrac{2}{3}\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} = \tfrac{2}{3}u(t) - \tfrac{2}{3}\cos 3t.u(t)$$

(b) Let us choose $F(s) = \dfrac{2}{s}$ and $G(s) = \dfrac{3}{s^2+9}$ then

$f(t) = \mathcal{L}^{-1}\{F(s)\} = 2u(t)$ and $g(t) = \mathcal{L}^{-1}\{G(s)\} = \sin 3t.u(t)$

So

$$\begin{aligned}\mathcal{L}^{-1}\{F(s)G(s)\} &= (f * g)(t) \qquad \text{(by the convolution theorem)}\\ &= \int_0^t 2u(t-x)\sin 3x.u(x)\,dx\end{aligned}$$

Now the variable t can take any value from $-\infty$ to $+\infty$. If $t < 0$ then the variable of integration, x, is negative and so $u(x) = 0$. We conclude that

$(f * g)(t) = 0$ if $t < 0$

that is, $(f * g)(t)$ is a **causal function**. Let us now consider the other possibility for t, that is the range $t \geq 0$. Now, in the range of integration $0 \leq x \leq t$ and so

$u(t-x) = 1 \qquad u(x) = 1$

since both $t - x$ and x are non-negative. Therefore

$$\begin{aligned}\mathcal{L}^{-1}\{F(s)G(s)\} &= \int_0^t 2\sin 3x\,dx\\ &= \left[-\frac{2}{3}\cos 3x\right]_0^t = -\frac{2}{3}(\cos 3t - 1) \qquad t \geq 0\end{aligned}$$

Hence

$$\mathcal{L}^{-1}\{\frac{6}{s(s^2+9)}\} = -\frac{2}{3}(\cos 3t - 1)u(t)$$

which agrees with the value obtained above using the partial fraction approach.

Task

Use the convolution theorem to find the inverse transform of $H(s) = \dfrac{s}{(s-1)(s^2+1)}$.

Begin by choosing two functions of s, that is, $F(s)$ and $G(s)$:

Your solution

Answer

Although there are many possibilities it would seem sensible to choose

$$F(s) = \frac{1}{s-1} \qquad \text{and} \qquad G(s) = \frac{s}{s^2+1}$$

since, by inspection, we can write down their inverse Laplace transforms:

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathrm{e}^t u(t) \qquad \text{and} \qquad g(t) = \mathcal{L}^{-1}\{G(s)\} = \cos t.u(t)$$

Now construct the convolution integral:

Your solution

$h(t) =$

Answer

$$\begin{aligned} h(t) &= \mathcal{L}^{-1}\{H(s)\} \\ &= \mathcal{L}^{-1}\{F(s)G(s)\} \\ &= \int_0^t f(t-x)g(x)\,dx = \int_0^t \mathrm{e}^{t-x}u(t-x)\cos x.u(x)\,dx \end{aligned}$$

Now complete the evaluation of the integral, treating the cases $t < 0$ and $t \geq 0$ separately:

Your solution

Answer

You should find $h(t) = \frac{1}{2}(\sin t - \cos t + \mathrm{e}^t)u(t)$ since $h(t) = 0$ if $t < 0$ and

$$\begin{aligned}
h(t) &= \int_0^t \mathrm{e}^{t-x} \cos x \, dx \qquad \text{if} \quad t \geq 0 \\
&= \left[\mathrm{e}^{t-x} \sin x\right]_0^t - \int_0^t (-1)\mathrm{e}^{t-x} \sin x \, dx \qquad \text{(integrating by parts)} \\
&= \sin t + \left[-\mathrm{e}^{t-x} \cos x\right]_0^t - \int_0^t (-\mathrm{e}^{t-x})(-\cos x) \, dx \\
&= \sin t - \cos t + \mathrm{e}^t - h(t) \\
\text{or} \quad 2h(t) &= \sin t - \cos t + \mathrm{e}^t \qquad t \geq 0
\end{aligned}$$

Finally $h(t) = \frac{1}{2}(\sin t - \cos t + \mathrm{e}^t)u(t)$

Exercises

1. Find the convolution of

   (a) $2tu(t)$ and $t^3u(t)$ (b) $\mathrm{e}^t u(t)$ and $tu(t)$ (c) $\mathrm{e}^{-2t}u(t)$ and $\mathrm{e}^{-t}u(t)$.

   In each case reverse the order to check that $(f * g)(t) = (g * f)(t)$.

2. Use the convolution theorem to determine the inverse Laplace transforms of

   (a) $\dfrac{1}{s^2(s+1)}$ (b) $\dfrac{1}{(s-1)(s-2)}$ (c) $\dfrac{1}{(s^2+1)^2}$

Answers

1. (a) $\frac{1}{10}t^5$ (b) $-t - 1 + \mathrm{e}^t$ (c) $\mathrm{e}^{-t} - \mathrm{e}^{-2t}$
2. (a) $(t - 1 + \mathrm{e}^{-t})u(t)$ (b) $(-\mathrm{e}^t + \mathrm{e}^{2t})u(t)$ (c) $\frac{1}{2}(\sin t - t\cos t)u(t)$

Transfer Functions 20.6

Introduction

In this Section we introduce the concept of a transfer function and then use this to obtain a Laplace transform model of a linear engineering system. (A linear engineering system is one modelled by a constant coefficient ordinary differential equation.)

We shall also see how to obtain the impulse response of a linear system and hence to construct the general response by use of the convolution theorem.

Prerequisites

Before starting this Section you should ...

- be able to use the convolution theorem
- be familiar with taking Laplace transforms and inverse Laplace transforms
- be familiar with the delta (impulse) function and its Laplace transform

Learning Outcomes

On completion you should be able to ...

- find a transfer function of a linear system
- show how some linear systems may be combined together by combining appropriate transfer functions
- obtain the impulse response and the general response to a linear engineering system

1. Transfer functions and linear systems

Linear engineering systems are those that can be modelled by linear differential equations. We shall only consider those sytems that can be modelled by constant coefficient ordinary differential equations.

Consider a system modelled by the second order differential equation.

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = f(t)$$

in which a, b, c are given constants and $f(t)$ is a given function. In this context $f(t)$ is often called the **input signal** or **forcing function** and the solution $y(t)$ is often called the **output signal**.

We shall assume that the initial conditions are zero (in this case $y(0) = 0,\ y'(0) = 0$).

Now, taking the Laplace transform of the differential equation, gives:

$$(as^2 + bs + c)Y(s) = F(s)$$

in which we have used $y(0) = y'(0) = 0$ and where we have designated $\mathcal{L}\{y(t)\} = Y(s)$ and $\mathcal{L}\{f(t)\} = F(s)$.

We define the **transfer function** of a system to be the ratio of the Laplace transform of the output signal to the Laplace transform of the input signal with the initial conditions as zero. The transfer function (a function of s), is denoted by $H(s)$. In this case

$$H(s) \equiv \frac{Y(s)}{F(s)} = \frac{1}{as^2 + bs + c}$$

Now, in the *special case* in which the input signal is the delta function, $f(t) = \delta(t)$, we have $F(s) = 1$ and so,

$$H(s) = Y(s)$$

We call the solution to the differential equation *in this special case* the **unit impulse response function** and denote it by $h(t)u(t)$ (we include the step function $u(t)$ to emphasize its causality). So

$$h(t)u(t) = \mathcal{L}^{-1}\{H(s)\} \qquad \text{when} \quad f(t) = \delta(t)$$

Now, keeping this in mind and returning to the general case in which the input signal $f(t)$ is not necessarily the impulse function $\delta(t)$, we have:

$$Y(s) = H(s)F(s)$$

and so the solution for the output signal is, as usual, obtained by taking the inverse Laplace transform:

$$\begin{aligned} y(t) = \mathcal{L}^{-1}\{Y(s)\} &= \mathcal{L}^{-1}\{H(s)F(s)\} \\ &= (h * f)(t) \end{aligned}$$

using the convolution theorem.

Key Point 17

Linear System Solution

The solution to a linear system, modelled by a constant coefficient ordinary differential equation, is given by the convolution of the unit impulse response function $h(t)u(t)$ with the input function $f(t)$.

This approach provides yet another method of solving a linear system as Example 6 illustrates.

Example 6

Find the impulse response function $h(t)$ to a linear engineering system modelled by the differential equation:

$$\frac{d^2y}{dt^2} + 4y = \mathsf{e}^{-t} \qquad y(0) = 0 \quad y'(0) = 0$$

and hence solve the system.

Solution

Here

$$H(s) = \frac{1}{s^2+4} \qquad \Big(= \frac{1}{as^2+bs+c} \quad \text{with} \quad a = 1,\ b = 0,\ c = 4\Big)$$

This is obtained by replacing the forcing function e^{-t} by the impulse function $\delta(t)$ and then taking the Laplace transform. Using this:

$$h(t) = \mathcal{L}^{-1}\{H(s)\} = \mathcal{L}^{-1}\{\frac{1}{s^2+4}\} = \tfrac{1}{2}\sin 2t.u(t)$$

Then the output $y(t)$ corresponding to the input e^{-t} is given by the convolution of e^{-t} and $h(t)$. That is,

$$\begin{aligned} y(t) = (h * \mathsf{e}^{-t})(t) &= \int_0^t \tfrac{1}{2}\sin 2(t-x)\mathsf{e}^{-x}\,dx \\ &= \tfrac{1}{10}\left[\sin 2t - 2\cos 2t + 2\mathsf{e}^{-t}\right] \end{aligned}$$

(Note: the last integral can be determined by integrating by parts (twice), or by use of a computer algebra system such as Matlab.)

Task

Use the transfer function approach to solve

$$\frac{dx}{dt} - 4x = \sin t \qquad x(0) = 0.$$

Begin by finding the transfer function $H(s)$:

Your solution

Answer

You should find $H(s) = 1/(s-4)$ since the transfer function is the Laplace transform of the output $X(s)$ when the input is a delta function $\delta(t)$.

Now obtain an expression for the solution $x(t)$ in terms of the convolution:

Your solution

Answer

You should obtain $x(t) = (\sin t * h)(t)$ where

$$h(t) = \mathcal{L}^{-1}\{H(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s-4}\right\} = \mathsf{e}^{4t}u(t) \qquad \text{and} \qquad x(t) = \int_0^t (\sin x)\mathsf{e}^{4(t-x)}u(t-x)\,dx$$

Now complete the evaluation of this integral:

Your solution

Answer

If $t > 0$ then $u(t-x) = 1$ and so

$$\begin{aligned} x(t) = \int_0^t \sin x\ \mathsf{e}^{4(t-x)}\,dx &= \mathsf{e}^{4t}\left\{\left[-\frac{\sin x}{4}\mathsf{e}^{-4x}\right]_0^t - \int_0^t -\frac{\cos x}{4}\mathsf{e}^{-4x}\,dx\right\} \\ &= \mathsf{e}^{4t}\left\{-\frac{\sin t}{4}\mathsf{e}^{-4t} + \frac{1}{4}\left(\left[-\frac{\cos x}{4}\mathsf{e}^{-4x}\right]_0^t - \int_0^t \frac{\sin x}{4}\mathsf{e}^{-4x}\,dt\right)\right\} \end{aligned}$$

Therefore $\quad x(t) = -\frac{1}{4}\sin t - \frac{1}{16}\cos t + \frac{1}{16}\mathsf{e}^{4t} - \frac{1}{16}x(t)$

Hence $\quad x(t) = \dfrac{1}{17}\left(-4\sin t - \cos t + \mathsf{e}^{4t}\right)$

2. Modelling linear systems by transfer functions

We have seen previously that an engineering system can be modelled by one or more differential equations. However, with the introduction of the transfer function we have an alternative model which we examine in this Section.

It will be helpful to develop a pictorial approach to system modelling. To begin, we can imagine a differential equation:

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = f(t)$$

as being a model of the engineering system which transforms the input signal $f(t)$ into an output signal $y(t)$ (the solution of the differential equation). The **system** is characterised by the values of the coefficients a, b, c. A different engineering system will be characterised by a different set of coefficients. These coefficients are *independent* of the input signal. Changing the input signal does not change the system. It is the system that changes the input signal into the output signal. This is easy to describe pictorially (Figure 21).

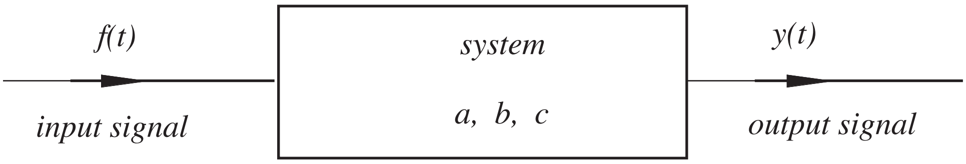


Figure 21: Block diagram describing the system in the t-domain

In a block diagram the system is represented by a rectangular box and the input and output signals represented by lines with an arrow to indicate the 'flow'.

After the Laplace transform of the differential equation is taken, the differential equation is transformed into

$$Y(s) \equiv H(s)F(s) \qquad H(s) \equiv \frac{1}{as^2 + bs + c}$$

in which $H(s)$ is the transfer function. The latter characterises (in Laplace transform terms) the engineering system from which it was derived. The relation, connecting the Laplace transform of the output $Y(s)$ to the Laplace transform of the input $F(s)$, can also be described schematically (Figure 22).

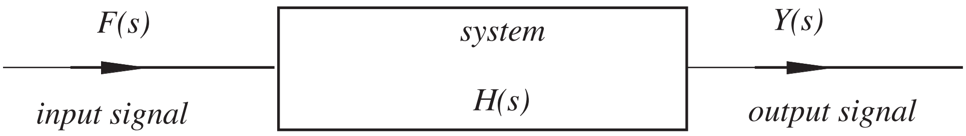


Figure 22: Block diagram describing the system in the $s-$domain

We can begin to model an engineering system directly in terms of transfer functions. In order to do this effectively we need to know how transfer functions are to be combined together. Before we do this we first extend our block diagrams to allow for 'interactions'.

There are three basic components occurring in block diagrams which we now describe.

The **first component**, we have already met: the block relating the **input to the output** (Figure 23):

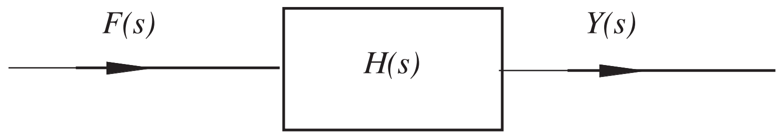


Figure 23: Input to Output

The **second component** is called a **summing point** (Figure 24):

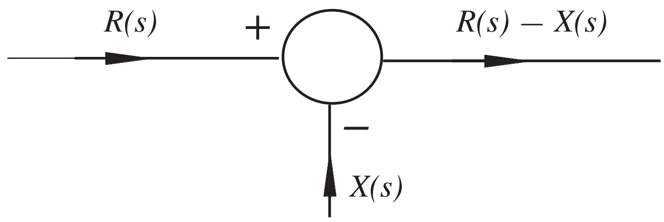


Figure 24: Summing point

Here we have shown two incoming signals $R(s), X(s)$ (but at a general summing point there may be *many* incoming signals) and one outgoing signal (there should never be more than *one* outgoing signal). The sign attached to the incoming signal defines whether the signal is adding to $(+)$ or subtracting from $(-)$ the summing point. The outgoing signal is then calculated in an obvious way, taking these signs into account.

The **third component** is a **take-off point** (Figure 25):

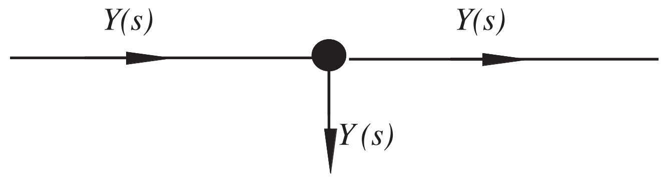


Figure 25: Take-off point

Here the value of the signal $Y(s)$ is found in such a way as not to affect the signal that is being transmitted. (This situation can never be precisely realised in practice, but using sensitive measuring devices it can be well approximated. As a simple example consider the problem of measuring the temperature of a certain volume of liquid. The act of putting a thermometer in the liquid will usually slightly affect the temperature we are trying to observe.)

An example of a block diagram is the so-called negative feedback loop, shown in Figure 26 (we are using $G(s)$ to denote the transfer function):

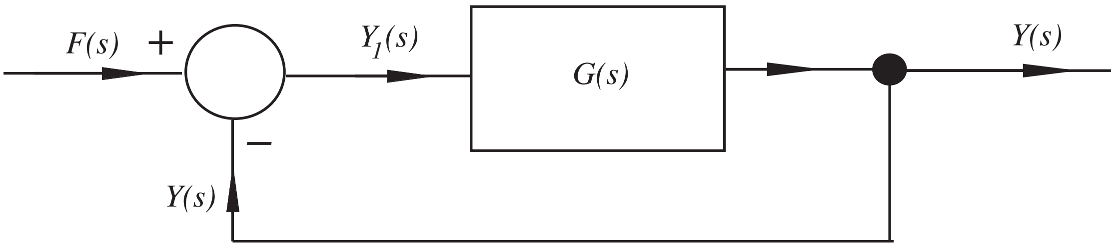


Figure 26: Negative feedback loop

Here, the output signal is tapped and subtracted from the input signal. Hence

$$Y(s) = G(s)Y_1(s)$$

because $Y_1(s)$ is the input signal to the system characterised by transfer function $G(s)$. However, at the summing point $Y_1(s) = F(s) - Y(s)$ and so

$$Y(s) = G(s)(F(s) - Y(s))$$

from which we easily obtain:

$$Y(s) = \left[\frac{G(s)}{1 + G(s)}\right] F(s)$$

so that, in terms of input and output signals, the feedback loop is characterised by a transfer function

$$\frac{G(s)}{1 + G(s)}.$$

In some feedback loops the tapped signal $Y(s)$ may be modified in some way before feedback. Using the overall transfer function we can now picture the feedback loop in a simpler way (Figure 27):

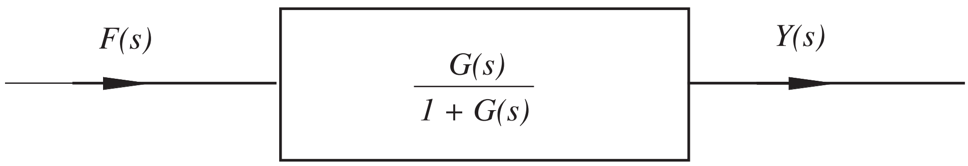


Figure 27: Feedback loop transfer function

Another type of block diagram occurs when the output from one system becomes the input to another system. For example consider the system of coupled differential equations:

$$\begin{aligned} \frac{dx}{dt} + x &= f(t) \\ 3\frac{dy}{dt} - y &= x(t) \\ x(0) = 0 \qquad & y(0) = 0 \end{aligned}$$

in which $f(t)$ is a given input signal.

In terms of Laplace transforms we have, as usual

$$sX(s) + X(s) = F(s) \qquad 3sY(s) - Y(s) = X(s)$$

so the transfer function for the first equation ($G_1(s)$ say) satisfies

$$G_1(s) \equiv \frac{X(s)}{F(s)} = \frac{1}{s+1}$$

whilst the transfer function for the second equation $G_2(s)$ satisfies

$$G_2(s) \equiv \frac{Y(s)}{X(s)} = \frac{1}{3s-1}$$

In pictorial terms this is shown in Figure 28:

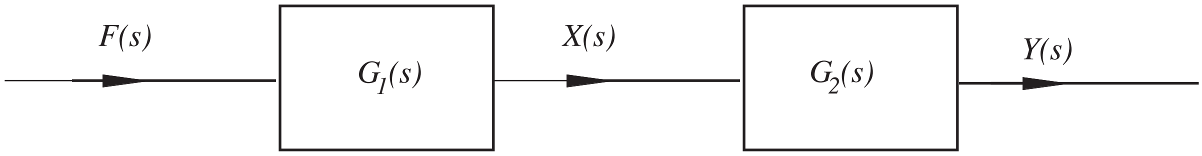


Figure 28

So we have two transfer functions 'in series'. To find how they combine we simply find an expression connecting the final output $Y(s)$ to the initial input $F(s)$. Clearly

$$X(s) = G_1(s)F(s) \quad \text{and so} \quad Y(s) = G_2(s)X(s) = [G_2(s)G_1(s)]\,F(s)$$

So transfer functions in series are simply multiplied together. In this case the overall transfer function $H(s)$ is:

$$H(s) = G_1(s)G_2(s) = \frac{1}{(s+1)(3s-1)}$$

Note that this result could be found directly from the differential equations used to model this system. If we differentiate the second differential equation of the original pair we get:

$$3\frac{d^2y}{dt^2} - \frac{dy}{dt} = \frac{dx}{dt}$$

Rearranging the first equation gives $\quad \dfrac{dx}{dt} = f(t) - x$

Substituting gives: $\quad 3\dfrac{d^2y}{dt^2} - \dfrac{dy}{dt} = f(t) - x = f(t) - \left[3\dfrac{dy}{dt} - y\right]$

or

$$3\frac{d^2y}{dt^2} + 2\frac{dy}{dt} - y = f(t)$$

which, on taking Laplace transforms, gives the s-relation $(3s^2 + 2s - 1)Y(s) = F(s)$ implying a transfer function:

$$H(s) = \frac{1}{3s^2 + 2s - 1} = \frac{1}{(s+1)(3s-1)}$$

as obtained above.

Engineering Example 3

System response

An engineering system is modelled by the block diagram in Figure 29:

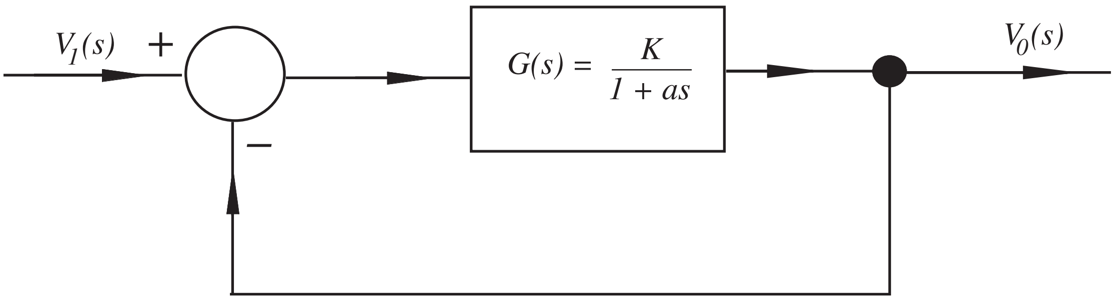


Figure 29

Determine the system response $v_0(t)$ when the input function is a unit step function when $K = 2.5$ and $a = 0$.

Solution

If the system has an overall transfer function $H(s)$ then $V_0(s) = H(s)V_1(s)$. But this particular system is the negative feedback loop described earlier and so

$$H(s) = \frac{G(s)}{1 + G(s)} = \frac{\dfrac{K}{1 + as}}{1 + \dfrac{K}{1 + as}} = \frac{K}{K + 1 + as}$$

In this particular case

$$H(s) = \frac{2.5}{3.5 + 0.5s} = \frac{5}{7 + s}$$

Thus the impulse response $h(t)$ is

$$h(t) = \mathcal{L}^{-1}\{H(s)\} = \mathcal{L}^{-1}\left\{\frac{5}{(7 + s)}\right\} = 5\mathrm{e}^{-7t}u(t)$$

and so the response to a step input $u(t)$ is given by the convolution of $h(t)$ with $u(t)$

$$\begin{aligned} v_0(t) &= \int_0^t u(t - x)5\mathrm{e}^{-7x}u(t)\,dx \\ &= \int_0^t 5\mathrm{e}^{-7x}\,dx \qquad t > 0 \\ &= \left[-\frac{5}{7}\mathrm{e}^{-7x} \right]_0^t = -\frac{5}{7}[\mathrm{e}^{-7t} - 1] \end{aligned}$$

Table of Laplace Transforms

Rule	Causal function	Laplace transform
1	$f(t)$	$F(s)$
2	$u(t)$	$\dfrac{1}{s}$
3	$t^n u(t)$	$\dfrac{n!}{s^{n+1}}$
4	$\mathsf{e}^{-at}u(t)$	$\dfrac{1}{s+a}$
5	$\sin at\,.\,u(t)$	$\dfrac{a}{s^2+a^2}$
6	$\cos at\,.\,u(t)$	$\dfrac{s}{s^2+a^2}$
7	$\mathsf{e}^{-at}\sin bt\,.\,u(t)$	$\dfrac{b}{(s+a)^2+b^2}$
8	$\mathsf{e}^{-at}\cos bt\,u(t)$	$\dfrac{s+a}{(s+a)^2+b^2}$