Contents 12

Applications of Differentiation

Learning outcomes

In this Workbook you will learn to apply your knowledge of differentiation to solve some basic problems connected with curves. First you will learn how to obtain the equation of the tangent line and the normal line to any point of interest on a curve. Secondly, you will learn how to find the positions of maxima and minima on a given curve. Thirdly, you will learn how, given an approximate position of the root of a function, a better estimate of the position can be obtained using the Newton-Raphson technique. Lastly you will learn how to characterise how sharply a curve is turning by calculating its curvature.

Tangents and Normals 12.1

Introduction

In this Section we see how the equations of the tangent line and the normal line at a particular point on the curve $y = f(x)$ can be obtained. The equations of tangent and normal lines are often written as

$$y = mx + c, \qquad y = nx + d$$

respectively. We shall show that the product of their gradients m and n is such that mn is -1 which is the condition for perpendicularity.

Prerequisites

Before starting this Section you should ...

- be able to differentiate standard functions
- understand the geometrical interpretation of a derivative
- know the trigonometric expansions of $\sin(A + B),\ \cos(A + B)$

Learning Outcomes

On completion you should be able to ...

- obtain the condition that two given lines are perpendicular
- obtain the equation of the tangent line to a curve
- obtain the equation of the normal line to a curve

1. Perpendicular lines

One form for the equation of a straight line is

$$y = mx + c$$

where m and c are constants. We remember that m is the gradient of the line and its value is the tangent of the angle θ that the line makes with the positive x-axis. The constant c is the value obtained where the line intersects the y-axis. See Figure 1:

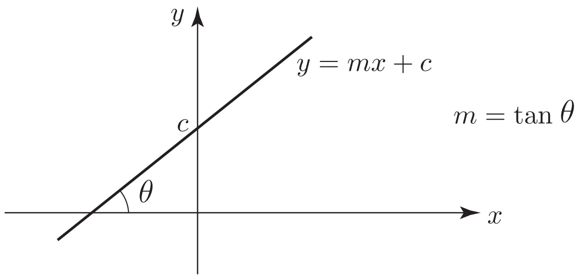


Figure 1

If we have a second line, with equation

$$y = nx + d$$

then, unless $m = n$, the two lines will intersect at one point. These are drawn together in Figure 2. The second line makes an angle ψ with the positive x-axis.

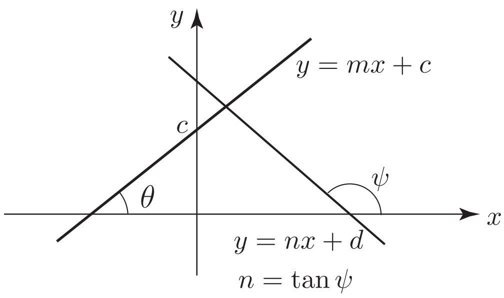


Figure 2

A simple question to ask is "what is the relation between m and n if the lines are perpendicular?" If the lines are perpendicular, as shown in Figure 3, the angles θ and ψ must satisfy the relation:

$$\psi - \theta = 90^\circ$$

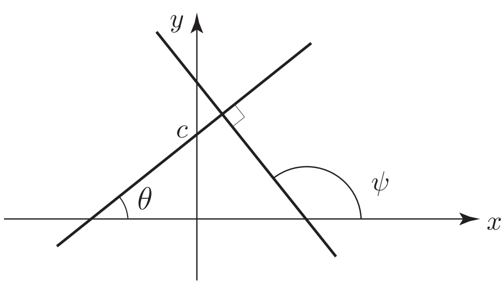


Figure 3

This is true since the angles in a triangle add up to $180°$. According to the figure the three angles are $90°$, θ and $180° - \psi$. Therefore

$$180° = 90° + \theta + (180° - \psi) \qquad \text{implying} \qquad \psi - \theta = 90°$$

In this special case that the lines are perpendicular or **normal** to each other the relation between the gradients m and n is easily obtained. In this deduction we use the following basic trigonometric relations and identities:

$$\sin(A - B) \equiv \sin A \cos B - \cos A \sin B \qquad \cos(A - B) \equiv \cos A \cos B + \sin A \sin B$$

$$\tan A \equiv \frac{\sin A}{\cos A} \qquad \sin 90° = 1 \qquad \cos 90° = 0$$

Now

$$\begin{aligned} m &= \tan\theta \\ &= \tan(\psi - 90^o) \qquad \text{(see Figure 3)} \\ &= \frac{\sin(\psi - 90^o)}{\cos(\psi - 90^o)} \\ &= \frac{-\cos\psi}{\sin\psi} = -\frac{1}{\tan\psi} = -\frac{1}{n} \end{aligned}$$

So $\quad mn = -1$

Key Point 1

Two straight lines $y = mx + c$, $y = nx + d$ are perpendicular if

$$m = -\frac{1}{n} \qquad \text{or equivalently} \qquad mn = -1$$

This result assumes that neither of the lines are parallel to the x-axis or to the y-axis, as in such cases one gradient will be zero and the other infinite.

Exercise

Which of the following pairs of lines are perpendicular?

(a) $y = -x + 1, \quad y = x + 1$

(b) $y + x - 1 = 0, \quad y + x - 2 = 0$

(c) $2y = 8x + 3, \quad y = -0.25x - 1$

Answer

(a) perpendicular (b) not perpendicular (c) perpendicular

2. Tangents and normals to a curve

As we know, the relationship between an independent variable x and a dependent variable y is denoted by

$$y = f(x)$$

As we also know, the geometrical interpretation of this relation takes the form of a curve in an xy plane as illustrated in Figure 4.

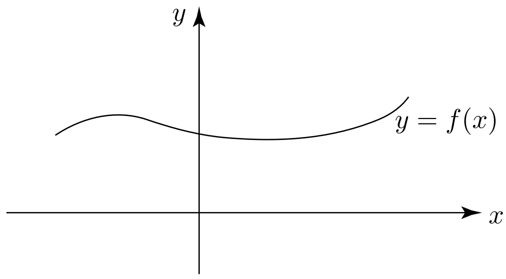


Figure 4

We know how to calculate a value of y given a value of x. We can either do this graphically (which is inaccurate) or else use the function itself. So, at an x value of x_0 the corresponding y value is y_0 where

$$y_0 = f(x_0)$$

Let us examine the curve in the neighbourhood of the point (x_0, y_0). There are two important constructions of interest

- the tangent line at (x_0, y_0)
- the normal line at (x_0, y_0)

These are shown in Figure 5.

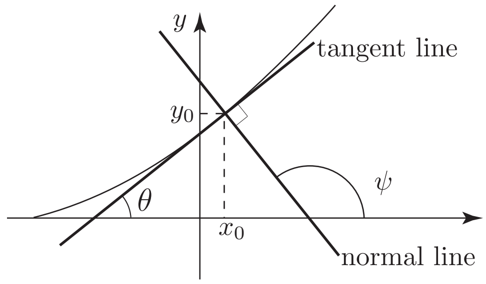
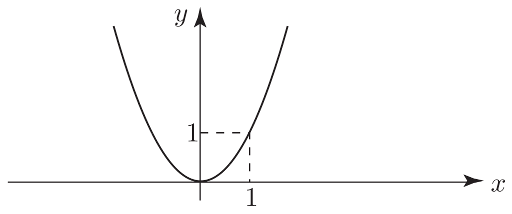


Figure 5

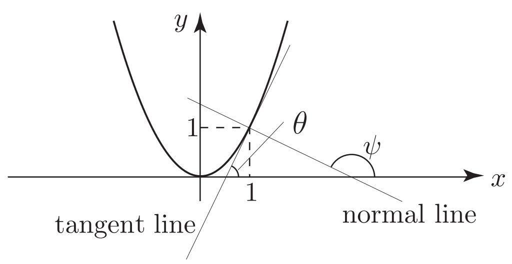
We note the geometrically obvious fact: the tangent and normal lines at any given point on a curve are perpendicular to each other.

The curve $y = x^2$ is drawn below. On this graph draw the tangent line and the normal line at the point $(x_0 = 1, y_0 = 1)$:

Your solution



Answer



From your graph, estimate the values of θ and ψ in degrees. (You will need a protractor.)

Your solution

$\theta \simeq$ $\qquad\qquad$ $\psi \simeq$

Answer

$\theta \approx 63.4^o$ $\qquad\qquad$ $\psi \approx 153.4^o$

Returning to the curve $y = f(x)$: we know, from the geometrical interpretation of the derivative that

$$\left.\frac{df}{dx}\right|_{x_0} = \tan\theta$$

(the notation $\left.\dfrac{df}{dx}\right|_{x_0}$ means evaluate $\dfrac{df}{dx}$ at the value $x = x_0$)

Here θ is the angle the tangent line to the curve $y = f(x)$ makes with the positive x-axis. This is highlighted in Figure 6:

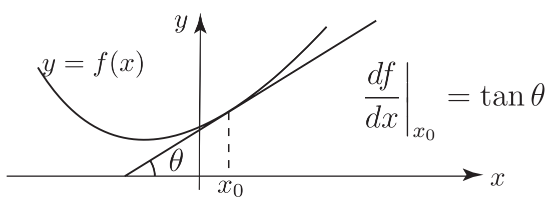


Figure 6

3. The tangent line to a curve

Let the equation of the tangent line to the curve $y = f(x)$ at the point (x_0, y_0) be:

$$y = mx + c$$

where m and c are constants to be found. The line just touches the curve $y = f(x)$ at the point (x_0, y_0) so, at this point both must have the **same value** for the derivative. That is:

$$m = \left.\frac{df}{dx}\right|_{x_0}$$

Since we know (in any particular case) $f(x)$ and the value x_0 we can readily calculate the value for m. The value of c is found by using the fact that the tangent line and the curve pass through the same point (x_0, y_0).

$$y_0 = mx_0 + c \qquad \text{and} \qquad y_0 = f(x_0)$$

Thus $\qquad mx_0 + c = f(x_0) \qquad$ leading to $\qquad c = f(x_0) - mx_0$

Key Point 2

The equation of the tangent line to the curve $y = f(x)$ at the point (x_0, y_0) is

$$y = mx + c \quad \text{where} \quad m = \left.\frac{df}{dx}\right|_{x_0} \quad \text{and} \quad c = f(x_0) - mx_0$$

Alternatively, the equation is $\quad y - y_0 = m(x - x_0) \quad$ where $\quad m = \left.\frac{df}{dx}\right|_{x_0} \quad$ and $\quad y_0 = f(x_0)$

Example 1

Find the equation of the tangent line to the curve $y = x^2$ at the point (1,1).

Solution

Method 1

Here $f(x) = x^2$ and $x_0 = 1$ thus $\dfrac{df}{dx} = 2x \quad \therefore \quad m = \left.\dfrac{df}{dx}\right|_{x_0} = 2$

Also $c = f(x_0) - mx_0 = f(1) - m = 1 - 2 = -1$. The tangent line has equation $y = 2x - 1$.

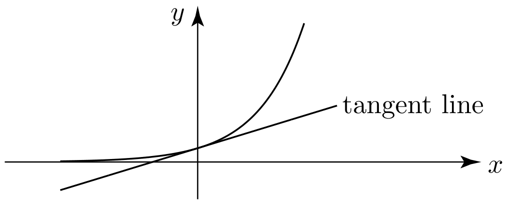
Method 2

$$y_0 = f(x_0) = f(1) = 1^2 = 1$$

The tangent line has equation $\qquad y - 1 = 2(x - 1) \qquad \rightarrow \qquad y = 2x - 1$

Task

Find the equation of the tangent line to the curve $y = \mathrm{e}^x$ at the point $x = 0$. The curve and the line are displayed in the following figure:



First specify x_0 and f:

Your solution

$x_0 =$

$f(x) =$

Answer

$x_0 = 0 \qquad f(x) = \mathrm{e}^x$

Now obtain the values of $\left.\dfrac{df}{dx}\right|_{x_0}$ and $f(x_0) - mx_0$:

Your solution

$\left.\dfrac{df}{dx}\right|_{x_0} =$

$f(x_0) - mx_0 =$

Answer

$\dfrac{df}{dx} = \mathrm{e}^x \qquad \therefore \qquad \left.\dfrac{df}{dx}\right|_0 = 1$ and $f(0) - 1(0) = \mathrm{e}^0 - 0 = 1$

Now obtain the equation of the tangent line:

Your solution

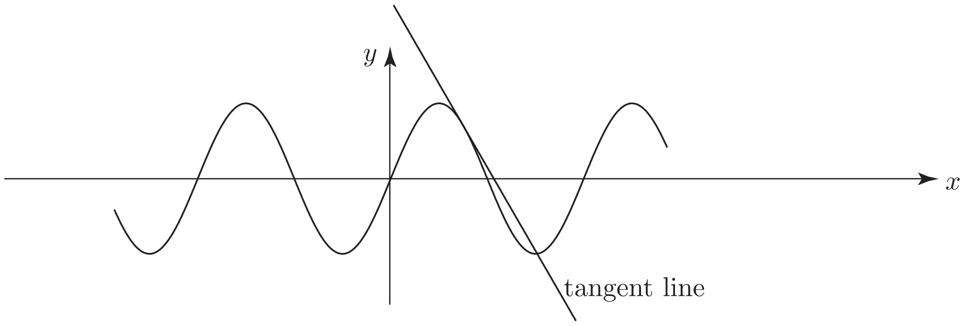
$y =$

Answer

$y = x + 1$

Task

Find the equation of the tangent line to the curve $y = \sin 3x$ at the point $x = \dfrac{\pi}{4}$ and find where the tangent line intersects the x-axis. See the following figure:



First specify x_0 and f:

Your solution

$x_0 =$ $\qquad$ $f(x) =$

Answer

$x_0 = \dfrac{\pi}{4} \qquad f(x) = \sin 3x$

Now obtain the values of $\left.\dfrac{df}{dx}\right|_{x_0}$ and $f(x_0) - mx_0$ correct to 2 d.p.:

Your solution

$\left.\dfrac{df}{dx}\right|_{x_0} =$ $\qquad$ $f(x_0) - mx_0 =$

Answer

$$\frac{df}{dx} = 3\cos 3x \qquad \therefore \qquad \left.\frac{df}{dx}\right|_{\frac{\pi}{4}} = 3\cos\frac{3\pi}{4} = -\frac{3}{\sqrt{2}} = -2.12$$

and $f\left(\dfrac{\pi}{4}\right) - \dfrac{m\pi}{4} = \sin\dfrac{3\pi}{4} - \left(\dfrac{-3}{\sqrt{2}}\right)\dfrac{\pi}{4} = \dfrac{1}{\sqrt{2}} + \dfrac{3}{\sqrt{2}}\dfrac{\pi}{4} = 2.37$ to 2 d.p.

Now obtain the equation of the tangent line:

Your solution

$y =$

Answer

$y = \dfrac{-3}{\sqrt{2}}x + \dfrac{1}{4\sqrt{2}}(4 + 3\pi)$ so $y = -2.12x + 2.37$ (to 2 d.p.)

Where does the line intersect the x-axis?

Your solution

$x =$

Answer

When $y = 0 \quad \therefore \quad -2.12x + 2.37 = 0 \quad \therefore \quad x = 1.12$ to 2 d.p.

4. The normal line to a curve

We have already noted that, at any point (x_0, y_0) on a curve $y = f(x)$, the tangent and normal lines are perpendicular. Thus if the equations of the tangent and normal lines are, respectively

$$y = mx + c \qquad y = nx + d$$

then $m = -\dfrac{1}{n}$ or, equivalently $n = -\dfrac{1}{m}$.
We have also noted, for the tangent line

$$m = \left.\frac{df}{dx}\right|_{x_0}$$

so n can easily be obtained. To find d, we again use the fact that the normal line $y = nx + d$ and the curve have a point in common:

$$y_0 = nx_0 + d \qquad \text{and} \qquad y_0 = f(x_0)$$

so $nx_0 + d = f(x_0)$ leading to $d = f(x_0) - nx_0$.

Task

Find the equation of the normal line to curve $y = \sin 3x$ at the point $x = \dfrac{\pi}{4}$.
[The equation of the tangent line was found in the previous Task.]

First find the value of m:

Your solution

$m = \left.\dfrac{df}{dx}\right|_{\frac{\pi}{4}} =$

Answer

$m = \dfrac{-3}{\sqrt{2}}$

Hence find the value of n:

Your solution

$n =$

Answer

$nm = -1 \qquad \therefore \qquad n = \dfrac{\sqrt{2}}{3}$

The equation of the normal line is $y = \frac{\sqrt{2}}{3}x + d$. Now find the value of d to 2 d.p.. (Remember the normal line must pass through the curve at the point $x = \frac{\pi}{4}$.)

Your solution

Answer

$$\frac{\sqrt{2}}{3}\left(\frac{\pi}{4}\right) + d = \sin\frac{\pi}{4} \quad \therefore \quad d = \frac{1}{\sqrt{2}} - \frac{\sqrt{2}}{3}\frac{\pi}{4} \simeq 0.34$$

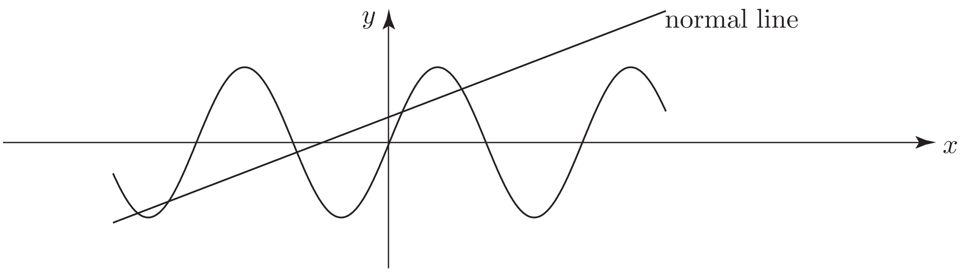
Now obtain the equation of the normal line to 2 d.p.:

Your solution

$y =$

Answer

$y = 0.47x + 0.34$. The curve and the normal line are shown in the following figure:



Task

Find the equation of the normal line to the curve $y = x^3$ at $x = 1$.

First find $f(x),\ x_0,\ \left.\frac{df}{dx}\right|_{x_0},\ m,\ n$:

Your solution

Answer

$$f(x) = x^3,\ x_0 = 1,\ \left.\frac{df}{dx}\right|_1 = \left.3x^2\right|_1 = 3 \qquad \therefore \qquad m = 3 \text{ and } n = -\frac{1}{3}$$

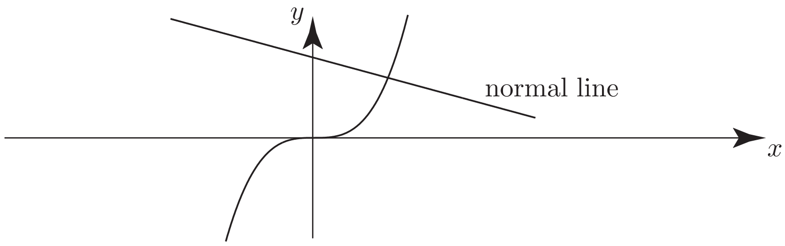
Now use the property that the normal line $y = nx + d$ and the curve $y = x^3$ pass through the point $(1, 1)$ to find d and so obtain the equation of the normal line:

Your solution

$d =$ $\qquad\qquad$ $y =$

Answer

$1 = n + d \quad \therefore \quad d = 1 - n = 1 + \dfrac{1}{3} = \dfrac{4}{3}$. Thus the equation of the normal line is $y = -\dfrac{1}{3}x + \dfrac{4}{3}$. The curve and the normal line through $(1, 1)$ are shown below:



Exercises

1. Find the equations of the tangent and normal lines to the following curves at the points indicated

   (a) $y = x^4 + 2x^2, \quad (1, 3)$

   (b) $y = \sqrt{1 - x^2}, \quad (\dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2})$ $\quad$ What would be obtained if the point was $(1, 0)$?

   (c) $y = x^{1/2}, \quad (1, 1)$

2. Find the value of a if the two curves $y = e^{-x}$ and $y = e^{ax}$ are to intersect at right-angles.

Answers

1. (a) $f(x) = x^4 + 2x^2 \qquad \dfrac{df}{dx} = 4x^3 + 4x, \qquad \left.\dfrac{df}{dx}\right|_{x=1} = 8$

tangent line $y = 8x + c$. This passes through $(1, 3)$ so $\quad y = 8x - 5$

normal line $y = -\dfrac{1}{8}x +$ d. This passes through $(1, 3)$ so $\quad y = -\dfrac{1}{8}x + \dfrac{25}{8}$.

(b) $f(x) = \sqrt{1 - x^2} \quad \dfrac{df}{dx} = \dfrac{-x}{\sqrt{1 - x^2}} \quad \left.\dfrac{df}{dx}\right|_{x=\frac{\sqrt{2}}{2}} = -1$

tangent line $y = -x + c$. This passes through $\left(\dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2}\right)$ so $\quad y = -x + \sqrt{2}$

normal line $y = x + d$. This passes through $\left(\dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2}\right)$ so $\quad y = x$.

At $(1, 0)$ the tangent line is $x = 1$ and the gradient is infinite (the line is vertical), and the normal line is $y = 0$.

(c) $f(x) = x^{\frac{1}{2}} \qquad \dfrac{df}{dx} = \dfrac{1}{2}x^{-\frac{1}{2}} \qquad \left.\dfrac{df}{dx}\right|_{x=1} = \dfrac{1}{2}$

tangent line: $y = \dfrac{1}{2}x + c$. This passes through $(1, 1)$ so $\quad y = \dfrac{1}{2}x + \dfrac{1}{2}$

normal line: $y = -2x + d$. This passes through $(1, 1)$ so $\quad y = -2x + 3$.

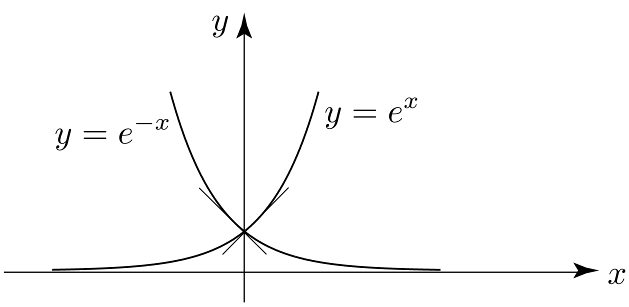
2. The curves will intersect at right-angles if their tangent lines, at the point of intersection, are perpendicular.

Point of intersection: $\quad e^{-x} = e^{ax} \quad$ i.e. $\quad -x = ax \quad \therefore \quad x = 0 \quad (a = -1$ not sensible$)$

The tangent line to $y = e^{ax}$ is $y = mx + c$ where $\quad m = ae^{ax}\big|_{x=0} = a$

The tangent line to $y = e^{-x}$ is $y = kx + g$ where $\quad k = -e^{-x}\big|_{x=0} = -1$

These two lines are perpendicular if $\quad a(-1) = -1 \qquad$ i.e. $\qquad a = 1$.



Maxima and Minima 12.2

Introduction

In this Section we analyse curves in the 'local neighbourhood' of a stationary point and, from this analysis, deduce necessary conditions satisfied by local maxima and local minima. Locating the maxima and minima of a function is an important task which arises often in applications of mathematics to problems in engineering and science. It is a task which can often be carried out using only a knowledge of the derivatives of the function concerned. The problem breaks into two parts

- finding the stationary points of the given functions
- distinguishing whether these stationary points are maxima, minima or, exceptionally, points of inflection.

This Section ends with maximum and minimum problems from engineering contexts.

Prerequisites

Before starting this Section you should ...

- be able to obtain first and second derivatives of simple functions
- be able to find the roots of simple equations

Learning Outcomes

On completion you should be able to ...

- explain the difference between local and global maxima and minima
- describe how a tangent line changes near a maximum or a minimum
- locate the position of stationary points
- use knowledge of the second derivative to distinguish between maxima and minima

1. Maxima and minima

Consider the curve

$$y = f(x) \qquad a \leq x \leq b$$

shown in Figure 7:

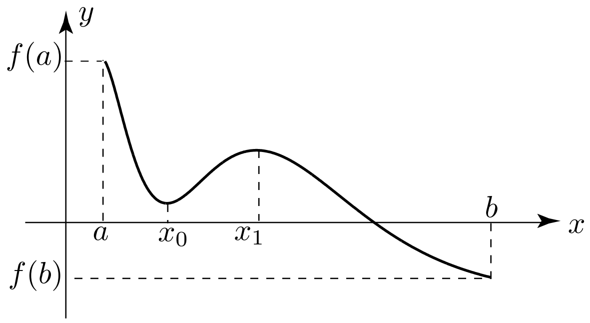


Figure 7

By inspection we see that there is no y-value greater than that at $x = a$ (i.e. $f(a)$) and there is no value smaller than that at $x = b$ (i.e. $f(b)$). However, the points on the curve at x_0 and x_1 merit comment. It is clear that in the **near neighbourhood** of x_0 all the y-values are greater than the y-value at x_0 and, similarly, in the near neighbourhood of x_1 all the y-values are less than the y-value at x_1.

We say $f(x)$ has a **global maximum** at $x = a$ and a **global minimum** at $x = b$ but also has a **local minimum** at $x = x_0$ and a **local maximum** at $x = x_1$.

Our primary purpose in this Section is to see how we might locate the position of the local maxima and the local minima for a smooth function $f(x)$.

A **stationary point** on a curve is one at which the derivative has a zero value. In Figure 8 we have sketched a curve with a maximum and a curve with a minimum.

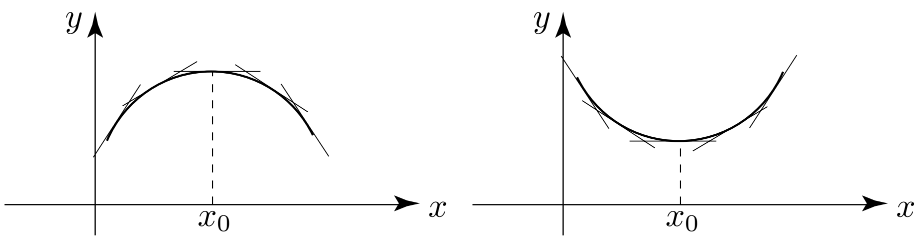


Figure 8

By drawing tangent lines to these curves in the near neighbourhood of the local maximum and the local minimum it is obvious that at these points the tangent line is parallel to the x-axis so that

$$\left.\frac{df}{dx}\right|_{x_0} = 0$$

Key Point 3

Points on the curve $y = f(x)$ at which $\dfrac{df}{dx} = 0$ are called **stationary points** of the function.

However, be careful! A stationary point is not necessarily a local maximum or minimum of the function but may be an exceptional point called a point of inflection, illustrated in Figure 9.

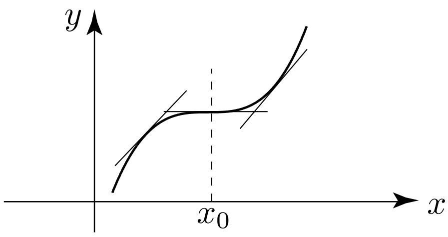


Figure 9

Example 2

Sketch the curve $y = (x-2)^2 + 2$ and locate the stationary points on the curve.

Solution

Here $f(x) = (x-2)^2 + 2$ so $\dfrac{df}{dx} = 2(x-2)$.

At a stationary point $\dfrac{df}{dx} = 0$ so we have $2(x-2) = 0$ so $x = 2$. We conclude that this function has just one stationary point located at $x = 2$ (where $y = 2$).

By sketching the curve $y = f(x)$ it is clear that this stationary point is a local **minimum**.

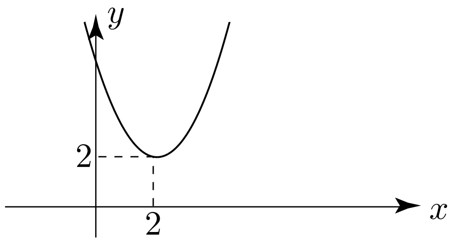


Figure 10

Task

Locate the position of the stationary points of $f(x) = x^3 - 1.5x^2 - 6x + 10.$

First find $\dfrac{df}{dx}$:

Your solution

$\dfrac{df}{dx} =$

Answer

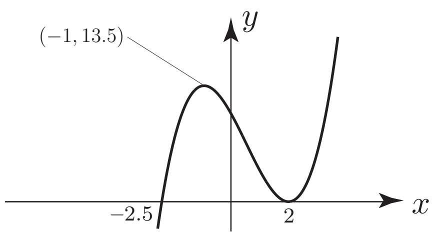
$\dfrac{df}{dx} = 3x^2 - 3x - 6$

Now locate the stationary points by solving $\dfrac{df}{dx} = 0$:

Your solution

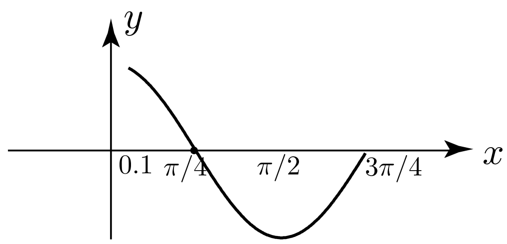
Answer

$3x^2 - 3x - 6 = 3(x + 1)(x - 2) = 0$ so $x = -1$ or $x = 2$. When $x = -1$, $f(x) = 13.5$ and when $x = 2$, $f(x) = 0$, so the stationary points are $(-1, 13.5)$ and $(2, 0)$. We have, in the figure, sketched the curve which confirms our deductions.

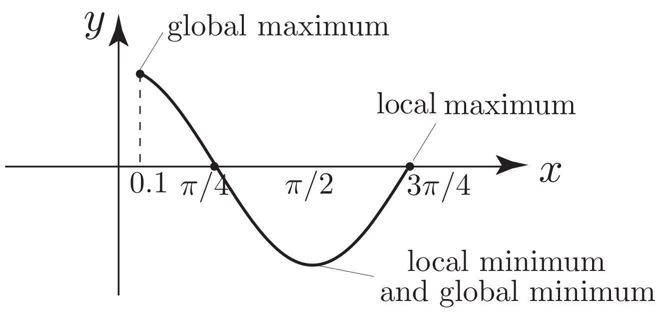


Sketch the curve $y = \cos 2x \qquad 0.1 \leq x \leq \dfrac{3\pi}{4}$ and on it locate the position of the global maximum, global minimum and any local maxima or minima.

Your solution



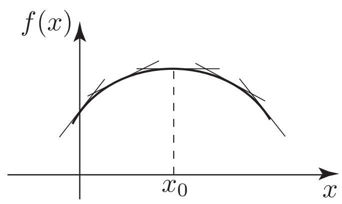
Answer



2. Distinguishing between local maxima and minima

We might ask if it is possible to predict when a stationary point is a local maximum, a local minimum or a point of inflection without the necessity of drawing the curve. To do this we highlight the general characteristics of curves in the neighbourhood of local maxima and minima.

For example: at a local maximum (located at x_0 say) Figure 11 describes the situation:



to the left of the maximum $\dfrac{df}{dx} > 0$

to the right of the maximum $\dfrac{df}{dx} < 0$

Figure 11

If we draw a graph of the **derivative** $\dfrac{df}{dx}$ against x then, near a local maximum, it **must** take one of two basic shapes described in Figure 12:

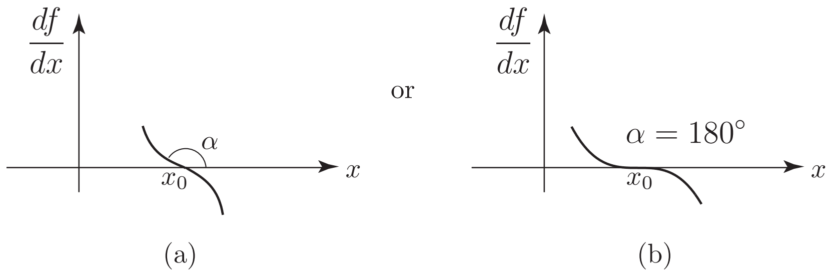
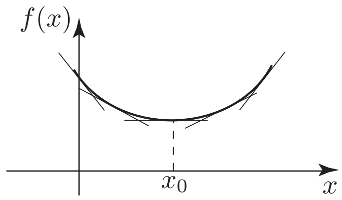


Figure 12

In case (a) $\left.\dfrac{d}{dx}\left(\dfrac{df}{dx}\right)\right|_{x_0} \equiv \tan\alpha < 0$ whilst in case (b) $\left.\dfrac{d}{dx}\left(\dfrac{df}{dx}\right)\right|_{x_0} = 0$

We reach the conclusion that at a stationary point which is a maximum the value of the second derivative $\dfrac{d^2f}{dx^2}$ is either negative or zero.

Near a local minimum the above graphs are inverted. Figure 13 shows a local minimum.



to the left of the minimum $\dfrac{df}{dx} < 0$

to the right of the minimum $\dfrac{df}{dx} > 0$

Figure 13

Figure 134 shows the two possible graphs of the derivative:

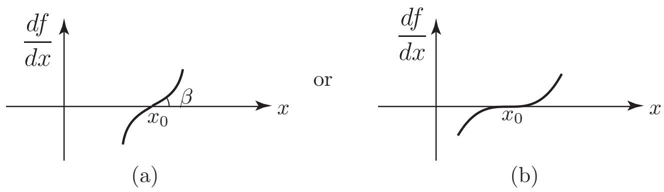
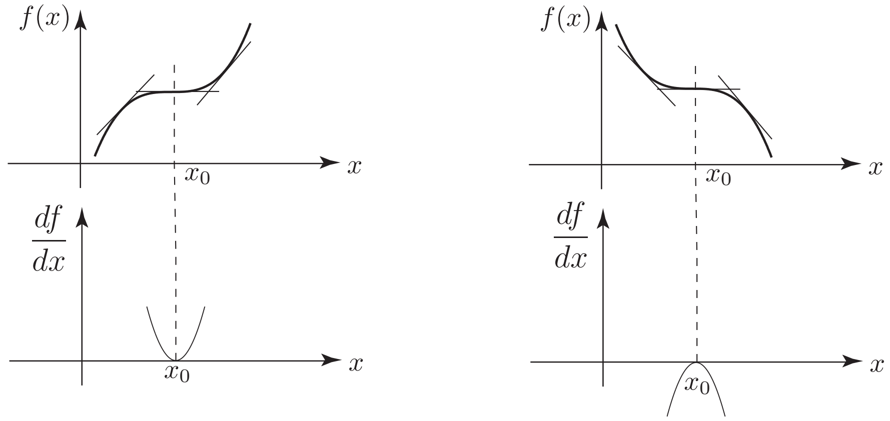


Figure 14

Here, for case (a) $\left.\dfrac{d}{dx}\left(\dfrac{df}{dx}\right)\right|_{x_0} = \tan\beta > 0$ whilst in (b) $\left.\dfrac{d}{dx}\left(\dfrac{df}{dx}\right)\right|_{x_0} = 0.$

In this case we conclude that at a stationary point which is a minimum the value of the second derivative $\dfrac{d^2f}{dx^2}$ is either positive or zero.

For the third possibility for a stationary point - a point of inflection - the graph of $f(x)$ against x and of $\dfrac{df}{dx}$ against x take one of two forms as shown in Figure 15.



to the left of x_0	$\dfrac{df}{dx} > 0$	to the left of x_0	$\dfrac{df}{dx} < 0$
to the right of x_0	$\dfrac{df}{dx} > 0$	to the right of x_0	$\dfrac{df}{dx} < 0$

Figure 15

For either of these cases $\quad \dfrac{d}{dx}\left(\dfrac{df}{dx}\right)\Bigg|_{x_0} = 0$

The sketches and analysis of the shape of a curve $y = f(x)$ in the near neighbourhood of stationary points allow us to make the following important deduction:

Key Point 4

If x_0 locates a stationary point of $f(x)$, so that $\left.\dfrac{df}{dx}\right|_{x_0} = 0$, then the stationary point

is a local minimum if $\quad \left.\dfrac{d^2f}{dx^2}\right|_{x_0} > 0$

is a local maximum if $\quad \left.\dfrac{d^2f}{dx^2}\right|_{x_0} < 0$

is inconclusive if $\quad \left.\dfrac{d^2f}{dx^2}\right|_{x_0} = 0$

Example 3

Find the stationary points of the function $f(x) = x^3 - 6x$.

Are these stationary points local maxima or local minima?

Solution

$\dfrac{df}{dx} = 3x^2 - 6$. At a stationary point $\dfrac{df}{dx} = 0$ so $3x^2 - 6 = 0$, implying $x = \pm\sqrt{2}$.

Thus $f(x)$ has stationary points at $x = \sqrt{2}$ and $x = -\sqrt{2}$. To decide if these are maxima or minima we examine the value of the second derivative of $f(x)$ at the stationary points.

$\dfrac{d^2f}{dx^2} = 6x$ so $\left.\dfrac{d^2f}{dx^2}\right|_{x=\sqrt{2}} = 6\sqrt{2} > 0$. Hence $x = \sqrt{2}$ locates a local minimum.

Similarly $\left.\dfrac{d^2f}{dx^2}\right|_{x=-\sqrt{2}} = -6\sqrt{2} < 0$. Hence $x = -\sqrt{2}$ locates a local maximum.

A sketch of the curve confirms this analysis:

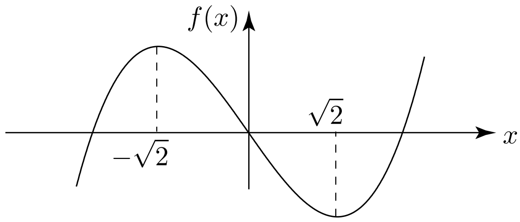


Figure 16

Task

For the function $f(x) = \cos 2x$, $0.1 \leq x \leq 6$, find the positions of any local minima or maxima and distinguish between them.

Calculate the first derivative and locate stationary points:

Your solution

$\dfrac{df}{dx} =$

Stationary points are located at:

Answer

$\dfrac{df}{dx} = -2\sin 2x.$

Hence stationary points are at values of x in the range specified for which $\sin 2x = 0$ i.e. at $2x = \pi$ or $2x = 2\pi$ or $2x = 3\pi$ (making sure x is within the range $0.1 \leq x \leq 6$)

$\therefore$ Stationary points at $x = \dfrac{\pi}{2},\ x = \pi,\ x = \dfrac{3\pi}{2}$

Now calculate the second derivative:

Your solution

$$\frac{d^2f}{dx^2} =$$

Answer

$\dfrac{d^2f}{dx^2} = -4\cos 2x$

Finally: evaluate the second derivative at each stationary points and draw appropriate conclusions:

Your solution

$$\left.\frac{d^2f}{dx^2}\right|_{x=\frac{\pi}{2}} =$$

$$\left.\frac{d^2f}{dx^2}\right|_{x=\pi} =$$

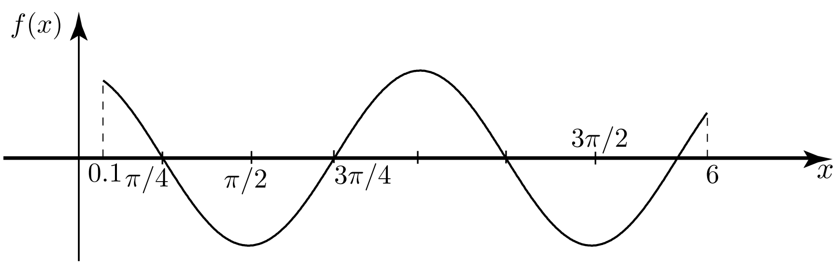
$$\left.\frac{d^2f}{dx^2}\right|_{x=\frac{3\pi}{2}} =$$

Answer

$\left.\dfrac{d^2f}{dx^2}\right|_{x=\frac{\pi}{2}} = -4\cos\pi = 4 > 0 \qquad \therefore \qquad x = \dfrac{\pi}{2}$ locates a local minimum.

$\left.\dfrac{d^2f}{dx^2}\right|_{x=\pi} = -4\cos 2\pi = -4 < 0 \qquad \therefore \qquad x = \pi$ locates a local maximum.

$\left.\dfrac{d^2f}{dx^2}\right|_{x=\frac{3\pi}{2}} = -4\cos 3\pi = 4 > 0 \qquad \therefore \qquad x = \dfrac{3\pi}{2}$ locates a local minimum.



Task

Determine the local maxima and/or minima of the function $y = x^4 - \frac{1}{3}x^3$

First obtain the positions of the stationary points:

Your solution

$$f(x) = x^4 - \frac{1}{3}x^3 \qquad \frac{df}{dx} =$$

Thus $\frac{df}{dx} = 0$ when:

Answer

$$\frac{df}{dx} = 4x^3 - x^2 = x^2(4x-1) \qquad \frac{df}{dx} = 0 \text{ when } x = 0 \text{ or when } x = 1/4$$

Now obtain the value of the second derivatives at the stationary points:

Your solution

$$\frac{d^2f}{dx^2} = \qquad \therefore \qquad \left.\frac{d^2f}{dx^2}\right|_{x=0} =$$

$$\left.\frac{d^2f}{dx^2}\right|_{x=1/4} =$$

Answer

$$\frac{d^2f}{dx^2} = 12x^2 - 2x \qquad \left.\frac{d^2f}{dx^2}\right|_{x=0} = 0, \quad \text{which is inconclusive.}$$

$$\left.\frac{d^2f}{dx^2}\right|_{x=1/4} = \frac{12}{16} - \frac{1}{2} = \frac{1}{4} > 0 \quad \text{Hence } x = \frac{1}{4} \text{ locates a local minimum.}$$

Using this analysis we cannot decide whether the stationary point at $x = 0$ is a local maximum, minimum or a point of inflection. However, just to the left of $x = 0$ the value of $\frac{df}{dx}$ (which equals $x^2(4x-1)$) is negative whilst just to the right of $x = 0$ the value of $\frac{df}{dx}$ is negative again. Hence the stationary point at $x = 0$ is a **point of inflection**. This is confirmed by sketching the curve as in Figure 17.

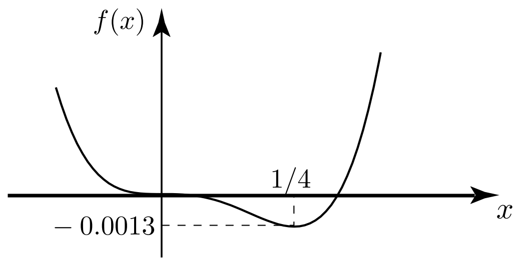
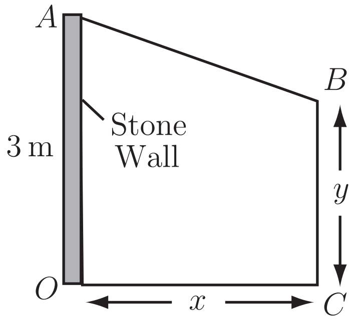


Figure 17

A materials store is to be constructed next to a 3 metre high stone wall (shown as OA in the cross section in the diagram). The roof (AB) and front (BC) are to be constructed from corrugated metal sheeting. Only 6 metre length sheets are available. Each of them is to be cut into two parts such that one part is used for the roof and the other is used for the front. Find the dimensions x, y of the store that result in the maximum cross-sectional area. Hence determine the maximum cross-sectional area.



Your solution

Answer

Note that the store has the shape of a trapezium. So the cross-sectional area (A) of the store is given by the formula: Area = average length of parallel sides × distance between parallel sides:

$$A = \frac{1}{2}(y+3)x \qquad (1)$$

The lengths x and y are related through the fact that $AB + BC = 6$, where $BC = y$ and $AB = \sqrt{x^2 + (3-y)^2}$. Hence $\sqrt{x^2 + (3-y)^2} + y = 6$. This equation can be rearranged in the following way:

$$\sqrt{x^2 + (3-y)^2} = 6 - y \quad \Leftrightarrow \quad x^2 + (3-y)^2 = (6-y)^2 \text{ i.e. } \quad x^2 + 9 - 6y + y^2 = 36 - 12y + y^2$$

which implies that $\qquad x^2 + 6y = 27 \qquad (2)$

It is necessary to eliminate either x or y from (1) and (2) to obtain an equation in a single variable. Using y instead of x as the variable will avoid having square roots appearing in the expression for the cross-sectional area. Hence from Equation (2)

$$y = \frac{27 - x^2}{6} \qquad (3)$$

Substituting for y from Equation (3) into Equation (1) gives

$$A = \frac{1}{2}\left(\frac{27 - x^2}{6} + 3\right)x = \frac{1}{2}\left(\frac{27 - x^2 + 18}{6}\right)x = \frac{1}{12}\left(45x - x^3\right) \qquad (4)$$

To find turning points, we evaluate $\dfrac{dA}{dx}$ from Equation (4) to get

$$\frac{dA}{dx} = \frac{1}{12}(45 - 3x^2) \qquad (5)$$

Solving the equation $\dfrac{dA}{dx} = 0$ gives $\dfrac{1}{12}(45 - 3x^2) = 0 \Rightarrow \qquad 45 - 3x^2 = 0$

Hence $\quad x = \pm\sqrt{15} = \pm\, 3.8730.$ Only $x > 0$ is of interest, so

$$x = \sqrt{15} = 3.87306 \qquad (6)$$

gives the required turning point.

Check: Differentiating Equation (5) and using the positive x solution (6) gives

$$\frac{d^2A}{dx^2} = -\frac{6x}{12} = -\frac{x}{2} = -\frac{3.8730}{2} < 0$$

Since the second derivative is negative then the cross-sectional area is a maximum. This is the only turning point identified for $A > 0$ and it is identified as a maximum. To find the corresponding value of y, substitute $x = 3.8730$ into Equation (3) to get $y = \dfrac{27 - 3.8730^2}{6} = 2.0000$

So the values of x and y that yield the maximum cross-sectional area are 3.8730 m and 2.00000 m respectively. To find the maximum cross-sectional area, substitute for $x = 3.8730$ into Equation (5) to get

$$A = \frac{1}{2}(45 \times 3.8730 - 3.8730^3) = 9.6825$$

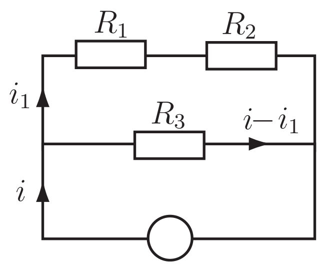
So the maximum cross-sectional area of the store is 9.68 m^2 to 2 d.p.

Task

Equivalent resistance in an electrical circuit

Current distributes itself in the wires of an electrical circuit so as to minimise the total power consumption i.e. the rate at which heat is produced. The power (p) dissipated in an electrical circuit is given by the product of voltage (v) and current (i) flowing in the circuit, i.e. $p = vi$. The voltage across a resistor is the product of current and resistance (r). This means that the power dissipated in a resistor is given by $p = i^2 r$.

Suppose that an electrical circuit contains three resistors r_1, r_2, r_3 and i_1 represents the current flowing through both r_1 and r_2, and that $(i - i_1)$ represents the current flowing through r_3 (see diagram):



(a) Write down an expression for the power dissipated in the circuit:

Your solution

Answer

$p = i_1^2 r_1 + i_1^2 r_2 + (i - i_1)^2 r_3$

(b) Show that the power dissipated is a minimum when $\quad i_1 = \dfrac{r_3}{r_1 + r_2 + r_3}\, i$:

Your solution

Answer

Differentiate result (a) with respect to i_1:

$$\begin{aligned}\frac{dp}{di_1} &= 2i_1r_1 + 2i_1r_2 + 2(i-i_1)(-1)r_3 \\ &= 2i_1(r_1+r_2+r_3) - 2ir_3\end{aligned}$$

This is zero when

$$i_1 = \frac{r_3}{r_1+r_2+r_3}\ i.$$

To check if this represents a minimum, differentiate again:

$$\frac{d^2p}{di_1^2} = 2(r_1+r_2+r_3)$$

This is positive, so the previous result represents a minimum.

(c) If R is the equivalent resistance of the circuit, i.e. of $r_1,\ r_2$ and r_3, for minimum power dissipation and the corresponding voltage V across the circuit is given by $V = iR = i_1(r_1+r_2)$, show that

$$R = \frac{(r_1+r_2)r_3}{r_1+r_2+r_3}.$$

Your solution

Answer

Substituting for i_1 in $iR = i_1(r_1+r_2)$ gives

$$iR = \frac{r_3(r_1+r_2)}{r_1+r_2+r_3}\ i.$$

So

$$R = \frac{(r_1+r_2)r_3}{r_1+r_2+r_3}.$$

Note In this problem R_1 and R_2 could be replaced by a single resistor. However, treating them as separate allows the possibility of considering more general situations (variable resistors or temperature dependent resistors).

Engineering Example 1

Water wheel efficiency

Introduction

A water wheel is constructed with symmetrical curved vanes of angle of curvature θ. Assuming that friction can be taken as negligible, the efficiency, η, i.e. the ratio of output power to input power, is calculated as

$$\eta = \frac{2(V-v)(1+\cos\theta)v}{V^2}$$

where V is the velocity of the jet of water as it strikes the vane, v is the velocity of the vane in the direction of the jet and θ is constant. Find the ratio, v/V, which gives maximum efficiency and find the maximum efficiency.

Mathematical statement of the problem

We need to express the efficiency in terms of a single variable so that we can find the maximum value.

$$\text{Efficiency} = \frac{2(V-v)(1+\cos\theta)v}{V^2} = 2\left(1-\frac{v}{V}\right)\frac{v}{V}(1+\cos\theta).$$

Let $\eta =$ Efficiency and $x = \dfrac{v}{V}$ then $\eta = 2x(1-x)(1+\cos\theta)$.

We must find the value of x which maximises η and we must find the maximum value of η. To do this we differentiate η with respect to x and solve $\dfrac{d\eta}{dx} = 0$ in order to find the stationary points.

Mathematical analysis

Now $\eta = 2x(1-x)(1+\cos\theta) = (2x-2x^2)(1+\cos\theta)$

So $\dfrac{d\eta}{dx} = (2-4x)(1+\cos\theta)$

Now $\dfrac{d\eta}{dx} = 0 \Rightarrow 2-4x = 0 \Rightarrow x = \dfrac{1}{2}$ and the value of η when $x = \dfrac{1}{2}$ is

$$\eta = 2\left(\frac{1}{2}\right)\left(1-\frac{1}{2}\right)(1+\cos\theta) = \frac{1}{2}(1+\cos\theta).$$

This is clearly a maximum not a minimum, but to check we calculate $\dfrac{d^2\eta}{dx^2} = -4(1+\cos\theta)$ which is negative which provides confirmation.

Interpretation

Maximum efficiency occurs when $\dfrac{v}{V} = \dfrac{1}{2}$ and the maximum efficiency is given by

$$\eta = \frac{1}{2}(1+\cos\theta).$$

Engineering Example 2

Refraction

The problem

A light ray is travelling in a medium (A) at speed c_A. The ray encounters an interface with a medium (B) where the velocity of light is c_B . Between two fixed points P in media A and Q in media B, find the path through the interface point O that minimizes the time of light travel (see Figure 18). Express the result in terms of the angles of incidence and refraction at the interface and the velocities of light in the two media.

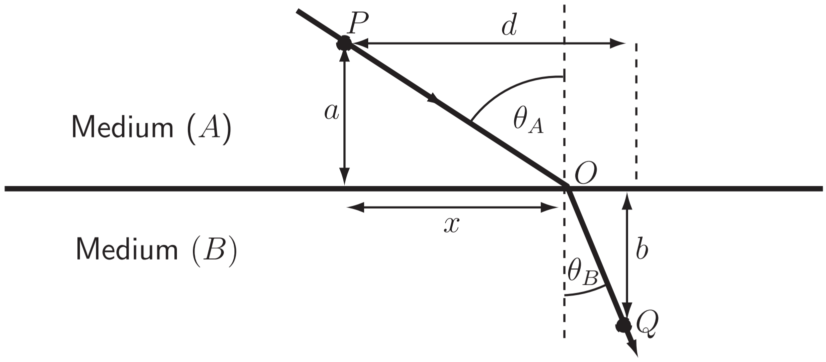


Figure 18: Geometry of light rays at an interface

The solution

The light ray path is shown as POQ in the above figure where O is a point with variable horizontal position x. The points P and Q are fixed and their positions are determined by the constants $a,\ b,\ d$ indicated in the figure. The total path length can be decomposed as $PO + OQ$ so the total time of travel $T(x)$ is given by

$$T(x) = PO/c_A + OQ/c_B. \qquad (1)$$

Expressing the distances PO and OQ in terms of the fixed coordinates $a,\ b,\ d$, and in terms of the unknown position x, Equation (1) becomes

$$T(x) = \frac{\sqrt{a^2 + x^2}}{c_A} + \frac{\sqrt{b^2 + (d-x)^2}}{c_B} \qquad (2)$$

It is assumed that the minimum of the travel time is given by the stationary point of $T(x)$ such that

$$\frac{dT}{dx} = 0. \qquad (3)$$

Using the chain rule in (HELM 11.5) to compute (3) given (2) leads to

$$\frac{1}{2}\ \frac{2x}{c_A\sqrt{a^2 + x^2}} + \frac{1}{2}\ \frac{2x - 2d}{c_B\sqrt{b^2 + (d-x)^2}} = 0.$$

After simplification and rearrangement

$$\frac{x}{c_A\sqrt{a^2 + x^2}} = \frac{d - x}{c_B\sqrt{b^2 + (d-x)^2}}.$$

Using the definitions $\sin\theta_A = \dfrac{x}{\sqrt{a^2+x^2}}$ and $\sin\theta_B = \dfrac{d-x}{\sqrt{b^2+(d-x)^2}}$ this can be written as

$$\frac{\sin\theta_A}{c_A} = \frac{\sin\theta_B}{c_B}. \tag{4}$$

Note that θ_A andθ_B are the incidence angles measured from the interface normal as shown in the figure. Equation (4) can be expressed as

$$\frac{\sin\theta_A}{\sin\theta_B} = \frac{c_A}{c_B}$$

which is the well-known law of refraction for geometrical optics and applies to many other kinds of waves. The ratio $\dfrac{c_A}{c_B}$ is a constant called the **refractive index** of medium (B) with respect to medium (A).

Engineering Example 3

Fluid power transmission

Introduction

Power transmitted through fluid-filled pipes is the basis of hydraulic braking systems and other hydraulic control systems. Suppose that power associated with a piston motion at one end of a pipeline is transmitted by a fluid of density ρ moving with positive velocity V along a cylindrical pipeline of constant cross-sectional area A. Assuming that the loss of power is mainly attributable to friction and that the friction coefficient f can be taken to be a constant, then the power transmitted, P is given by

$$P = \rho g A(hV - cV^3),$$

where g is the acceleration due to gravity and h is the head which is the height of the fluid above some reference level (= the potential energy per unit weight of the fluid). The constant $c = \dfrac{4fl}{2gd}$ where l is the length of the pipe and d is the diameter of the pipe. The power transmission efficiency is the ratio of power output to power input.

Problem in words

Assuming that the head of the fluid, h, is a constant find the value of the fluid velocity, V, which gives a maximum value for the output power P. Given that the input power is $P_i = \rho g A V h$, find the maximum power transmission efficiency obtainable.

Mathematical statement of the problem

We are given that $P = \rho g A(hV - cV^3)$ and we want to find its maximum value and hence maximum efficiency.

To find stationary points for P we solve $\dfrac{dP}{dV} = 0$.

To classify the stationary points we can differentiate again to find the value of $\dfrac{d^2P}{dV^2}$ at each stationary point and if this is negative then we have found a local maximum point. The maximum efficiency is given by the ratio P/P_i at this value of V and where $P_i = \rho g A V h$. Finally we should check that this is the only maximum in the range of P that is of interest.

Mathematical analysis

$$P = \rho g A(hV - cV^3)$$

$$\frac{dP}{dV} = \rho g A(h - 3cV^2)$$

$$\frac{dP}{dV} = 0 \text{ gives } \rho g A(h - 3cV^2) = 0$$

$$\Rightarrow \quad V^2 = \frac{h}{3c} \quad \Rightarrow \quad V = \pm\sqrt{\frac{h}{3c}} \text{ and as } V \text{ is positive} \quad \Rightarrow \quad V = \sqrt{\frac{h}{3c}}.$$

To show this is a maximum we differentiate $\dfrac{dP}{dV}$ again giving $\dfrac{d^2P}{dV^2} = \rho g A(-6cV)$. Clearly this is negative, or zero if $V = 0$. Thus $V = \sqrt{\dfrac{h}{3c}}$ gives a local maximum value for P.

We note that $P = 0$ when $E = \rho g A(hV - cV^3) = 0$, i.e. when $hV - cV^3 = 0$, so $V = 0$ or $V = \sqrt{\dfrac{h}{C}}$. So the maximum at $V = \sqrt{\dfrac{h}{3C}}$ is the only max in this range between 0 and $V = \sqrt{\dfrac{h}{C}}$.

The efficiency E, is given by (input power/output power), so here

$$E = \frac{\rho g A(hV - cV^3)}{\rho g A V h} = 1 - \frac{cV^2}{h}$$

At $V = \sqrt{\dfrac{h}{3c}}$ then $V^2 = \dfrac{h}{3c}$ and therefore $E = 1 - \dfrac{c\dfrac{h}{3c}}{c} = 1 - \dfrac{1}{3} = \dfrac{2}{3}$ or $66\frac{2}{3}\%$.

Interpretation

Maximum power transmitted through the fluid when the velocity $V = \sqrt{\dfrac{h}{3c}}$ and the maximum efficiency is $66\frac{2}{3}\%$. Note that this result is independent of the friction and the maximum efficiency is independent of the velocity and (static) pressure in the pipe.

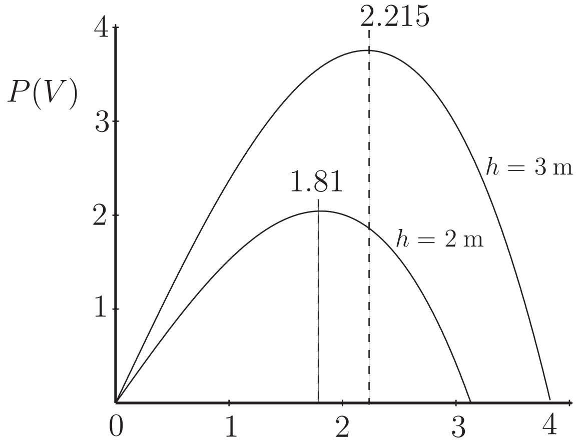


Figure 19: Graphs of transmitted power as a function of fluid velocity for two values of the head

Figure 19 shows the maxima in the power transmission for two different values of the head in an oil filled pipe (oil density 1100 kg m^{-3}) of inner diameter 0.01 m and coefficient of friction 0.01 and pipe length 1 m.

Engineering Example 4

Crank used to drive a piston

Introduction

A crank is used to drive a piston as in Figure 20.

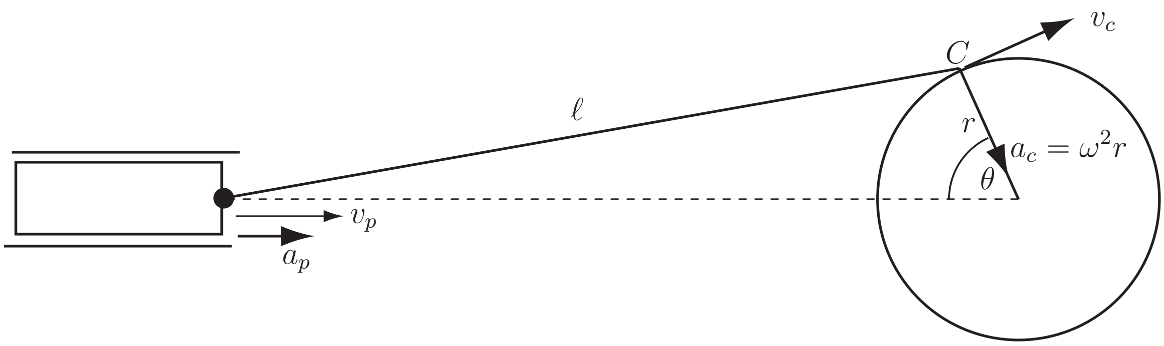


Figure 20: Crank used to drive a piston

Problem

The angular velocity of the crankshaft is the rate of change of the angle $\theta,\ \omega = d\theta/dt$. The piston moves horizontally with velocity v_p and acceleration a_p; r is the length of the crank and l is the length of the connecting rod. The crankpin performs circular motion with a velocity of v_c and centripetal acceleration of $\omega^2 r$. The acceleration a_p of the piston varies with θ and is related by

$$a_p = \omega^2 r\left(\cos\theta + \frac{r\cos 2\theta}{l}\right)$$

Find the maximum and minimum values of the acceleration a_p when $r = 150$ mm and $l = 375$ mm.

Mathematical statement of the problem

We need to find the stationary values of $a_p = \omega^2 r\left(\cos\theta + \dfrac{r\cos 2\theta}{l}\right)$ when $r = 150$ mm and $l = 375$ mm. We do this by solving $\dfrac{da_p}{d\theta} = 0$ and then analysing the stationary points to decide whether they are a maximum, minimum or point of inflexion.

Mathematical analysis.

$$a_p = \omega^2 r\left(\cos\theta + \frac{r\cos 2\theta}{l}\right) \text{ so } \frac{da_p}{d\theta} = \omega^2 r\left(-\sin\theta - \frac{2r\sin 2\theta}{l}\right).$$

To find the maximum and minimum acceleration we need to solve

$$\frac{da_p}{d\theta} = 0 \Leftrightarrow \omega^2 r\left(-\sin\theta - \frac{2r\sin 2\theta}{l}\right) = 0.$$

$$\sin\theta + \frac{2r}{l}\sin 2\theta = 0 \Leftrightarrow \sin\theta + \frac{4r}{l}\sin\theta\cos\theta = 0$$

$$\Leftrightarrow \quad \sin\theta\left(1+\frac{4r}{l}\cos\theta\right)=0$$

$$\Leftrightarrow \quad \sin\theta = 0 \text{ or } \cos\theta = -\frac{l}{4r} \text{ and as } r = 150 \text{ mm and } l = 375 \text{ mm}$$

$$\Leftrightarrow \quad \sin\theta = 0 \text{ or } \cos\theta = -\frac{5}{8}$$

CASE 1: $\sin\theta = 0$

If $\sin\theta = 0$ then $\theta = 0$ or $\theta = \pi$. If $\theta = 0$ then $\cos\theta = \cos 2\theta = 1$

so $\quad a_p = \omega^2 r\left(\cos\theta + \dfrac{r\cos 2\theta}{l}\right) = \omega^2 r\left(1+\dfrac{r}{l}\right) = \omega^2 r\left(1+\dfrac{2}{5}\right) = \dfrac{7}{5}\omega^2 r$

If $\theta = \pi$ then $\cos\theta = -1,\ \cos 2\theta = 1$ so

$$a_p = \omega^2 r\left(\cos\theta + \frac{r\cos 2\theta}{l}\right) = \omega^2 r\left(-1+\frac{r}{l}\right) = \omega^2 r\left(-1+\frac{2}{5}\right) = -\frac{3}{5}\omega^2 r$$

In order to classify the stationary points, we differentiate $\dfrac{da_p}{d\theta}$ with respect to θ to find the second derivative:

$$\frac{d^2a_p}{d\theta^2} = \omega^2 r\left(-\cos\theta - \frac{4r\cos 2\theta}{l}\right) = -\omega^2 r\left(\cos\theta + \frac{4r\cos 2\theta}{l}\right)$$

At $\theta = 0$ we get $\dfrac{d^2a_p}{d\theta^2} = -\omega^2 r\left(1+\dfrac{4r}{l}\right)$ which is negative.

So $\theta = 0$ gives a **maximum** value and $a_p = \dfrac{7}{5}\omega^2 r$ is the value at the maximum.

At $\theta = \pi$ we get $\dfrac{d^2a_p}{d\theta^2} = -\omega^2 r\left(-1+\dfrac{4}{l}\right) = -\omega^2 r\left(\dfrac{3}{5}\right)$ which is negative.

So $\theta = \pi$ gives a **maximum** value and $a_p = -\dfrac{3}{5}\omega^2 r$

CASE 2: $\cos\theta = -\dfrac{5}{8}$

If $\cos\theta = -\dfrac{5}{8}$ then $\cos 2\theta = 2\cos^2\theta - 1 = 2\left(\dfrac{5}{8}\right)^2 - 1$ so $\cos 2\theta = -\dfrac{7}{32}$.

$$a_p = \omega^2 r\left(\cos\theta + \frac{r\cos 2\theta}{l}\right) = \omega^2 r\left(-\frac{5}{8} + -\frac{7}{32}\times\frac{2}{5}\right) = \frac{57}{80}\omega^2 r.$$

At $\cos\theta = -\dfrac{5}{8}$ we get $\dfrac{d^2a_p}{d\theta^2} = \omega^2 r\left(-\cos\theta - \dfrac{4r\cos 2\theta}{l}\right) = \omega^2 r\left(\dfrac{5}{8} + \dfrac{4r}{l}\ \dfrac{7}{32}\right)$ which is positive.

So $\cos\theta = -\dfrac{5}{8}$ gives a **minimum** value and $a_p = -\dfrac{57}{80}\omega^2 r$

Thus the values of a_p at the stationary points are:-

$\dfrac{7}{5}\omega^2 r$ (maximum), $-\dfrac{3}{5}\omega^2 r$ (maximum) and $-\dfrac{57}{80}\omega^2 r$ (minimum).

So the overall maximum value is $1.4\omega^2 r = 0.21\omega^2$ and the minimum value is $-0.7125\omega^2 r = -0.106875\omega^2$ where we have substituted $r = 150$ mm (= 0.15 m) and $l = 375$ mm ($= 0.375$ m).

Interpretation

The maximum acceleration occurs when $\theta = 0$ and $a_p = 0.21\omega^2$.

The minimum acceleration occurs when $\cos\theta = -\dfrac{5}{8}$ and $a_p = -0.106875\omega^2$.

Exercises

1. Locate the stationary points of the following functions and distinguish among them as maxima, minima and points of inflection.

   (a) $f(x) = x - \ln|x|$. [Remember $\dfrac{d}{dx}(\ln|x|) = \dfrac{1}{x}$]

   (b) $f(x) = x^3$

   (c) $f(x) = \dfrac{(x-1)}{(x+1)(x-2)} \qquad -1 < x < 2$

2. A perturbation in the temperature of a stream leaving a chemical reactor follows a decaying sinusoidal variation, according to

   $$T(t) = 5\text{exp}(-at)\sin(\omega t)$$

   where a and ω are positive constants.

   (a) Sketch the variation of temperature with time.

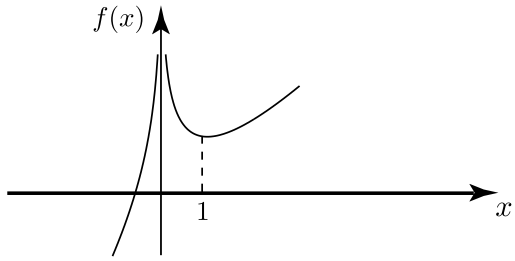
   (b) By examining the behaviour of $\dfrac{dT}{dt}$, show that the maximum temperatures occur at times of $\left(\tan^{-1}(\dfrac{\omega}{a}) + 2\pi n\right)/\omega$.

Answers

1. (a) $\dfrac{df}{dx} = 1 - \dfrac{1}{x} = 0$ when $x = 1$

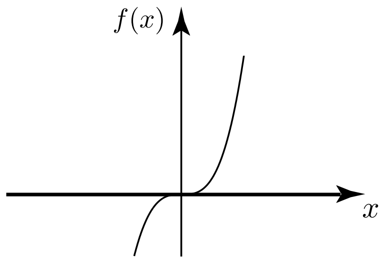
$\dfrac{d^2f}{dx^2} = \dfrac{1}{x^2} \qquad \left.\dfrac{d^2f}{dx^2}\right|_{x=1} = 1 > 0$

$\therefore \quad x = 1, y = 1$ locates a local minimum.



(b) $\dfrac{df}{dx} = 3x^2 = 0$ when $x = 0 \qquad \dfrac{d^2f}{dx^2} = 6x = 0$ when $x = 0$

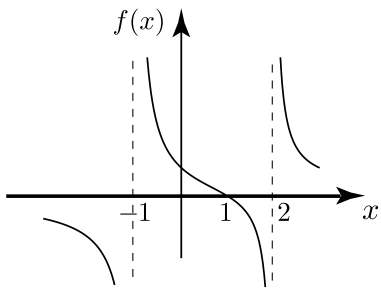
However, $\dfrac{df}{dx} > 0$ on either side of $x = 0$ so $(0, 0)$ is a point of inflection.



(c) $\dfrac{df}{dx} = \dfrac{(x+1)(x-2) - (x-1)(2x-1)}{(x+1)(x-2)}$

This is zero when $(x+1)(x-2) - (x-1)(2x-1) = 0 \quad$ i.e. $x^2 - 2x + 3 = 0$

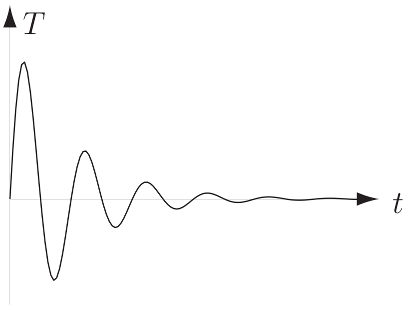
However, this equation has no real roots (since $b^2 < 4ac$) and so $f(x)$ has no stationary points. The graph of this function confirms this:



Nevertheless $f(x)$ **does** have a point of inflection at $x = 1, y = 0$ as the graph shows, although at that point $\dfrac{dy}{dx} \neq 0$.

Answer

2. (a)



(b) $\dfrac{dT}{dt} = 0$ implies $\tan \omega t = \dfrac{\omega}{a}$, so $\tan \omega t > 0$ and

$$\omega t = \tan^{-1}\left(\frac{\omega}{a}\right) + k\pi, \qquad k \text{ integer}$$

Examination of $\dfrac{d^2T}{dt^2}$ reveals that only even values of k give $\dfrac{d^2T}{dt^2} < 0$ for a maximum so setting $k = 2n$ gives the required answer.

The Newton-Raphson Method

12.3

Introduction

This Section is concerned with the problem of "root location"; i.e. finding those values of x which satisfy an equation of the form $f(x) = 0$. An initial estimate of the root is found (for example by drawing a graph of the function). This estimate is then improved using a technique known as the Newton-Raphson method, which is based upon a knowledge of the tangent to the curve near the root. It is an "iterative" method in that it can be used repeatedly to continually improve the accuracy of the root.

Prerequisites

Before starting this Section you should ...

- be able to differentiate simple functions
- be able to sketch graphs

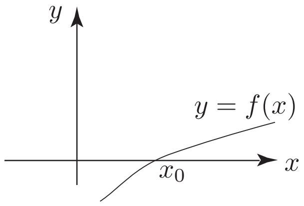
Learning Outcomes

On completion you should be able to ...

- distinguish between simple and multiple roots
- estimate the root of an equation by drawing a graph
- employ the Newton-Raphson method to improve the accuracy of a root

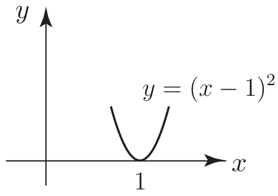
1. The Newton-Raphson method

We first remind the reader of some basic notation: If $f(x)$ is a given function the value of x for which $f(x) = 0$ is called a **root** of the equation or **zero** of the function. We also distinguish between various types of roots: simple roots and multiple roots. Figures 21 - 23 illustrate some common examples.



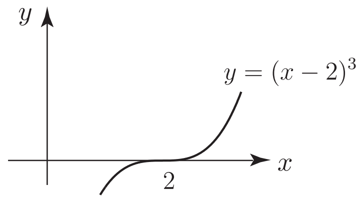
simple root

Figure 21



double root

Figure 22



triple root

Figure 23

More precisely; a root x_0 is said to be:

a **simple root** if $\quad f(x_0) = 0 \quad$ and $\quad \left.\dfrac{df}{dx}\right|_{x_0} \neq 0.$

a **double root** if $\quad f(x_0) = 0, \quad \left.\dfrac{df}{dx}\right|_{x_0} = 0 \quad$ and $\quad \left.\dfrac{d^2f}{dx^2}\right|_{x_0} \neq 0,\quad$ and so on.

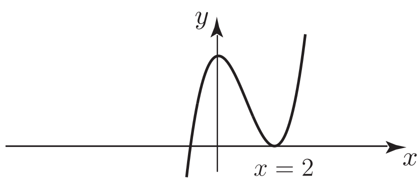
In this Section we shall concentrate on the location of simple roots of a given function $f(x)$.

Task

Given graphs of the functions (a) $f(x) = x^3 - 3x^2 + 4$, (b) $f(x) = 1 + \sin x$ classify the roots into simple or multiple.

Your solution

(a) $f(x) = x^3 - 3x^2 + 4$: The negative root is: and the positive root is:

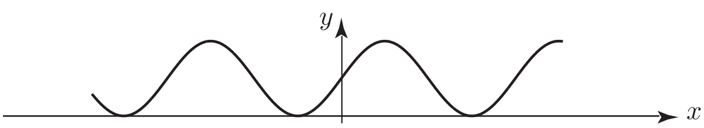


Answer

The negative root is simple and the positive root is double.

Your solution

(b) $f(x) = 1 + \sin x$: Each root is a root



Answer

Each root is a double root.

2. Finding roots of the equation $f(x) = 0$

A first investigation into the roots of $f(x)$ might be graphical. Such an analysis will supply information as to the approximate location of the roots.

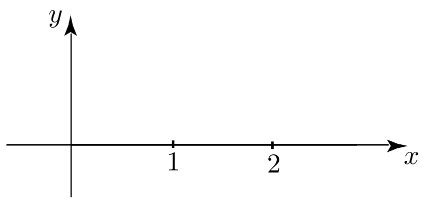
Task

Sketch the function

$$f(x) = x - 2 + \ln x \qquad x > 0$$

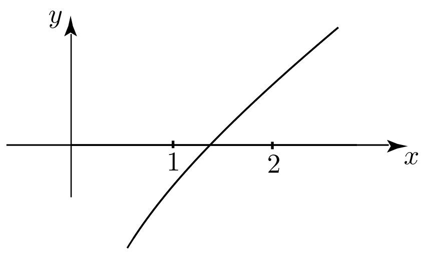
and estimate the value of the root.

Your solution



An estimate of the root is:

Answer



A simple root is located near 1.5

One method of obtaining a better approximation is to halve the interval $1 \leq x \leq 2$ into $1 \leq x \leq 1.5$ and $1.5 \leq x \leq 2$ and test the **sign** of the function at the end-points of these new regions. We find

x	$f(x)$
1	< 0
1.5	< 0
2	> 0

so a root must lie between $x = 1.5$ and $x = 2$ because the sign of $f(x)$ changes between these values and $f(x)$ is a continuous curve. We can repeat this procedure and divide the interval $(1.5, 2)$ into the two new intervals $(1.5, 1.75)$ and $(1.75, 2)$ and test again. This time we find

x	$f(x)$
1.5	< 0
1.75	> 0
2.0	> 0

so a root lies in the interval $(1.5, 1.75)$. It is obvious that proceeding in this way will give a smaller and smaller interval in which the root must lie. But can we do better than this rather laborious bisection procedure? In fact there are many ways to improve this numerical search for the root. In this Section we examine one of the best methods: the **Newton-Raphson** method.

To derive the method we examine the general characteristics of a curve in the neighbourhood of a simple root. Consider Figure 24 showing a function $f(x)$ with a simple root at $x = x^*$ whose value is required. Initial analysis has indicated that the root is approximately located at $x = x_0$. The aim is to provide a better estimate to the location of the root.

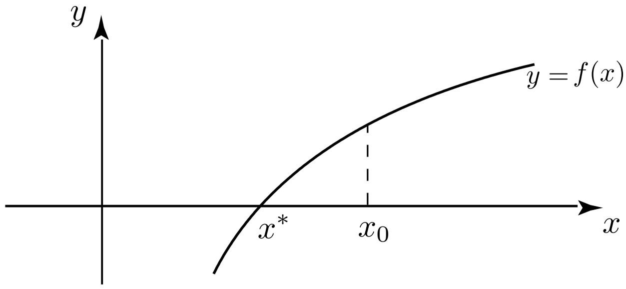


Figure 24

The basic premise of the Newton-Raphson method is the assumption that the curve in the close neighbourhood of the simple root at x^* is approximately a straight line. Hence if we draw the tangent to the curve at x_0, this tangent will intersect the x-axis at a point closer to x^* than is x_0: see Figure 25.

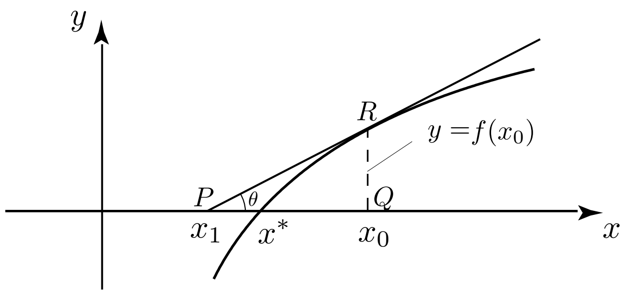


Figure 25

From the geometry of this diagram we see that

$$x_1 = x_0 - PQ$$

But from the right-angled triangle PQR we have

$$\frac{RQ}{PQ} = \tan\theta = f'(x_0)$$

and so $\quad PQ = \dfrac{RQ}{f'(x_0)} = \dfrac{f(x_0)}{f'(x_0)} \qquad \therefore \qquad x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$

If $f(x)$ has a simple root near x_0 then a closer estimate to the root is x_1 where

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

This formula can be used iteratively to get closer and closer to the root, as summarised in Key Point 5:

Key Point 5

Newton-Raphson Method

If $f(x)$ has a simple root near x_n then a closer estimate to the root is x_{n+1} where

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

This is the **Newton-Raphson iterative formula**. The iteration is begun with an initial estimate of the root, x_0, and continued to find $x_1, x_2, \ldots$ until a suitably accurate estimate of the position of the root is obtained. This is judged by the convergence of $x_1, x_2, \ldots$ to a fixed value.

Example 4

$f(x) = x - 2 + \ln x$ has a root near $x = 1.5$. Use the Newton-Raphson method to obtain a better estimate.

Solution

Here $x_0 = 1.5$, $\quad f(1.5) = -0.5 + \ln(1.5) = -0.0945$

$f'(x) = 1 + \dfrac{1}{x} \qquad \therefore \qquad f'(1.5) = 1 + \dfrac{1}{1.5} = \dfrac{5}{3}$

Hence using the formula:

$$x_1 = 1.5 - \frac{(-0.0945)}{(1.6667)} = 1.5567$$

The Newton-Raphson formula can be used again: this time beginning with 1.5567 as our estimate:

$$\begin{aligned} x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.5567 - \frac{f(1.5567)}{f'(1.5567)} &= 1.5567 - \frac{\{1.5567 - 2 + \ln(1.5567)\}}{\left\{1 + \dfrac{1}{1.5567}\right\}} \\ &= 1.5567 - \frac{\{-0.0007\}}{\{1.6424\}} = 1.5571 \end{aligned}$$

This is in fact the correct value of the root to 4 d.p., which calculating x_3 would confirm.

Task

The function $f(x) = x - \tan x$ has a simple root near $x = 4.5$. Use one iteration of the Newton-Raphson method to find a more accurate value for the root.

First find $\dfrac{df}{dx}$:

Your solution

$$\frac{df}{dx} =$$

Answer

$$\frac{df}{dx} = 1 - \sec^2 x = -\tan^2 x$$

Now use the formula $x_1 = x_0 - f(x_0)/f'(x_0)$ with $x_0 = 4.5$ to obtain x_1:

Your solution

$$f(4.5) = 4.5 - \tan(4.5) =$$

$$f'(4.5) = 1 - \sec^2(4.5) = -\tan^2(4.5) =$$

$$x_1 = 4.5 - \frac{f(4.5)}{f'(4.5)} =$$

Answer

$f(4.5) = -0.1373, \quad f'(4.5) = -21.5048$

$$\therefore \quad x_1 = 4.5 - \frac{0.1373}{21.5048} = 4.4936.$$

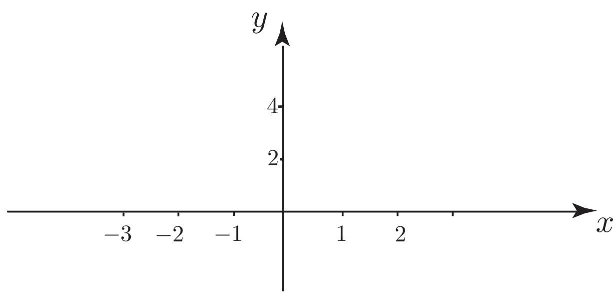
As the value of x_1 has changed little from $x_0 = 4.5$ we can expect the root to be 4.49 to 3 d.p.

Task

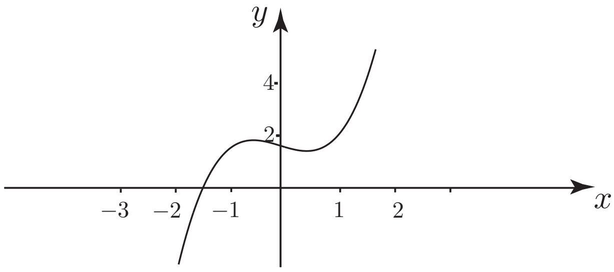
Sketch the function $f(x) = x^3 - x + 3$ and confirm that there is a simple root between $x = -2$ and $x = -1$. Use $x_0 = -2$ as an initial estimate to obtain the value to 2 d.p.

First sketch $f(x) = x^3 - x + 3$ and identify a root:

Your solution



Answer



Clearly a simple root lies between $x = -2$ and $x = -1$.

Now use one iteration of Newton-Raphson to improve the estimate of the root using $x_0 = -2$:

Your solution

$f(x) =$ $\qquad$ $f'(x) =$ $\qquad$ $x_0 =$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} =$$

Answer

$f(x) = x^3 - x + 3, \quad f'(x) = 3x^2 - 1 \quad x_0 = -2$

$$\therefore \quad x_1 = -2 - \frac{\{-8+2+3\}}{11} = -2 + \frac{3}{11} = -1.727$$

Now repeat this process for a second iteration using $x_1 = -1.727$:

Your solution

$$x_2 = x_1 - f(x_1)/f'(x_1) =$$

Answer

$$\begin{aligned} x_2 &= -1.727 - \{-(1.727)^3 + 1.727 + 3\}/\{3(1.727)^2 - 1\} \\ &= -1.727 + \{(0.424)/(7.948) = -1.674 \end{aligned}$$

Repeat for a third iteration and state the root to 2 d.p.:

Your solution

$$x_3 = x_2 - f(x_2)/f'(x_2) =$$

Answer

$$\begin{aligned} x_3 &= -1.674 - \{-(1.674)^3 + 1.674 + 3\}/\{3(1.674)^2 - 1\} \\ &= -1.674 + \{0.017\}/\{7.407\} = -1.672 \end{aligned}$$

We conclude the value of the simple root is -1.67 correct to 2 d.p.

Engineering Example 5

Buckling of a strut

The equation governing the buckling load P of a strut with one end fixed and the other end simply supported is given by $\tan \mu L = \mu L$ where $\mu = \sqrt{\dfrac{P}{EI}}$, L is the length of the strut and EI is the flexural rigidity of the strut. For safe design it is important that the load applied to the strut is less than the lowest buckling load. This equation has no exact solution and we must therefore use the method described in this Workbook to find the lowest buckling load$P$.

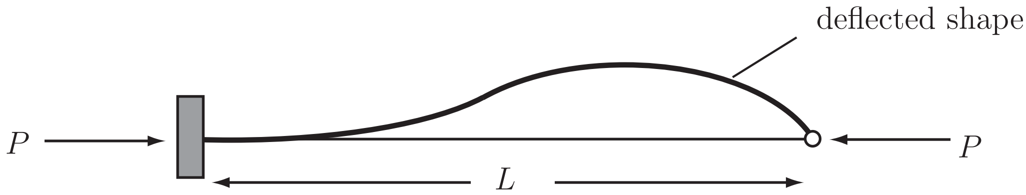
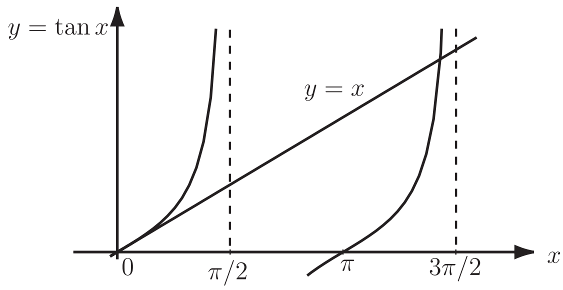


Figure 26

We let $\mu L = x$ and so we need to solve the equation $\tan x = x$. Before starting to apply the Newton-Raphson iteration we must first obtain an approximate solution by plotting graphs of $y = \tan x$ and $y = x$ using the same axes.



From the graph it can be seen that the solution is near to but below $x = 3\pi/2$ (~ 4.7). We therefore start the Newton-Raphson iteration with a value $x_0 = 4.5$.

The equation is rewritten as $\tan x - x = 0$. Let $f(x) = \tan x - x$ then $f'(x) = \sec^2 x - 1 = \tan^2 x$

The Newton-Raphson iteration is $x_{n+1} = x_n - \dfrac{\tan x_n - x_n}{\tan^2 x_n}, \quad x_0 = 4.5$

so $x_1 = 4.5 - \dfrac{\tan(4.5) - 4.5}{\tan^2 4.5} = 4.5 - \dfrac{0.137332}{21.504847} = 4.493614$ to 7 sig.fig.

Rounding to 4 sig.fig. and iterating:

$$x_2 = 4.494 - \frac{\tan(4.494) - 4.494}{\tan^2 4.494} = 4.494 - \frac{0.004132}{20.229717} = 4.493410 \text{ to 7 sig.fig.}$$

So we conclude that the value of x is 4.493 to 4 sig.fig. As $x = \mu L = \left(\sqrt{P/EI}\right) L$ we find, after re-arrangement, that the smallest buckling load is given by $P = 20.19\dfrac{EI}{L^2}$.

Exercises

1. By sketching the function $f(x) = x - 1 - \sin x$ show that there is a simple root near $x = 2$. Use two iterations of the Newton-Raphson method to obtain a better estimate of the root.

2. Obtain an estimation accurate to 2 d.p. of the point of intersection of the curves $y = x - 1$ and $y = \cos x$.

Answers

1. $x_0 = 2, \quad x_1 = 1.936, \quad x_2 = 1.935$

2. The curves intersect when $x - 1 - \cos x = 0$. Solve this using the Newton-Raphson method with initial estimate (say) $x_0 = 1.2$.

   The point of intersection is $(1.28342,\ 0.283437)$ to 6 significant figures.

Curvature

12.4

Introduction

Curvature is a measure of how sharply a curve is turning. At a particular point along the curve a tangent line can be drawn; this tangent line making an angle ψ with the positive x-axis. Curvature is then defined as the magnitude of the rate of change of ψ with respect to the measure of length on the curve - the arc length s. That is

$$\text{Curvature } = \left|\frac{d\psi}{ds}\right|$$

In this Section we examine the concept of curvature and, from its definition, obtain more useful expressions for curvature when the equation of the curve is expressed either in Cartesian form $y = f(x)$ or in parametric form $x = x(t) \quad y = y(t)$. We show that a circle has a constant value for the curvature, which is to be expected, as the tangent line to a circle turns equally quickly irrespective of the position on the circle. For all curves, except circles, other than a circle, the curvature will depend upon position, changing its value as the curve twists and turns.

Prerequisites

Before starting this Section you should . . .

- understand the geometrical interpretation of the derivative
- be able to differentiate standard functions
- be able to use the parametric description of a curve

Learning Outcomes

On completion you should be able to . . .

- understand the concept of curvature
- calculate curvature when the curve is defined in Cartesian form or in parametric form

1. Curvature

Curvature is a measure of how quickly a tangent line turns as the contact point moves along a curve. For example, consider a simple parabola, with equation $y = x^2$. Its graph is shown in Figure 27.

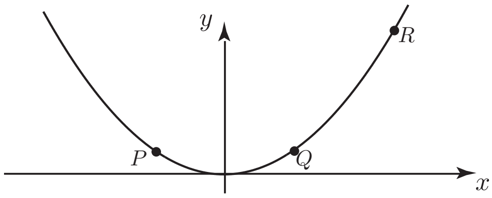


Figure 27

It is obvious, geometrically, that the tangent lines to this curve turn 'more quickly' between P and Q than between Q and R. It is the purpose of this Section to give, a quantitative measure of this rate of 'turning'.

If we change from a parabola to a circle, (centred on the origin, of radius 1), we can again consider how quickly the tangent lines turn as we move along the curve. See Figure 28. It is immediately clear that the tangent lines to a circle turn equally quickly no matter where located on the circle.

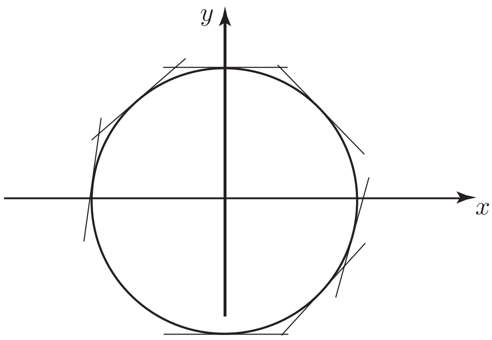


Figure 28

However, if we consider two circles with the same centre but different radii, as in Figure 29, it is again obvious that the smaller circle 'bends' more tightly than the larger circle and we say it has a larger curvature. Athletes who run the 200 metres find it easier to run in the outside lanes (where the curve turns less sharply) than in the inside lanes.

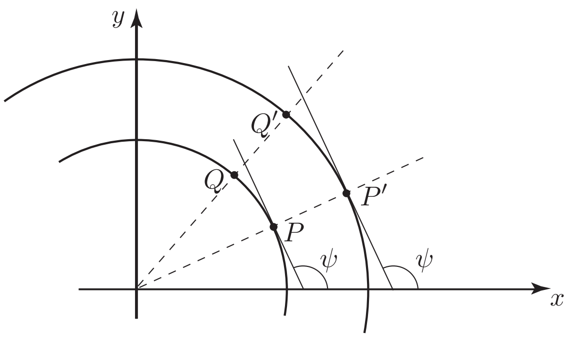


Figure 29

On the two circle diagram (Figure 29) we have drawn tangent lines at P and P'; both lines make an angle ψ (greek letter psi) with the positive x-axis. We need to measure how quickly the angle

ψ changes as we move along the curve. As we move from P to Q (inner circle), or from P' to Q' (outer circle), the angle ψ changes by the same amount. However, the distance traversed on the inner circle is less than the distance traversed on the outer circle. This suggests that a measure of curvature is:

curvature is the magnitude of the rate of change of ψ with respect to the distance moved along the curve.

We shall denote the curvature by the Greek letter κ (kappa).
So

$$\kappa = \left|\frac{d\psi}{ds}\right|$$

where s is the measure of arc-length along a curve. This rather odd-looking derivative needs converting to involve the variable x if the equation of the curve is given in the usual form $y = f(x)$. As a preliminary we note that

$$\frac{d\psi}{ds} = \frac{d\psi}{dx}\bigg/\frac{ds}{dx}$$

We now obtain expressions for the derivatives $\dfrac{d\psi}{dx}$ and $\dfrac{ds}{dx}$ in terms of the derivatives of $f(x)$.
Consider Figure 30 below.

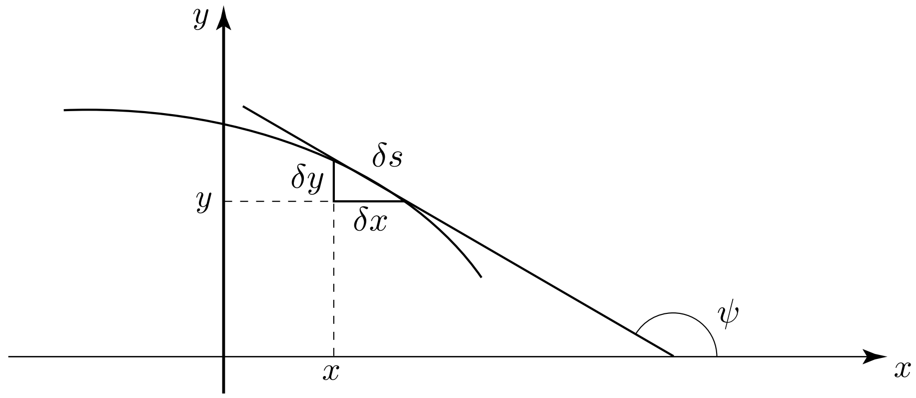


Figure 30

Small increments in the x- and y-directions have been denoted by δx and δy respectively. The hypotenuse on this 'small triangle' is δs which is the change in arc-length along the curve.
From Pythagoras' theorem:

$$\delta s^2 = \delta x^2 + \delta y^2$$

so

$$\left(\frac{\delta s}{\delta x}\right)^2 = 1 + \left(\frac{\delta y}{\delta x}\right)^2 \qquad \text{so that} \qquad \frac{\delta s}{\delta x} = \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}$$

In the limit as the increments get smaller and smaller, we write this relation in derivative form:

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

However, as $y = f(x)$ is the equation of the curve we obtain

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{df}{dx}\right)^2} = (1 + [f'(x)]^2)^{1/2}$$

We also know the relation between the angle ψ and the derivative $\dfrac{df}{dx}$:

$$\frac{df}{dx} = \tan\psi$$

so differentiating again:

$$\frac{d^2 f}{dx^2} = \sec^2\psi \frac{d\psi}{dx} = (1 + \tan^2\psi)\frac{d\psi}{dx}$$

$$= (1 + [f'(x)]^2)\frac{d\psi}{dx}$$

Inverting this relation:

$$\frac{d\psi}{dx} = \frac{f''(x)}{(1 + [f'(x)]^2)}$$

and so, finally, the curvature is given by

$$\kappa = \left|\frac{d\psi}{ds}\right| = \left|\frac{d\psi}{dx}\bigg/\frac{ds}{dx}\right| = \left|\frac{f''(x)}{(1 + [f'(x)]^2)^{3/2}}\right|$$

Key Point 6

Curvature

At each point on a curve, with equation $y = f(x)$, the tangent line turns at a certain rate.

A measure of this rate of turning is the **curvature** κ defined by

$$\kappa = \left|\frac{f''(x)}{(1 + [f'(x)]^2)^{3/2}}\right|$$

Task

Obtain the curvature of the parabola $y = x^2$.

First calculate the derivatives of $f(x)$:

Your solution

$f(x) =$ $\qquad \dfrac{df}{dx} =$ $\qquad \dfrac{d^2 f}{dx^2} =$

Answer

$f(x) = x^2 \qquad \dfrac{df}{dx} = 2x \qquad \dfrac{d^2 f}{dx^2} = 2$

Now find an expression for the curvature:

Your solution

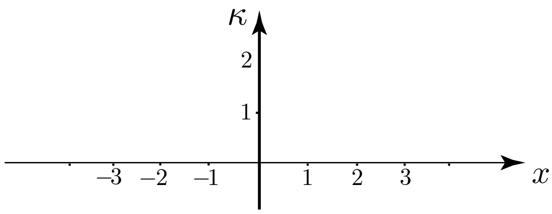
$\kappa =$

Answer

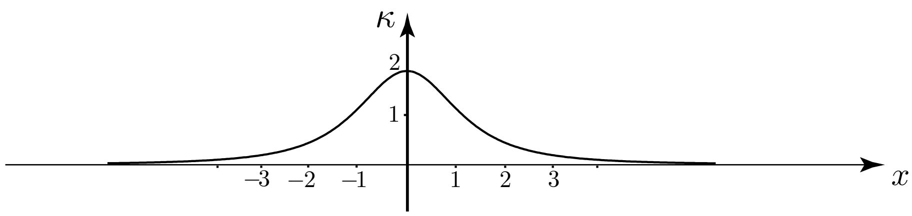
$$\kappa = \left| \frac{f''(x)}{[1 + [f'(x))]^2]^{3/2}} \right| = \frac{2}{[1 + 4x^2]^{3/2}}$$

Finally, plot the curvature κ as a function of x:

Your solution



Answer



The figure above supports what we have already argued:

- Close to $x = 0$ the parabola turns sharply (near $x = 0$ the curvature κ is relatively, large).
- Further away from $x = 0$ the curve is more 'gentle' (in these regions κ is small).

In general, the curvature κ is a function of position. However, from what we have said earlier, we expect the curvature to be a constant for a given circle but to increase as the radius of the circle decreases. This can now be checked directly.

Example 5

Find the curvature of $y = (a^2 - x^2)^{1/2}$ (this is the equation of the upper half of a circle centred at the origin of radius a).

Solution

Here $f(x) = (a^2 - x^2)^{\frac{1}{2}}$

$$\frac{df}{dx} = \frac{-x}{(a^2 - x^2)^{\frac{1}{2}}} \qquad \frac{d^2 f}{dx^2} = \frac{-a^2}{(a^2 - x^2)^{\frac{3}{2}}}$$

$$\therefore \qquad 1 + [f'(x)]^2 = 1 + \frac{x^2}{a^2 - x^2} = \frac{r^2}{a^2 - x^2}$$

$$\therefore \qquad \kappa = \left| \frac{\dfrac{-a^2}{(a^2 - x^2)^{3/2}}}{\left[\dfrac{a^2}{a^2 - x^2}\right]^{3/2}} \right| = \frac{1}{a}$$

For a circle of radius a, the curvature is constant, with value $\dfrac{1}{a}$.

The value of κ (at any particular point on the curve, i.e. at a particular value of x) indicates how sharply the curve is turning. What this result states is that, for a circle, the curvature is inversely related to the radius. The bigger the radius, the smaller the curvature; precisely what we predicted.

2. Curvature for parametrically defined curves

An expression for the curvature is also available if the curve is described parametrically:

$$x = g(t) \qquad y = h(t) \qquad t_0 \leq t \leq t_1$$

We remember the basic formulae connecting derivatives

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} \qquad \frac{d^2 y}{dx^2} = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^3}$$

where, as usual $\dot{x} \equiv \dfrac{dx}{dt}, \quad \ddot{x} \equiv \dfrac{d^2 x}{dt^2}$ etc.

Then

$$\kappa = \left| \frac{f''(x)}{\{1 + [f'(x)]^2\}^{3/2}} \right| = \left| \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^3 \left[1 + \left(\dfrac{\dot{y}}{\dot{x}}\right)^2\right]^{3/2}} \right|$$

$$= \left| \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{[\dot{x}^2 + \dot{y}^2]^{3/2}} \right|$$

Key Point 7

The formula for curvature in parametric form is $\kappa = \left| \dfrac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{[\dot{x}^2 + \dot{y}^2]^{3/2}} \right|$

Task

An ellipse is described parametrically by the equations

$$x = 2\cos t \qquad y = \sin t \qquad 0 \le t \le 2\pi$$

Obtain an expression for the curvature κ and find where the curvature is a maximum or a minimum.

First find $\dot{x},\ \dot{y},\ \ddot{x},\ \ddot{y}$:

Your solution

$\dot{x} =$ $\qquad$ $\dot{y} =$ $\qquad$ $\ddot{x} =$ $\qquad$ $\ddot{y} =$

Answer

$\dot{x} = -2\sin t \qquad \dot{y} = \cos t \qquad \ddot{x} = -2\cos t \qquad \ddot{y} = -\sin t$

Now find κ:

Your solution

$\kappa =$

Answer

$$\kappa = \left| \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{[\dot{x}^2 + \dot{y}^2]^{3/2}} \right| = \left| \frac{2\sin^2 t + 2\cos^2 t}{[4\sin^2 t + \cos^2 t]^{3/2}} \right| = \frac{2}{[1 + 3\sin^2 t]^{3/2}}$$

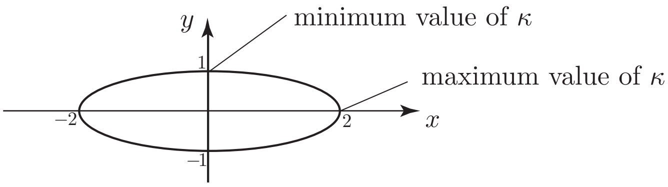
Find maximum and minimum values of κ **by inspection** of the expression for κ:

Your solution

max $\kappa =$ $\qquad$ min $\kappa =$

Answer

Denominator is max when $t = \pi/2$. This gives minimum value of $\kappa = 1/4$,
Denominator is min when $t = 0$. This gives maximum value of $\kappa = 2$.



Differentiation of Vectors

12.5

Introduction

The area of mathematics known as vector calculus is used to model mathematically a vast range of engineering phenomena including electrostatics, electromagnetic fields, air flow around aircraft and heat flow in nuclear reactors. In this Section we introduce briefly the differential calculus of vectors.

Prerequisites

Before starting this Section you should ...

- have a knowledge of vectors, in Cartesian form
- be able to calculate the scalar and vector products of two vectors
- be able to differentiate and integrate scalar functions

Learning Outcomes

On completion you should be able to ...

- differentiate vectors

1. Differentiation of vectors

Consider Figure 31.

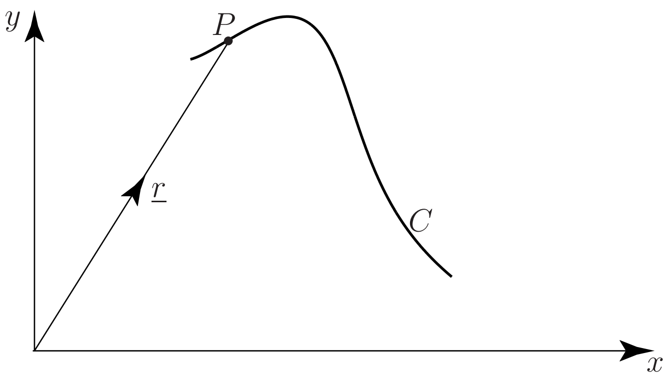


Figure 31

If $\underline{r}$ represents the **position vector** of an object which is moving along a curve C, then the position vector will be dependent upon the time, t. We write $\underline{r} = \underline{r}(t)$ to show the dependence upon time. Suppose that the object is at the point P, with position vector $\underline{\mathrm{r}}$ at time t and at the point Q, with position vector $\underline{r}(t + \delta t)$, at the later time $t + \delta t$, as shown in Figure 32.

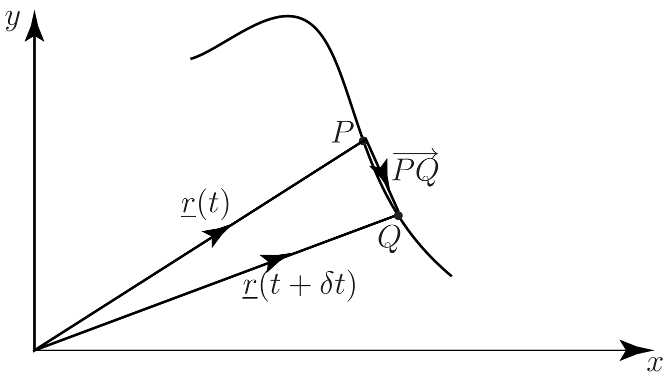


Figure 32

Then $\overrightarrow{PQ}$ represents the **displacement vector** of the object during the interval of time δt. The length of the displacement vector represents the distance travelled, and its direction gives the direction of motion. The average velocity during the time from t to $t + \delta t$ is defined as the displacement vector divided by the time interval δt, that is,

$$\text{average velocity } = \frac{\overrightarrow{PQ}}{\delta t} = \frac{\underline{r}(t + \delta t) - \underline{r}(t)}{\delta t}$$

If we now take the limit as the interval of time δt tends to zero then the expression on the right hand side is the **derivative** of $\underline{r}$ with respect to t. Not surprisingly we refer to this derivative as **the instantaneous velocity**, $\underline{v}$. By its very construction we see that the velocity vector is always tangential to the curve as the object moves along it. We have:

$$\underline{v} = \lim_{\delta t \to 0} \frac{\underline{r}(t + \delta t) - \underline{r}(t)}{\delta t} = \frac{d\underline{r}}{dt}$$

Now, since the x and y coordinates of the object depend upon time, we can write the position vector $\underline{r}$ in Cartesian coordinates as:

$$\underline{r}(t) = x(t)\underline{i} + y(t)\underline{j}$$

Therefore,

$$\underline{r}(t + \delta t) = x(t + \delta t)\underline{i} + y(t + \delta t)\underline{j}$$

so that,

$$\begin{aligned} \underline{v}(t) &= \lim_{\delta t \to 0} \frac{x(t+\delta t)\underline{i} + y(t+\delta t)\underline{j} - x(t)\underline{i} - y(t)\underline{j}}{\delta t} \\ &= \lim_{\delta t \to 0} \left\{ \frac{x(t+\delta t) - x(t)}{\delta t}\underline{i} + \frac{y(t+\delta t) - y(t)}{\delta t}\underline{j} \right\} \\ &= \frac{dx}{dt}\underline{i} + \frac{dy}{dt}\underline{j} \end{aligned}$$

This is often abbreviated to $\underline{v} = \dot{\underline{r}} = \dot{x}\underline{i} + \dot{y}\underline{j}$, using notation for derivatives with respect to time. So we see that the velocity vector is the derivative of the position vector with respect to time. This result generalizes in an obvious way to three dimensions as summarized in the following Key Point.

Key Point 8

Given $\quad \underline{r}(t) = x(t)\underline{i} + y(t)\underline{j} + z(t)\underline{k}$

then the velocity vector is

$$\underline{v} = \dot{\underline{r}}(t) = \dot{x}(t)\underline{i} + \dot{y}(t)\underline{j} + \dot{z}(t)\underline{k}$$

The magnitude of the velocity vector gives the speed of the object.

We can define the acceleration vector in a similar way, as the rate of change (i.e. the derivative) of the velocity with respect to the time:

$$\underline{a} = \frac{d\underline{v}}{dt} = \frac{d^2\underline{r}}{dt^2} = \ddot{\underline{r}} = \ddot{x}\underline{i} + \ddot{y}\underline{j} + \ddot{z}\underline{k}$$

Example 6

If $\underline{w} = 3t^2\underline{i} + \cos 2t\underline{j}$, find

(a) $\dfrac{d\underline{w}}{dt}$ (b) $\left|\dfrac{d\underline{w}}{dt}\right|$ (c) $\dfrac{d^2\underline{w}}{dt^2}$

Solution

(a) If $\underline{w} = 3t^2\underline{i} + \cos 2t\underline{j}$, then differentiation with respect to t yields: $\dfrac{d\underline{w}}{dt} = 6t\underline{i} - 2\sin 2t\underline{j}$

(b) $\left|\dfrac{d\underline{w}}{dt}\right| = \sqrt{(6t)^2 + (-2\sin 2t)^2} = \sqrt{36t^2 + 4\sin^2 2t}$

(c) $\dfrac{d^2\underline{w}}{dt^2} = 6\underline{i} - 4\cos 2t\underline{j}$

It is possible to differentiate more complicated expressions involving vectors provided certain rules are adhered to as summarized in the following Key Point.

Key Point 9

If $\underline{w}$ and $\underline{z}$ are vectors and c is a scalar, all these being functions of time t, then:

$$\begin{aligned}
\frac{d}{dt}(\underline{w} + \underline{z}) &= \frac{d\underline{w}}{dt} + \frac{d\underline{z}}{dt} \\
\frac{d}{dt}(c\underline{w}) &= c\frac{d\underline{w}}{dt} + \frac{dc}{dt}\underline{w} \\
\frac{d}{dt}(\underline{w} \cdot \underline{z}) &= \underline{w} \cdot \frac{d\underline{z}}{dt} + \frac{d\underline{w}}{dt} \cdot \underline{z} \\
\frac{d}{dt}(\underline{w} \times \underline{z}) &= \underline{w} \times \frac{d\underline{z}}{dt} + \frac{d\underline{w}}{dt} \times \underline{z}
\end{aligned}$$

Example 7

If $\underline{w} = 3t\underline{i} - t^2\underline{j}$ and $\underline{z} = 2t^2\underline{i} + 3\underline{j}$, verify the result

$$\frac{d}{dt}(\underline{w}\cdot\underline{z}) = \underline{w}\cdot\frac{d\underline{z}}{dt} + \frac{d\underline{w}}{dt}\cdot\underline{z}$$

Solution

$\underline{w}\cdot\underline{z} = (3t\underline{i} - t^2\underline{j})\cdot(2t^2\underline{i} + 3\underline{j}) = 6t^3 - 3t^2.$

Therefore $\quad \dfrac{d}{dt}(\underline{w}\cdot\underline{z}) = 18t^2 - 6t \qquad (1)$

Also $\quad \dfrac{d\underline{w}}{dt} = 3\underline{i} - 2t\underline{j} \quad$ and $\quad \dfrac{d\underline{z}}{dt} = 4t\underline{i}$

so

$$\begin{aligned}\underline{w}\cdot\frac{d\underline{z}}{dt} + \underline{z}\cdot\frac{d\underline{w}}{dt} &= (3t\underline{i} - t^2\underline{j})\cdot(4t\underline{i}) + (2t^2\underline{i} + 3\underline{j})\cdot(3\underline{i} - 2t\underline{j})\\ &= 12t^2 + 6t^2 - 6t\\ &= 18t^2 - 6t \qquad (2)\end{aligned}$$

We have verified $\dfrac{d}{dt}(\underline{w}\cdot\underline{z}) = \underline{w}\cdot\dfrac{d\underline{z}}{dt} + \dfrac{d\underline{w}}{dt}\cdot\underline{z}$ since (1) is the same as (2).

Example 8

If $\underline{w} = 3t\underline{i} - t^2\underline{j}$ and $\underline{z} = 2t^2\underline{i} + 3\underline{j}$, verify the result

$$\frac{d}{dt}(\underline{w}\times\underline{z}) = \underline{w}\times\frac{d\underline{z}}{dt} + \frac{d\underline{w}}{dt}\times\underline{z}$$

Solution

$$\underline{w}\times\underline{z} = \begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ 3t & -t^2 & 0\\ 2t^2 & 3 & 0\end{vmatrix} = (9t + 2t^4)\underline{k} \quad \text{implying} \quad \frac{d}{dt}(\underline{w}\times\underline{z}) = (9 + 8t^3)\underline{k} \qquad (1)$$

$$\underline{w}\times\frac{d\underline{z}}{dt} = \begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ 3t & -t^2 & 0\\ 4t & 0 & 0\end{vmatrix} = 4t^3\underline{k} \qquad (2)$$

$$\frac{d\underline{w}}{dt}\times\underline{z} = \begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ 3 & -2t & 0\\ 2t^2 & 3 & 0\end{vmatrix} = (9 + 4t^3)\underline{k} \qquad (3)$$

We can see that (1) is the same as (2) + (3) as required.

Exercises

1. If $r = 3t\underline{i} + 2t^2\underline{j} + t^3\underline{k}$, find (a) $\dfrac{d\underline{r}}{dt}$ (b) $\dfrac{d^2\underline{r}}{dt^2}$

2. Given $\underline{B} = te^{-t}\ \underline{i} + \cos t\ \underline{j}$ find (a) $\dfrac{d\underline{B}}{dt}$ (b) $\dfrac{d^2\underline{B}}{dt^2}$

3. If $\underline{r} = 4t^2\underline{i} + 2t\underline{j} - 7\underline{k}$ evaluate $\underline{r}$ and $\dfrac{d\underline{r}}{dt}$ when $t = 1$.

4. If $\underline{w} = t^3\underline{i} - 7t\underline{k}$ and $\underline{z} = (2+t)\underline{i} + t^2\underline{j} - 2\underline{k}$

   (a) find $\underline{w} \cdot \underline{z}$, (b) find $\dfrac{d\underline{w}}{dt}$, (c) find $\dfrac{d\underline{z}}{dt}$, (d) show that $\dfrac{d}{dt}(\underline{w} \cdot \underline{z}) = \underline{w} \cdot \dfrac{d\underline{z}}{dt} + \dfrac{d\underline{w}}{dt} \cdot \underline{z}$

5. Given $\underline{r} = \sin t\ \underline{i} + \cos t\ \underline{j}$

   (a) find $\dot{\underline{r}}$, (b) find $\ddot{\underline{r}}$, (c) find $|\underline{r}|$

   (d) Show that the position vector $\underline{r}$ and velocity vector $\dot{\underline{r}}$ are perpendicular.

Answers

1. (a) $3\underline{i} + 4t\underline{j} + 3t^2\underline{k}$ (b) $4\underline{j} + 6t\underline{k}$

2. (a) $(-te^{-t} + e^{-t})\underline{i} - \sin t\underline{j}$ (b) $e^{-t}(t-2)\underline{i} - \cos t\underline{j}$

3. $4\underline{i} + 2\underline{j} - 7\underline{k},\ 8\underline{i} + 2\underline{j}$

4. (a) $t(t^3 + 2t^2 + 14)$ (b) $3t^2\underline{i} - 7\underline{k}$ (c) $\underline{i} + 2t\underline{j}$

5. (a) $\cos t\underline{i} - \sin t\underline{j}$ (b) $-\sin t\underline{i} - \cos t\underline{j}$ (c) 1 (d) Follows by showing $\underline{r} \cdot \dot{\underline{r}} = 0$.

Case Study: Complex Impedance 12.6

Electronic Filters

Electronic filters are used widely, for example in audio equipment to correct for imperfections in microphones or loudspeakers, or to introduce special effects. The purpose of a filter is to produce an alternating current (a.c.) output voltage that varies with the frequency of the input voltage. A filter must have at least one component which has an impedance that varies with frequency. The **impedance** is given by the time dependent ratio of 'voltage across the component' to 'current through the component'. This means that a filter must contain at least one inductance or capacitance. An **inductor** consists of a large number of coils of wire. When the current i flowing through an inductor changes, the associated magnetic field changes and produces a voltage v across the inductor which is proportional to the rate of change of the current. The constant of proportionality (**inductance**) is given the symbol L.

In electronics, it is usual to use lower case symbols for the time varying quantities. The standard representations for a.c. electronic signals are

$$v = V_0 \mathrm{e}^{j\omega t} \text{ and } i = I_0 \mathrm{e}^{j\omega t}$$

where V_0 is the (real) amplitude of the a.c. voltage and I_0 is the (real) amplitude of the a.c. current and $j = \sqrt{-1}$.

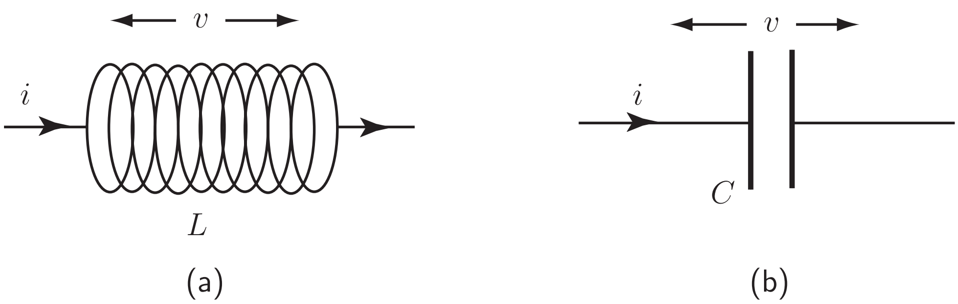


Figure 33: (a) an inductor (b) a capacitor

An inductor (see Figure 33) gives rise to an a.c. voltage

$$v = L\frac{di}{dt} = j\omega Li$$

Hence $v/i = jwL$ is the impedance of the inductor. The purely imaginary quantity, jwL, is called the **reactance** of the inductor. Usually a coil of wire forming an inductor also has resistance but this can be designed or assumed to be negligible. A **capacitor** consists of two conducting plates separated by a thin insulator. The charge (q) on the plates is proportional to the voltage (v) between the plates. The constant of proportionality (**capacitance**) is given the symbol C. So $q = Cv$. The current (i) into the capacitor is equal to the rate of change of the charge on the capacitor i.e.

$$i = \frac{dq}{dt} = C\frac{dv}{dt} = j\omega Cv.$$

Hence, for a capacitor, the impedance $Z_c = v/i = 1/jwC$. This purely imaginary quantity is also a reactance. Because of Ohm's law ($v = iR$), a resistance R provides a constant (real) contribution of R to the impedance of a circuit. If two resistors R_1 and R_2 are in series the same current passes through both of them and the combined resistance is $R_1 + R_2$. In the circuit shown in Figure 34 (consider the left-hand representation of this circuit first but note that the right-hand version is equivalent), the input voltage across both resistors and the output voltage across R_2 are related by

$$v_{in} = i(R_1 + R_2) \quad \text{and} \quad v_{out} = iR_2 \quad \text{so} \quad \frac{v_{out}}{v_{in}} = \frac{R_2}{R_1 + R_2}.$$

Such a circuit is called a **potential divider**.

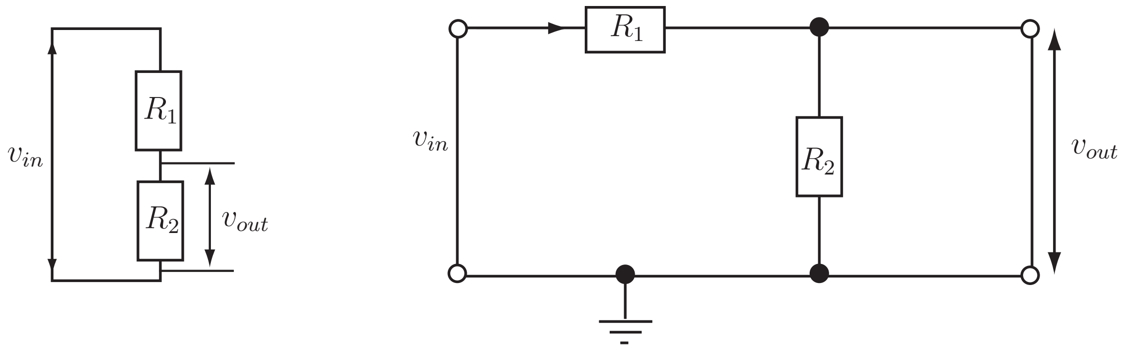


Figure 34: Two representations of a potential divider circuit

Now consider this circuit with the resistor R_2 replaced by a capacitor C as in Figure 35.

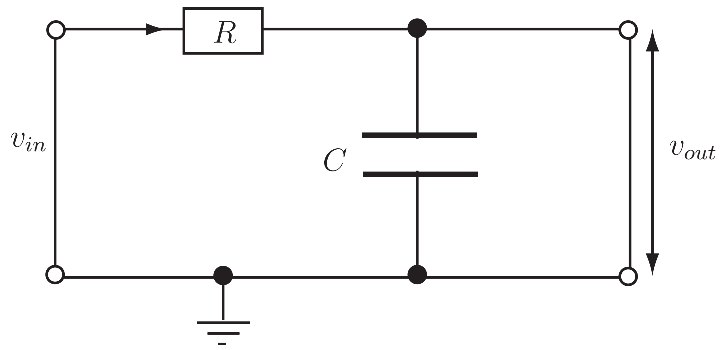


Figure 35: Low pass filter circuit containing a resistor and a capacitor

If R_1 is replaced by R and R_2 by $Z_C = 1/jwC$, in the relevant expression for the potential divider circuit, then

$$\frac{v_{out}}{v_{in}} = \frac{1/j\omega C}{R + 1/j\omega C} = \frac{1}{1 + j\omega RC}$$

The square of the magnitude of the voltage ratio is given by multiplying the existing complex expression by its complex conjugate, i.e.

$$\left|\frac{v_{out}}{v_{in}}\right|^2 = \frac{1}{(1 + j\omega RC)(1 - j\omega RC)} = \frac{1}{(1 + \omega^2 R^2 C^2)}$$

Figure 36 shows a plot of the magnitude of the voltage ratio as a function of ω, i.e. the **frequency response** for $R = 10\ \Omega$ and C =1 μF (i.e. 10^{-6}F). Note that the magnitude of the output voltage is close to that of the input voltage at low frequencies but decreases rapidly as frequency increases. This is an ideal low pass filter response.

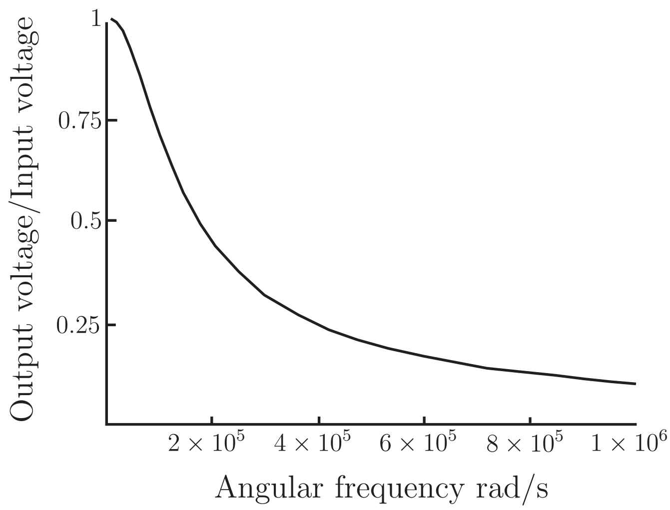


Figure 36: Frequency response of a low pass filter

Engineering problem stated in words

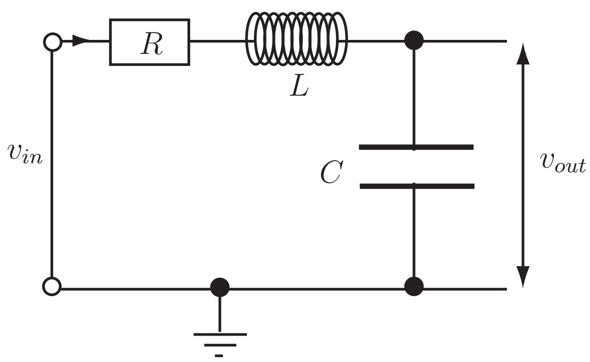


Figure 37: An LC filter circuit

Plot the frequency response of the LC filter circuit shown in Figure 37 if $R = 10\ \Omega$, $L = 0.1$ mH (i.e. 10^{-4}H) and $C = 1\ \mu$F. After plotting the response for two values of R below 10 Ω, comment on the way in which the response varies as R varies. Identify the frequency for which the response is maximum.

Engineering problem expressed mathematically

(a) Noting that the resistor and inductor are in series, replace R_1 by $(R + jwL)$ and $R2$ by $1/jwC$ in the equation $\dfrac{v_{out}}{v_{in}} = \dfrac{R_2}{R_1 + R_2}$

(b) Derive an expression for $\left|\dfrac{v_{out}}{v_{in}}\right|^2$

(c) Hence plot $\left|\dfrac{v_{out}}{v_{in}}\right|$ as a function of ω for $R = 10\ \Omega$.

(d) Plot $\left|\frac{v_{out}}{v_{in}}\right|$ for two further values of $R < 10\ \Omega$ (e.g. $5\ \Omega$ and $2\ \Omega$).

(e) Find an expression for the value of $\omega = \omega_{res}$ at which $\left|\frac{v_{out}}{v_{in}}\right|$ is maximum.

Mathematical analysis

(a) The substitutions $R_1 \to (R + jwL)$ and $R_2 \to 1/jwC$ in the equation

$$\frac{v_{out}}{v_{in}} = \frac{R_2}{R_1 + R_2} \quad \text{yield} \quad \frac{v_{out}}{v_{in}} = \frac{1/j\omega C}{R + j\omega L + 1/j\omega C} = \frac{1}{(1 - \omega^2 LC + j\omega RC)}$$

(b) Multiplying by the complex conjugate of the denominator

$$\left|\frac{v_{out}}{v_{in}}\right|^2 = \frac{1}{(1 - \omega^2 LC + j\omega RC)(1 - \omega^2 LC - j\omega RC)} = \frac{1}{(1 - \omega^2 LC)^2 + \omega^2 R^2 C^2}$$

(c) See the solid line in Figure 38.

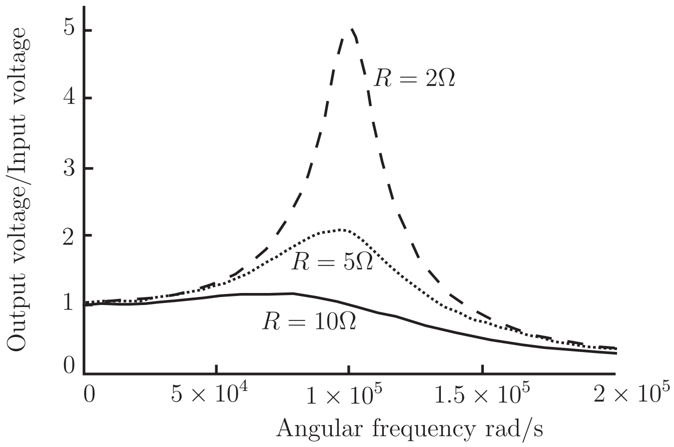


Figure 38: Frequency response of LC filter

(d) See the other broken lines in Figure 38.

There is a peak in the voltage output, which can exceed the voltage input by a considerable amount. It is particularly noticeable for small values of the resistance and decreases as the resistance increases.

(e) $\left|\frac{v_{out}}{v_{in}}\right|$ will be maximum when the first term in the denominator is zero (the other term is always positive for $\omega > 0$) i.e. when

$$\omega = \omega_{res} = \frac{1}{\sqrt{LC}} \quad \text{or} \quad f_{res} = \frac{\omega_{res}}{2\pi} = \frac{1}{2\pi\sqrt{LC}}$$

The corresponding frequency is known as the resonant frequency of the circuit.

Additional comment

The resonant behaviour depicted in Figure 38 is found in certain vibrating systems as well as electronic circuits. This gives rise to an electrical analogy for such mechanical systems and will be explored further after HELM 19 on differential equations.