Contents 11

Differentiation

Learning outcomes

In this Workbook you will learn what a derivative is and how to obtain the derivative of many commonly occurring functions. You will learn of the relationship between a derivative and the tangent line to a curve. You will learn something of the limiting process which arises in many areas of mathematics. You will learn how to use a table of derivatives to obtain the derivative of simple combinations of functions. Finally, you will learn how to take higher derivatives

Introducing Differentiation

11.1

Introduction

Differentiation is a technique which can be used for analysing the way in which functions change. In particular, it measures how rapidly a function is changing at any point. In engineering applications the function may, for example, represent the magnetic field strength of a motor, the voltage across a capacitor, the temperature of a chemical mix, and it is often important to know how quickly these quantities change.

In this Section we explain what is meant by the gradient of a curve and introduce differentiation as a method for finding the gradient at any point.

Prerequisites

Before starting this Section you should ...

- understand functional notation, e.g. $y = f(x)$
- be able to calculate the gradient of a straight line

Learning Outcomes

On completion you should be able to ...

- explain what is meant by the tangent to a curve
- explain what is meant by the gradient of a curve at a point
- calculate the derivative of a number of simple functions from first principles

1. Drawing tangents

Look at the graph shown in Figure 1a. A and B are two points on the graph, and they have been joined by a straight line. The straight line segment AB is known as a **chord**. We have lengthened the chord on both sides so that it extends beyond both A and B.

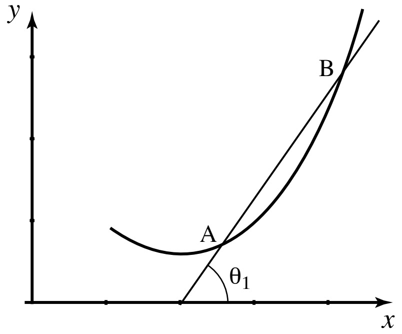


Figure 1a

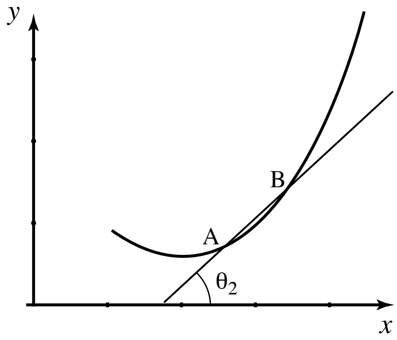


Figure 1b

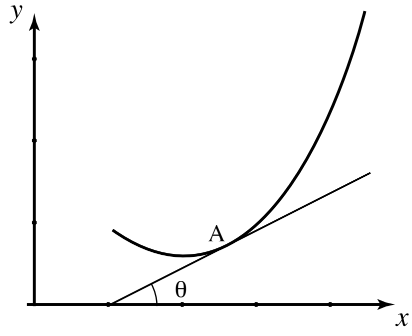
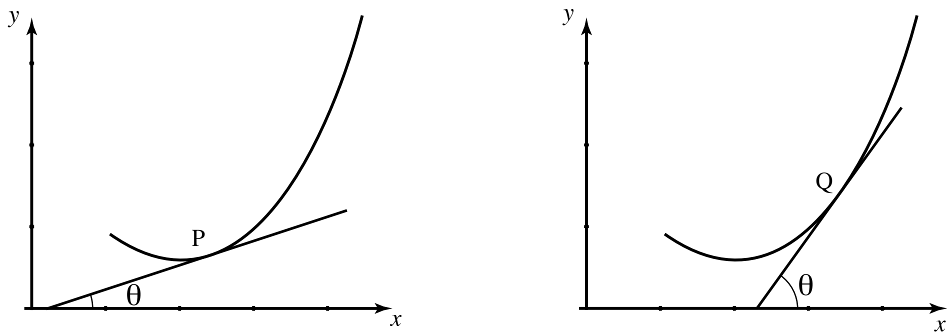


Figure 1c

In Figure 1b we have moved point B nearer to point A before drawing the extended chord. Imagine what would happen if we continue moving B nearer and nearer to A. You can do this for yourself by drawing additional points on the graph. Eventually, when B coincides with A, the extended chord is a straight line which just touches the curve at A. This line is now called the **tangent** to the curve at A, and is shown in Figure 1c.

If we know the position of two points on the line we can find the gradient of the straight line and can calculate the gradient of the tangent. We define the **gradient of the curve** at A to be the **gradient of the tangent** there. If this gradient is large at a particular point, the rate at which the function is changing is large too. If the gradient is small, the rate at which the function is changing is small. This is illustrated in Figure 2. Because of this, the gradient at A is also known as **the instantaneous rate of change** of the curve at A. Recall from your knowledge of the straight line, that if the line slopes upwards as we look from left to right, the gradient of the line is positive, whereas if the line slopes downwards, the gradient is negative.



The gradient of the tangent at P is small, so the rate at which the function is changing is small.

The gradient of the tangent at Q is large so the rate rate at which the function is changing is large.

Figure 2

Key Point 1

The gradient of the curve at a point, P, is equal to $\tan\theta$ where θ is the angle the tangent line at P makes with the positive x axis.

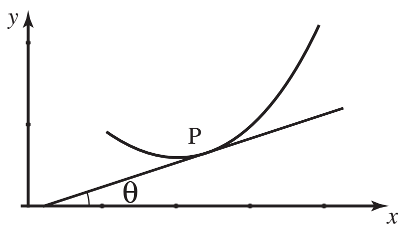
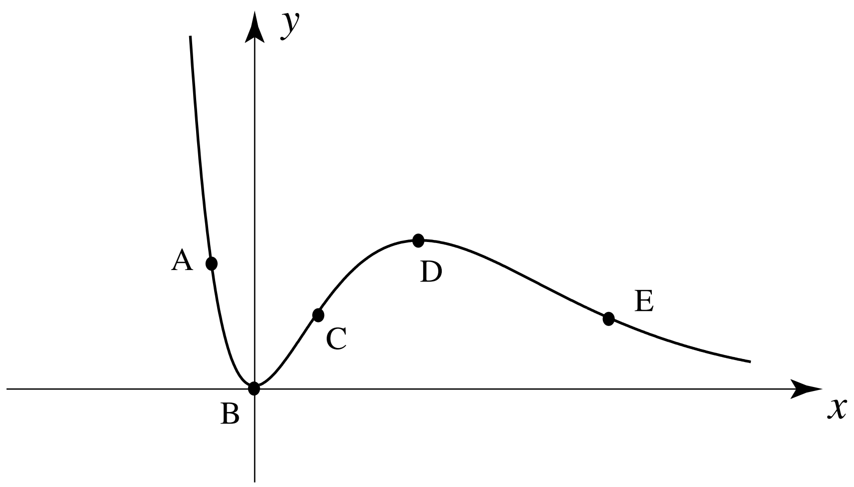


Figure 3

Task

Draw in, by eye, tangents to the curve shown below, at points A to E. State whether each tangent has positive, negative or zero gradient.



Your solution

Answer

A negative, B zero, C positive, D zero, E negative

In the following subsection we will see how to calculate the gradient of a curve precisely.

2. Finding the gradient at a specific point

In this subsection we shall consider a simple function to illustrate the calculation of a gradient. Look at the graph of the function $y(x) = x^2$ shown in Figure 4. Notice that the gradient of the graph changes as we move from point to point. In some places the gradient is positive; at others it is negative. The gradient is greater at some points than at others. In fact the gradient changes from point to point as we move along the curve.

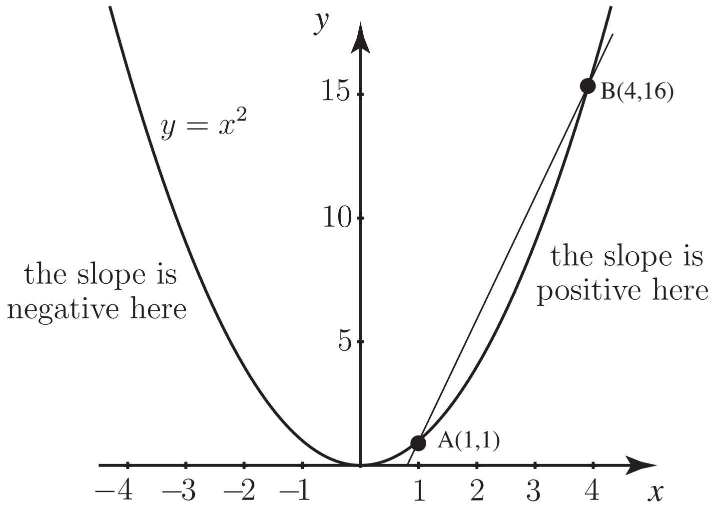


Figure 4

Inspect the graph carefully and make the following observations:

(a) A is the point with coordinates $(1, 1)$.
(b) B is the point with coordinates $(4, 16)$.
(c) We can calculate the gradient of the line AB from the formula

$$\text{gradient} = \frac{\text{difference between } y \text{ coordinates}}{\text{difference between } x \text{ coordinates}}$$

Therefore the gradient of chord AB is equal to $\dfrac{16-1}{4-1} = \dfrac{15}{3} = 5$. The gradient of AB is not the same as the gradient of the graph at A but we can regard it as an approximation, or estimate of the gradient at A. Is it an over-estimate or under-estimate ?

Task

Add the point C to the graph in Figure 4 where C has coordinates $(3, 9)$. Draw the line AC and calculate its gradient.

Your solution

gradient =

Answer
$\dfrac{9-1}{3-1} = 4$. Would you agree that this is a better estimate of the gradient at A than using AB?

We now carry the last task further by introducing point D at $(2, 4)$ and point E at $(1.5, 2.25)$ as shown in Figure 5. The gradient of AD is found to be 3 and the gradient of AE is 2.5.

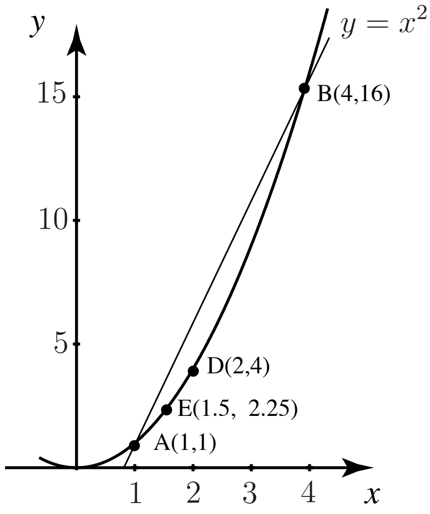


Figure 5

Observe that each time we carry out this procedure, and move the second point closer to A, the gradient of the line drawn is getting closer and closer to the gradient of the tangent at A. If we continue, the value we eventually obtain is the gradient of the tangent at A whose value is 2 as we will see shortly. This procedure illustrates how we define the gradient of the curve at A.

3. Finding the gradient at a general point

We now carry out the previous procedure more mathematically. Consider the graph of $y(x) = x^2$ in Figure 6. Let point A be any point with coordinates (a, a^2), and let point B be a second point with x coordinate $(a + h)$.

The y coordinate at A is a^2, because A lies on the graph $y = x^2$.

Similarly the y coordinate at B is $(a + h)^2$.

Therefore the gradient of the chord AB is

$$\frac{(a+h)^2 - a^2}{h}$$

This simplifies to

$$\frac{a^2 + 2ha + h^2 - a^2}{h} = \frac{2ha + h^2}{h} = \frac{h(2a+h)}{h} = 2a + h$$

This is the gradient of the line AB. As we let B move closer to A the value of h gets smaller and smaller and eventually tends to zero. We write this as $h \to 0$.

Now, as $h \to 0$, the gradient of AB tends to $2a$. Thus the gradient of the tangent to the curve at point A is $2a$. Because A is an arbitrary point, this result gives us a formula for finding the gradient of the graph of $y = x^2$ at any point: **the gradient is simply twice the x coordinate there**. For example when $x = 3$ the gradient is 2×3, that is 6, and when $x = 1$ the gradient is 2×1, that is 2 as we saw in the previous subsection.

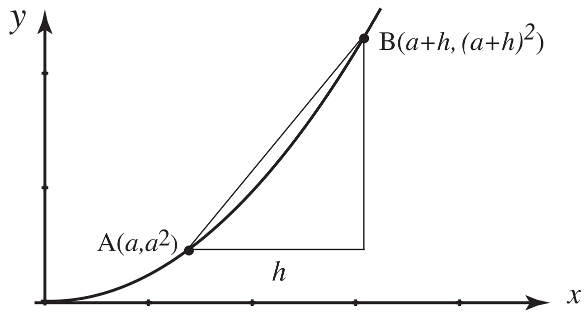


Figure 6

Generally, at a point whose coordinate is x the gradient is given by $2x$. The function, $2x$ which gives the gradient of $y = x^2$ is called the **derivative** of y with respect to x. It has other names too including the **rate of change** of y with respect to x.

A special notation is used to represent the derivative. It is not a particularly user-friendly notation but it is important to get used to it anyway. We write the derivative as $\dfrac{dy}{dx}$, pronounced 'dee y over dee x' or 'dee y by dee x' or even 'dee y, dee x'.

$\dfrac{dy}{dx}$ is not a fraction - so you can't do things like cancel the d's - just remember that it is the symbol or notation for the derivative. An alternative notation for the derivative is y'.

Key Point 2

The derivative of $y(x)$ is written $\dfrac{dy}{dx}$ or $y'(x)$ or simply y'

Exercises

1. Carry out the procedure above for the function $y = 3x^2$:
   (a) Let A be the point $(a, 3a^2)$.
   (b) Let B be the point $(a + h, 3(a + h)^2)$.
   (c) Find the gradient of the line AB.
   (d) Let $h \to 0$ to find the gradient of the curve at A.

2. Carry out the procedure above for the function $y = x^3$:
   (a) Let A be the point (a, a^3).
   (b) Let B be the point $(a + h, (a + h)^3)$.
   (c) Find the gradient of the line AB.
   (d) Let $h \to 0$ to find the gradient of the curve at A.

Answers

1. gradient $AB = 6a + 3h$, gradient at $A = 6a$. So, if $y = 3x^2$, $\dfrac{dy}{dx} = 6x$,

2. gradient $AB = 3a^2 + 3ah + h^2$, gradient at $A = 3a^2$. So, if $y = x^3$, $\dfrac{dy}{dx} = 3x^2$.

4. Differentiation of a general function from first principles

Consider the graph of $y = f(x)$ shown in Figure 7.

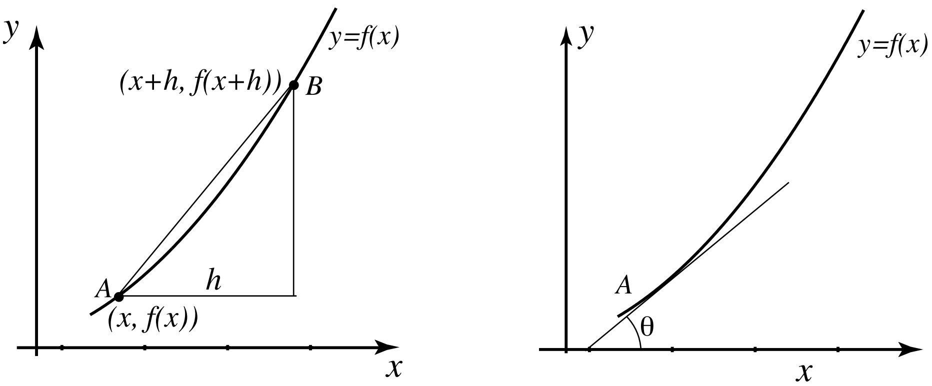


Figure 7: As $h \to 0$ the chord AB becomes the tangent at A

Carefully make the following observations:

(a) Point A has coordinates $(x, f(x))$.

(b) Point B has coordinates $(x + h, f(x + h))$.

(c) The straight line AB has gradient

$$\frac{f(x+h) - f(x)}{h}$$

(d) If we let $h \to 0$ we can find the gradient of the graph of $y = f(x)$ at the arbitrary point A, provided we can evaluate the appropriate limit on h. The resulting limit is the **derivative** of f with respect to x and is written $\dfrac{df}{dx}$ or $f'(x)$.

Key Point 3

Definition of Derivative

Given $y = f(x)$, its derivative is defined as

$$\frac{df}{dx} = \frac{f(x+h) - f(x)}{h} \qquad \text{in the limit as } h \text{ tends to 0.}$$

This is written

$$\frac{df}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

In a graphical context, the value of $\dfrac{df}{dx}$ at A is equal to $\tan\theta$ which is the tangent of the angle that the gradient line makes with the positive x-axis.

Example 1

Differentiate $f(x) = x^2 + 2x + 3$ from first principles.

Solution

$$\begin{aligned}
\frac{df}{dx} &= \lim_{h \to 0} \left\{ \frac{f(x+h) - f(x)}{h} \right\} \\
&= \lim_{h \to 0} \left\{ \frac{[(x+h)^2 + 2(x+h) + 3] - [x^2 + 2x + 3]}{h} \right\} \\
&= \lim_{h \to 0} \left\{ \frac{x^2 + 2xh + h^2 + 2x + 2h + 3 - x^2 - 2x - 3]}{h} \right\} \\
&= \lim_{h \to 0} \left\{ \frac{2xh + h^2 + 2h}{h} \right\} \\
&= \lim_{h \to 0} \{2x + h + 2\} \\
&= 2x + 2
\end{aligned}$$

Exercises

1. Use the definition of the derivative to find $\dfrac{df}{dx}$ when

   (a) $f(x) = 4x^2$, (b) $f(x) = 2x^3$, (c) $f(x) = 7x + 3$, (d) $f(x) = \dfrac{1}{x}$.
   (Harder: try (e) $f(x) = \sin x$ and use the small angle approximation $\sin\theta \approx \theta$ if θ is small and measured in radians.)

2. Using your results from Exercise 1 calculate the gradient of the following graphs at the given points:
   (a) $f(x) = 4x^2$ at $x = -2$, (b) $f(x) = 2x^3$ at $x = 2$, (c) $f(x) = 7x + 3$ at $x = -5$,
   (d) $f(x) = \dfrac{1}{x}$ at $x = 1/2$.

3. Find the rate of change of the function $y(x) = \dfrac{x}{x+3}$ at $x = 3$ by considering the interval $x = 3$ to $x = 3 + h$.

Answers

1. (a) $8x$, (b) $6x^2$, (c) 7, (d) $-1/x^2$, (e) $\cos x$.
2. (a) -16, (b) 24, (c) 7, (d) -4
3. $1/12$

Using a Table of Derivatives

11.2

Introduction

In Section 11.1 you were introduced to the idea of a derivative and you calculated some derivatives from first principles. Rather than calculating the derivative of a function from first principles it is common practice to use a **table of derivatives**. This Section provides such a table and shows you how to use it.

Prerequisites

Before starting this Section you should ...

- understand the meaning of the term 'derivative'
- understand what is meant by the notation $\frac{dy}{dx}$

Learning Outcomes

On completion you should be able to ...

- use a table of derivatives to perform differentiation

1. Table of derivatives

Table 1 lists some of the common functions used in engineering and their corresponding derivatives. Remember that in each case the function in the right-hand column gives the rate of change, or the gradient of the graph, of the function on the left at a particular value of x.

N.B. The angle must always be in radians when differentiating trigonometric functions.

Table 1
Common functions and their derivatives
(In this table k, n and c are constants)

Function	Derivative
constant	0
x	1
kx	k
x^n	nx^{n-1}
kx^n	knx^{n-1}
e^x	e^x
e^{kx}	ke^{kx}
$\ln x$	$1/x$
$\ln kx$	$1/x$
$\sin x$	$\cos x$
$\sin kx$	$k\cos kx$
$\sin(kx+c)$	$k\cos(kx+c)$
$\cos x$	$-\sin x$
$\cos kx$	$-k\sin kx$
$\cos(kx+c)$	$-k\sin(kx+c)$
$\tan x$	$\sec^2 x$
$\tan kx$	$k\sec^2 kx$
$\tan(kx+c)$	$k\sec^2(kx+c)$

In the trigonometric functions the angle is in radians.

Key Point 4

Particularly important is the rule for differentiating powers of functions:

$$\text{If} \quad y = x^n \quad \text{then} \quad \frac{dy}{dx} = nx^{n-1}$$

For example, if $y = x^3$ then $\dfrac{dy}{dx} = 3x^2$.

Example 2

Use Table 1 to find $\dfrac{dy}{dx}$ when y is given by (a) $7x$ (b) 14 (c) $5x^2$ (d) $4x^7$

Solution

(a) We note that $7x$ is of the form kx where $k = 7$. Using Table 1 we then have $\dfrac{dy}{dx} = 7$.

(b) Noting that 14 is a constant we see that $\dfrac{dy}{dx} = 0$.

(c) We see that $5x^2$ is of the form kx^n, with $k = 5$ and $n = 2$. The derivative, knx^{n-1}, is then $10x^1$, or more simply, $10x$. So if $y = 5x^2$, then $\dfrac{dy}{dx} = 10x$.

(d) We see that $4x^7$ is of the form kx^n, with $k = 4$ and $n = 7$. Hence the derivative, $\dfrac{dy}{dx}$, is given by $28x^6$.

Task

Use Table 1 to find $\dfrac{dy}{dx}$ when y is (a) $\sqrt{x}$ (b) $\dfrac{5}{x^3}$

(a) Write $\sqrt{x}$ as $x^{\frac{1}{2}}$, and use the result for differentiating x^n with $n = \dfrac{1}{2}$.

Your solution

Answer

$\dfrac{dy}{dx} = nx^{n-1} = \dfrac{1}{2}x^{\frac{1}{2}-1} = \dfrac{1}{2}x^{-\frac{1}{2}}$. This may be written as $\dfrac{1}{2\sqrt{x}}$.

(b) Write $\dfrac{5}{x^3}$ as $5x^{-3}$ and use the result for differentiating kx^n with $k = 5$ and $n = -3$.

Your solution

Answer

$5(-3)x^{-3-1} = -15x^{-4}$

Although Table 1 is written using x as the independent variable, the Table can be used for any variable.

Task

Use Table 1 to find

(a) $\dfrac{dz}{dt}$ given $z = e^t$ (b) $\dfrac{dp}{dt}$ given $p = e^{8t}$ (c) $\dfrac{dz}{dy}$ given $z = e^{-3y}$

(a)

Your solution

$\dfrac{dz}{dt} =$

Answer

From Table 1, if $y = e^x$, then $\dfrac{dy}{dx} = e^x$. Hence if $z = e^t$ then $\dfrac{dz}{dt} = e^t$.

(b)

Your solution

$\dfrac{dp}{dt} =$

Answer

$8e^{8t}$

(c)

Your solution

$\dfrac{dz}{dy} =$

Answer

$-3e^{-3y}$

Task

Find the derivative, $\dfrac{dy}{dx}$, when y is (a) $\sin 2x$ (b) $\cos \dfrac{x}{2}$ (c) $\tan 5x$

(a) Use the result for $\sin kx$ in Table 1, taking $k = 2$:

Your solution

$\dfrac{dy}{dx} =$

Answer

$2 \cos 2x$

(b) Note that $\cos\dfrac{x}{2}$ is the same as $\cos\dfrac{1}{2}x$. Use the result for $\cos kx$ in Table 1:

Your solution

$\dfrac{dy}{dx} =$

Answer

$-\dfrac{1}{2}\sin\dfrac{x}{2}$

(c) Use the result for $\tan kx$ in Table 1:

Your solution

$\dfrac{dy}{dx} =$

Answer

$5\sec^2 5x$

Exercises

1. Find the derivatives of the following functions with respect to x:

   (a) $9x^2$ (b) 5 (c) $6x^3$ (d) $-13x^4$

2. Find $\dfrac{dz}{dt}$ when z is given by:

   (a) $\dfrac{5}{t^3}$ (b) $\sqrt{t^3}$ (c) $5t^{-2}$ (d) $-\dfrac{3}{2}t^{\frac{3}{2}}$ (e) $\ln 5t$

3. Find the derivative of each of the following with respect to the appropriate variable:

   (a) $\sin 5x$ (b) $\cos 4t$ (c) $\tan 3r$ (d) e^{2v} (e) $\dfrac{1}{e^{3t}}$

4. Find the derivatives of the following with respect to x:

   (a) $\cos\dfrac{2x}{3}$ (b) $\sin(-2x)$ (c) $\tan \pi x$ (d) $e^{\frac{x}{2}}$ (e) $\ln \frac{2}{3}x$

Answers

1. (a) $18x$ (b) 0 (c) $18x^2$ (d) $-52x^3$

2. (a) $-15t^{-4}$ (b) $\dfrac{3}{2}t^{\frac{1}{2}}$ (c) $-10t^{-3}$ (d) $-\dfrac{9}{4}t^{\frac{1}{2}}$ (e) $\dfrac{1}{t}$

3. (a) $5\cos 5x$ (b) $-4\sin 4t$ (c) $3\sec^2 3r$ (d) $2e^{2v}$ (e) $-3e^{-3t}$

4. (a) $-\dfrac{2}{3}\sin\dfrac{2x}{3}$ (b) $-2\cos(-2x)$ (c) $\pi\sec^2 \pi x$ (d) $\dfrac{1}{2}e^{\frac{x}{2}}$ (e) $\dfrac{1}{x}$

Engineering Example 1

Electrostatic potential

Introduction

The electrostatic potential due to a point charge Q coulombs at a position r (m) from the charge is given by

$$V = \frac{Q}{4\pi\epsilon_0 r}$$

where ϵ_0, the permittivity of free space, $\approx 8.85 \times 10^{-12}$ F m^{-1} and $\pi \approx 3.14$.

The field strength at position r is given by $E = -\dfrac{dV}{dr}$.

Problem in words

Find the electric field strength at a distance of 5 m from a source with a charge of 1 coulomb.

Mathematical statement of the problem

$$V = \frac{Q}{4\pi\epsilon_0 r}$$

Substitute for ϵ_0 and π and use $Q = 1$, so that

$$V = \frac{1}{4 \times 3.14 \times 8.85 \times 10^{-12} r} \approx \frac{9 \times 10^9}{r} = 9 \times 10^9 r^{-1}$$

We need to differentiate V in order to find the electric field strength from the relationship $E = -\dfrac{dV}{dr}$

Mathematical analysis

$$E = -\frac{dV}{dr} = -9 \times 10^9(-r^{-2}) = 9 \times 10^9 r^{-2}$$

When $r = 5$,

$$E = \frac{9 \times 10^9}{25} = 3.6 \times 10^8 \text{ (V m}^{-1}.)$$

Interpretation

The electric field strength is 3.6×10^8 V m^{-1} at $r = 5$ m.

Note that the field potential varies with the reciprocal of distance (i.e. inverse linear law with distance) whereas the field strength obeys an inverse square law with distance.

2. Extending the table of derivatives

We now quote simple rules which enable us to extend the range of functions which we can differentiate. The first two rules are for differentiating **sums** or **differences** of functions. The reader should note that all of the rules quoted below can be obtained from first principles using the approach outlined in Section 11.1.

Key Point 5

Rule 1: The derivative of $f(x) + g(x)$ is $\dfrac{df}{dx} + \dfrac{dg}{dx}$

Rule 2: The derivative of $f(x) - g(x)$ is $\dfrac{df}{dx} - \dfrac{dg}{dx}$

These rules say that to find the derivative of the sum (or difference) of two functions, we simply calculate the sum (or difference) of the derivatives of each function.

Example 3

Find the derivative of $y = x^6 + x^4$.

Solution

We simply calculate the sum of the derivatives of each separate function:

$$\frac{dy}{dx} = 6x^5 + 4x^3$$

The third rule tells us how to differentiate a **multiple** of a function. We have already met and applied particular cases of this rule which appear in Table 1.

Key Point 6

Rule 3: If k is a constant, the derivative of $kf(x)$ is $k\dfrac{df}{dx}$

This rule tells us that if a function is multiplied by a constant, k, then the derivative is also multiplied by the same constant, k.

Example 4

Find the derivative of $y = 8e^{2x}$

Solution

Here we are interested in differentiating a multiple of the function e^{2x}. We differentiate e^{2x}, giving $2e^{2x}$, and multiply the result by 8. Thus

$$\frac{dy}{dx} = 8 \times 2e^{2x} = 16e^{2x}$$

Example 5

Find the derivative of $y = 6\sin 2x + 3x^2 - 5e^{3x}$

Solution

We differentiate each part of the function in turn.

$$\begin{aligned} y &= 6\sin 2x + 3x^2 - 5e^{3x} \\ \frac{dy}{dx} &= 6(2\cos 2x) + 3(2x) - 5(3e^{3x}) \\ &= 12\cos 2x + 6x - 15e^{3x} \end{aligned}$$

Task

Find $\dfrac{dy}{dx}$ where $y = 7x^5 - 3e^{5x}$.

First find the derivative of $7x^5$:

Your solution

Answer

$7(5x^4) = 35x^4$

Next find the derivative of $3e^{5x}$:

Your solution

Answer

$3(5e^{5x}) = 15e^{5x}$

Combine your results to find the derivative of $7x^5 - 3e^{5x}$:

Your solution

$$\frac{dy}{dx} =$$

Answer

$35x^4 - 15e^{5x}$

Task

Find $\dfrac{dy}{dx}$ where $y = 4\cos\dfrac{x}{2} + 17 - 9x^3$.

First find the derivative of $4\cos\dfrac{x}{2}$:

Your solution

Answer

$4(-\frac{1}{2}\sin\frac{x}{2}) = -2\sin\frac{x}{2}$

Next find the derivative of 17:

Your solution

Answer

0

Then find the derivative of $-9x^3$:

Your solution

Answer

$3(-9x^2) = -27x^2$

Finally state the derivative of $y = 4\cos\dfrac{x}{2} + 17 - 9x^3$:

Your solution

$$\frac{dy}{dx} =$$

Answer

$-2\sin\dfrac{x}{2} - 27x^2$

Exercises

1. Find $\dfrac{dy}{dx}$ when y is given by:

   (a) $3x^7 + 8x^3$ (b) $-3x^4 + 2x^{1.5}$ (c) $\dfrac{9}{x^2} + \dfrac{14}{x} - 3x$ (d) $\dfrac{3+2x}{4}$ (e) $(2+3x)^2$

2. Find the derivative of each of the following functions:

   (a) $z(t) = 5\sin t + \sin 5t$ (b) $h(v) = 3\cos 2v - 6\sin\frac{v}{2}$

   (c) $m(n) = 4e^{2n} + \dfrac{2}{e^{2n}} + \dfrac{n^2}{2}$ (d) $H(t) = \frac{e^{3t}}{2} + 2\tan 2t$ (e) $S(r) = (r^2+1)^2 - 4e^{-2r}$

3. Differentiate the following functions.

   (a) $A(t) = (3 + e^t)^2$ (b) $B(s) = \pi e^{2s} + \dfrac{1}{s} + 2\sin\pi s$

   (c) $V(r) = (1 + \dfrac{1}{r})^2 + (r+1)^2$ (d) $M(\theta) = 6\sin 2\theta - 2\cos\dfrac{\theta}{4} + 2\theta^2$

   (e) $H(t) = 4\tan 3t + 3\sin 2t - 2\cos 4t$

Answers

1. (a) $21x^6 + 24x^2$ (b) $-12x^3 + 3x^{0.5}$ (c) $-\dfrac{18}{x^3} - \dfrac{14}{x^2} - 3$ (d) $\dfrac{1}{2}$ (e) $12 + 18x$

2. (a) $z' = 5\cos t + 5\cos 5t$ (b) $h' = -6\sin 2v - 3\cos\dfrac{v}{2}$ (c) $m' = 8e^{2n} - 4e^{-2n} + n$

   (d) $H' = \dfrac{3e^{3t}}{2} + 4\sec^2 2t$ (e) $S' = 4r^3 + 4r + 8e^{-2r}$

3. (a) $A' = 6e^t + 2e^{2t}$ (b) $B' = 2\pi e^{2s} - \dfrac{1}{s^2} + 2\pi\cos(\pi s)$

   (c) $V' = -\dfrac{2}{r^2} - \dfrac{2}{r^3} + 2r + 2$ (d) $M' = 12\cos 2\theta + \dfrac{1}{2}\sin\dfrac{\theta}{4} + 4\theta$

   (e) $H' = 12\sec^2 3t + 6\cos 2t + 8\sin 4t$

3. Evaluating a derivative

The need to find the rate of change of a function at a particular point occurs often. We do this by finding the derivative of the function, and then evaluating the derivative at that point. When taking derivatives of trigonometric functions, any angles **must** be measured in radians. Consider a function, $y(x)$. We use the notation $\dfrac{dy}{dx}(a)$ or $y'(a)$ to denote the derivative of y evaluated at $x = a$. So $y'(0.5)$ means the value of the derivative of y when $x = 0.5$.

Example 6

Find the value of the derivative of $y = x^3$ where $x = 2$. Interpret your result.

Solution

We have $y = x^3$ and so $\dfrac{dy}{dx} = 3x^2$.

When $x = 2$, $\dfrac{dy}{dx} = 3(2)^2 = 12$, that is, $\dfrac{dy}{dx}(2) = 12$ (Equivalently, $y'(2) = 12$).

The derivative is positive when $x = 2$ and so y is increasing at this point. When $x = 2$, y is increasing at a rate of 12 vertical units per horizontal unit.

Engineering Example 2

Electromotive force

Introduction

Potential difference in an electrical circuit is produced by **electromotive force** (e.m.f.) which is measured in volts and describes the force that maintains current flow around a closed path. Every source of continuous electrical energy, including batteries, generators and thermocouples, consist essentially of an energy converter that produces an e.m.f. An electric current always produces a magnetic field. So the current i which flows round any closed path produces a magnetic flux ϕ which passes through that path. Conversely, if another closed path, i.e. another coil, is placed within the first path, then the magnetic field due to the first circuit can induce an e.m.f. and hence a current in the second coil. The simplest closed path is a single loop. More commonly, helical coils, known as search coils, with known area and number of turns, are used. The induced e.m.f. depends upon the number of turns in the coil. The search coil is used with a fluxmeter to measure the change of flux linkage.

Problem in words

A current i is travelling through a single turn loop of radius 1 m. A 4-turn search coil of effective area 0.03 m^2 is placed inside the loop. The magnetic flux in weber (Wb) linking the search coil is given by:

$$\phi = \mu_0 \frac{iA}{2r}$$

where r (m) is the radius of the current carrying loop, A (m^2) is the area of the search coil and μ_0 is the permeability of free space, 4×10^{-7} H m^{-1}.

Find the e.m.f. (in volts) induced in the search coil, given by $\varepsilon = -N\left(\dfrac{d\phi}{dt}\right)$ where N is the number of turns in the search coil, and the current is given by

$$i = 20\sin(20\pi t) + 50\sin(30\pi t)$$

Mathematical statement of the problem

Substitute $i = 20\sin(20\pi t) + 50\sin(30\pi t)$ into $\phi = \mu_0 \dfrac{iA}{2r}$ and find $\varepsilon = -N\left(\dfrac{d\phi}{dt}\right)$ when $r = 1$ m, $A = 0.03$ m^2, $\mu_0 = 4 \times 10^{-7}$ H m^{-1} and $N = 4$.

Mathematical analysis

$$\phi = \mu_0 \frac{iA}{2r} = \frac{\mu_0 A}{2r}(20\sin(20\pi t) + 50\sin(30\pi t))$$

So
$$\begin{aligned}
\frac{d\phi}{dt} &= \frac{\mu_0 A}{2r}(20\pi \times 20\cos(20\pi t) + 30\pi \times 50\cos(30\pi t)) \\
&= \frac{\mu_0 A}{2r}(400\pi\cos(20\pi t) + 1500\pi\cos(30\pi t))
\end{aligned}$$

so
$$\varepsilon = -N\frac{d\phi}{dt} = -4\frac{\mu_0 A}{2r}(400\pi\cos(20\pi t) + 1500\pi\cos(30\pi t))$$

Now, $\mu_0 = 4\pi \times 10^{-7},\ A = 0.03$ and $r = 1$

So
$$\begin{aligned}
\varepsilon &= \frac{-4 \times 4\pi \times 10^{-7} \times 0.03}{2 \times 1}(400\pi\cos(20\pi t) + 1500\pi\cos(30\pi t)) \\
&= -7.5398 \times 10^{-8}(1256.64\cos(20\pi t) + 4712.39\cos(30\pi t)) \\
&= -9.475 \times 10^{-5}\cos(20\pi t) + 3.553 \times 10^{-4}\cos(30\pi t)
\end{aligned}$$

Interpretation

The induced e.m.f. is $-9.475 \times 10^{-5}\cos(20\pi t) + 3.553 \times 10^{-4}\cos(30\pi t)$.

The graphs in Figure 8 show the initial current in the single loop and the e.m.f. induced in the search coil.

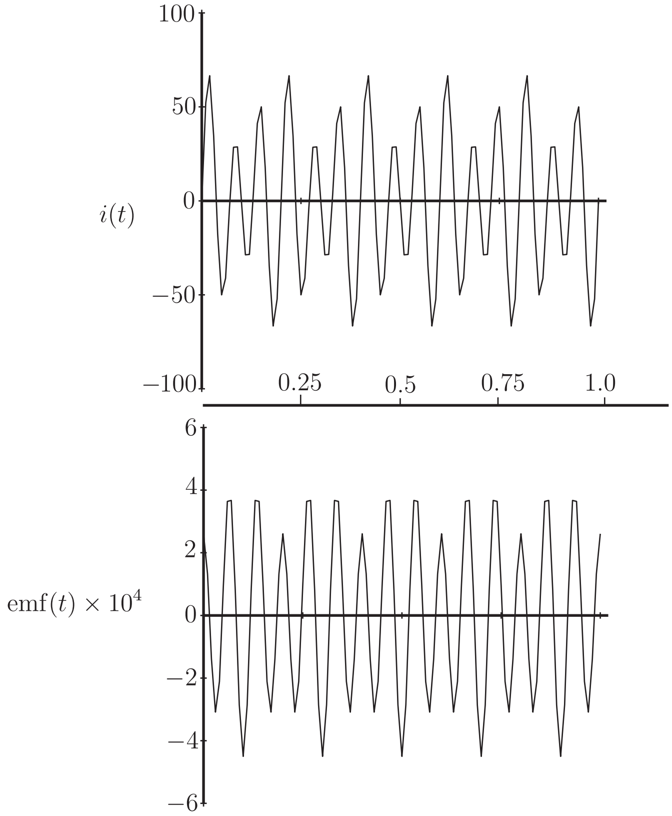


Figure 8

Note that the induced e.m.f. does not start at zero, which the initial current does, and has a different pattern of variation with time.

Exercises

1. Calculate the derivative of $y = x^2 + \sin x$ when $x = 0.2$ radians.

2. Calculate the rate of change of $i(t) = 4\sin 2t + 3t$ when

   (a) $t = \dfrac{\pi}{3}$ (b) $t = 0.6$ radians

3. Calculate the rate of change of $F(t) = 5\sin t - 3\cos 2t$ when

   (a) $t = 0$ radians (b) $t = 1.3$ radians

Answers

1. 1.380

2. (a) -1 (b) 5.8989

3. (a) 5 (b) 4.4305

Higher Derivatives 11.3

Introduction

The derivative, $\dfrac{dy}{dx}$, is more expressly called the **first derivative** of y. By differentiating the first derivative, we obtain the **second derivative**; by differentiating the second derivative we obtain the **third derivative** and so on. These second and subsequent derivatives are known as **higher derivatives**. Second derivatives in particular occur frequently in engineering contexts.

Prerequisites

Before starting this Section you should . . .

- be able to differentiate standard functions

Learning Outcomes

On completion you should be able to . . .

- obtain higher derivatives

1. The derivative of a derivative

You have already learnt how to calculate the derivative of a function using a table of derivatives and applying some basic rules. By differentiating the function, $y(x)$, we obtain the derivative, $\dfrac{dy}{dx}$. By repeating the process we can obtain higher derivatives.

Example 7

Calculate the first, second and third derivatives of $y = x^4 + 6x^2$.

Solution

The first derivative is $\dfrac{dy}{dx}$:

first derivative $= 4x^3 + 12x$

To obtain the second derivative we differentiate the first derivative.

second derivative $= 12x^2 + 12$

The third derivative is found by differentiating the second derivative.

third derivative $= 24x + 0 = 24x$

2. Notation for derivatives

Just as there is a notation for the first derivative so there is a similar notation for higher derivatives. Consider the function, $y(x)$. We know that the first derivative is $\dfrac{dy}{dx}$ or $\dfrac{d}{dx}(y)$ which is the instruction to differentiate the function $y(x)$. The second derivative is calculated by differentiating the first derivative, that is

$$\text{second derivative} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

So, using a fairly obvious adaptation of our derivative notation, the second derivative is denoted by $\dfrac{d^2y}{dx^2}$ and is read as 'dee two y by dee x squared'. This is often written more concisely as y''.

In similar manner, the third derivative is denoted by $\dfrac{d^3y}{dx^3}$ or y''' and so on. So, referring to Example 6 we could have written

$$\begin{aligned} \text{first derivative} = \frac{dy}{dx} &= 4x^3 + 12x \\ \text{second derivative} = \frac{d^2y}{dx^2} &= 12x^2 + 12 \\ \text{third derivative} = \frac{d^3y}{dx^3} &= 24x \end{aligned}$$

Key Point 7

If $y = y(x)$ then its first, second and third derivatives are denoted by:

$$\frac{dy}{dx} \qquad \frac{d^2y}{dx^2} \qquad \frac{d^3y}{dx^3}$$

$$\text{or} \qquad y' \qquad y'' \qquad y'''$$

In most examples we use x to denote the independent variable and y the dependent variable. However, in many applications, time t is the independent variable. In this case a special notation is used for derivatives. Derivatives with respect to t are often indicated using a **dot** notation, so $\dfrac{dy}{dt}$ can be written as $\dot{y}$, pronounced 'y dot'. Similarly, a second derivative with respect to t can be written as $\ddot{y}$, pronounced 'y double dot'.

Key Point 8

If $y = y(t)$ then

$$\dot{y} \quad \text{stands for} \quad \frac{dy}{dt}, \qquad \ddot{y} \quad \text{stands for} \quad \frac{d^2y}{dt^2} \quad \text{etc}$$

Task

Calculate $\dfrac{d^2y}{dt^2}$ and $\dfrac{d^3y}{dt^3}$ given $y = e^{2t} + \cos t$.

First find $\dfrac{dy}{dx}$:

Your solution

Answer

$\dfrac{dy}{dt} = 2e^{2t} - \sin t$

Now obtain the second derivative:

Your solution

$$\frac{d^2y}{dt^2} =$$

Answer

$4e^{2t} - \cos t$

Finally, obtain the third derivative:

Your solution

$$\frac{d^3y}{dt^3} = \frac{d}{dt}\left(\frac{d^2y}{dt^2}\right) =$$

Answer

$8e^{2t} + \sin t$

Note that in the last Task we could have used the dot notation and written $\dot{y} = 2e^{2t} - \sin t$, $\ddot{y} = 4e^{2t} - \cos t$ and $\dddot{y} = 8\mathrm{e}^{2t} + \sin t$

We may need to evaluate higher derivatives at specific points. We use an obvious notation.

The second derivative of $y(x)$, evaluated at say, $x = 2$, is written as $\dfrac{d^2y}{dx^2}(2)$, or more simply as $y''(2)$.

The third derivative evaluated at $x = -1$ is written as $\dfrac{d^3y}{dx^3}(-1)$ or $y'''(-1)$.

Task

Given $\quad y(x) = 2\sin x + 3x^2 \quad$ find (a) $y'(1)$ $\quad$ (b) $y''(-1)$ $\quad$ (c) $y'''(0)$

First find $y'(x), y''(x)$ and $y'''(x)$:

Your solution

$y'(x) =$ $\qquad$ $y''(x) =$ $\qquad$ $y''' =$

Answer

$y'(x) = 2\cos x + 6x \qquad y''(x) = -2\sin x + 6 \qquad y'''(x) = -2\cos x$

Now substitute $x = 1$ in $y'(x)$ to obtain $y'(1)$:

Your solution

(a) $\quad y'(1) =$

Answer

$y'(1) = 2\cos 1 + 6(1) = 7.0806$. Remember, in $\cos 1$ the '1' is 1 radian.

Now find $y''(-1)$:

Your solution

(b) $y''(-1) =$

Answer

$y''(-1) = -2\sin(-1) + 6 = 7.6829$

Finally, find $y'''(0)$:

Your solution

(c) $y'''(0) =$

Answer

$y'''(0) = -2\cos 0 = -2.$

Exercises

1. Find $\dfrac{d^2y}{dx^2}$ where $y(x)$ is defined by:

   (a) $3x^2 - e^{2x}$ (b) $\sin 3x + \cos x$ (c) $\sqrt{x}$ (d) $e^x + e^{-x}$ (e) $1 + x + x^2 + \ln x$

2. Find $\dfrac{d^3y}{dx^3}$ where y is given in Exercise 1.

3. Calculate $\ddot{y}(1)$ where $y(t)$ is given by:

   (a) $t(t^2+1)$ (b) $\sin(-2t)$ (c) $2e^t + e^{2t}$ (d) $\dfrac{1}{t}$ (e) $\cos\dfrac{t}{2}$

4. Calculate $\dddot{y}(-1)$ for the functions given in Exercise 3.

Answers

1. (a) $6 - 4e^{2x}$ (b) $-9\sin 3x - \cos x$ (c) $-\dfrac{1}{4}x^{-3/2}$ (d) $e^x + e^{-x}$ (e) $2 - \dfrac{1}{x^2}$

2. (a) $-8e^{2x}$ (b) $-27\cos 3x + \sin x$ (c) $\dfrac{3}{8}x^{-5/2}$ (d) $e^x - e^{-x}$ (e) $\dfrac{2}{x^3}$

3. (a) 6 (b) 3.6372 (c) 34.9927 (d) 2 (e) -0.2194

4. (a) 6 (b) -3.3292 (c) 1.8184 (d) -6 (e) -0.0599

Differentiating Products and Quotients 11.4

Introduction

We have seen, in the first three Sections, how standard functions like x^n, e^{ax}, $\sin ax$, $\cos ax$, $\ln ax$ may be differentiated.

In this Section we see how more complicated functions may be differentiated. We concentrate, for the moment, on products and quotients of standard functions like $x^n e^{ax}$, $\dfrac{e^{ax}\ln x}{\sin x}$.

We will see that two simple rules may be consistently employed to obtain the derivatives of such functions.

Prerequisites

Before starting this Section you should . . .

- be able to differentiate the standard functions: logarithms, polynomials, exponentials, and trigonometric functions
- be able to manipulate algebraic expressions

Learning Outcomes

On completion you should be able to . . .

- differentiate products and quotients of the standard functions
- differentiate a quotient using the product rule

1. Differentiating a product

In previous Sections we have examined the process of differentiating functions. We found how to obtain the derivative of many commonly occurring functions. These are recorded in the following table (remember, arguments of trigonometric functions are assumed to be in **radians**).

Table 2

y	$\dfrac{dy}{dx}$
x^n	nx^{n-1}
$\sin ax$	$a\cos ax$
$\cos ax$	$-a\sin ax$
$\tan ax$	$a\sec^2 ax$
$\sec ax$	$a\sec x\tan x$
$\ln ax$	$\dfrac{1}{x}$
e^{ax}	ae^{ax}
$\cosh ax$	$a\sinh ax$
$\sinh ax$	$a\cosh ax$

In this Section we consider how to differentiate non-standard functions - in particular those which can be written as the **product** of standard functions. Being able to differentiate such functions depends upon the following Key Point.

Key Point 9

Product Rule

If $y = f(x)g(x)$ then $\dfrac{dy}{dx} = \dfrac{df}{dx}g(x) + f(x)\dfrac{dg}{dx}$ (or $y' = f'g + fg'$)

If $y = u.v$ then $\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$ (or $y' = uv' + vu'$)

These versions are equivalent, and called the **product rule**.

We shall not prove this result, instead we shall concentrate on its use.

Example 8

Differentiate (a) $y = x^2 \sin x$ (b) $y = x \ln x$

Solution

(a) Here $f(x) = x^2$, $g(x) = \sin x$ $\quad \therefore \quad \dfrac{df}{dx} = 2x \, , \quad \dfrac{dg}{dx} = \cos x$

and so $\quad \dfrac{dy}{dx} = 2x(\sin x) + x^2(\cos x) = x(2\sin x + x\cos x)$

(b) Here $f(x) = x$, $g(x) = \ln x$ $\quad \therefore \quad \dfrac{df}{dx} = 1 \, , \quad \dfrac{dg}{dx} = \dfrac{1}{x}$

and so $\quad \dfrac{dy}{dx} = 1.(\ln x) + x.\left(\dfrac{1}{x}\right) = \ln x + 1$

Task

Determine the derivatives of the following functions (a) $y = e^x \ln x$, (b) $y = \dfrac{e^{2x}}{x^2}$

(a) Use the product rule:

Your solution

$f(x) =$ $\quad \dfrac{df}{dx} =$ $\quad g(x) =$ $\quad \dfrac{dg}{dx} =$

$\therefore \quad \dfrac{dy}{dx} =$

Answer

$$\frac{dy}{dx} = e^x \ln x + \frac{e^x}{x}$$

(b) Write $y = (x^{-2})e^{2x}$ and then differentiate:

Your solution

$f(x) =$ $\quad \dfrac{df}{dx} =$ $\quad g(x) =$ $\quad \dfrac{dg}{dx} =$

$\therefore \quad \dfrac{dy}{dx} =$

Answer

$$\frac{dy}{dx} = (-2x^{-3})e^{2x} + x^{-2}(2e^{2x}) = \frac{2e^{2x}}{x^3}(-1 + x)$$

The rule for differentiating a product can be extended to any number of products. If, for example, $y = f(x)g(x)h(x)$ then

$$\begin{aligned}\frac{dy}{dx} &= \frac{df}{dx}[g(x)h(x)] + f(x)\frac{d}{dx}[g(x)h(x)] \\ &= \frac{df}{dx}g(x)h(x) + f(x)\left\{\frac{dg}{dx}h(x) + g(x)\frac{dh}{dx}\right\} \\ &= \frac{df}{dx}g(x)h(x) + f(x)\frac{dg}{dx}h(x) + f(x)g(x)\frac{dh}{dx}\end{aligned}$$

That is, each function in the product is differentiated in turn and the three results added together.

Example 9

If $y = xe^{2x}\sin x$ then find $\dfrac{dy}{dx}$.

Solution

Here $f(x) = x$, $g(x) = e^{2x}$, $h(x) = \sin x$

$$\frac{df}{dx} = 1, \quad \frac{dg}{dx} = 2e^{2x}, \quad \frac{dh}{dx} = \cos x$$

$$\begin{aligned}\therefore \qquad \frac{dy}{dx} &= 1(e^{2x}\sin x) + x(2e^{2x})\sin x + xe^{2x}(\cos x) \\ &= e^{2x}(\sin x + 2x\sin x + x\cos x)\end{aligned}$$

Task

Obtain the first derivative of $y = x^2(\ln x)\sinh x$.

Firstly identify the three functions:

Your solution

$f(x) =$ $\qquad$ $g(x) =$ $\qquad$ $h(x) =$

Answer

$f(x) = x^2, \quad g(x) = \ln x, \quad h(x) = \sinh x$

Now find the derivative of each of these functions:

Your solution

$$\frac{df}{dx} = \qquad \frac{dg}{dx} = \qquad \frac{dh}{dx} =$$

Answer

$\dfrac{df}{dx} = 2x, \quad \dfrac{dg}{dx} = \dfrac{1}{x}, \quad \dfrac{dh}{dx} = \cosh x$

Finally obtain $\dfrac{dy}{dx}$:

Your solution

Answer

$$\begin{aligned} \frac{dy}{dx} &= 2x(\ln x)\sinh x + x^2\left(\frac{1}{x}\right)\sinh x + x^2 \ln x(\cosh x) \\ &= 2x \ln x \sinh x + x \sinh x + x^2 \ln x \cosh x \end{aligned}$$

Task

Find the second derivative of $y = x^2(\ln x)\sinh x$ by differentiating each of the three terms making up $\dfrac{dy}{dx}$ found in the previous Task ($2x \ln x \sinh x$, $x \sinh x$, $x^2 \ln x \cosh x$), and finally, simplify your answer by collecting like terms together:

Your solution

$\dfrac{d}{dx}(2x \ln x \sinh x) =$

$\dfrac{d}{dx}(x \sinh x) =$

$\dfrac{d}{dx}(x^2 \ln x \cosh x) =$

$\dfrac{d^2y}{dx^2} =$

Answer

$\dfrac{d^2y}{dx^2} = (2 + x^2)\ln x \sinh x + 3\sinh x + 2x \cosh x + 4x \ln x \cosh x$

Exercises

1. In each case find the derivative of the function

   (a) $y = x\tan x$

   (b) $y = x^4\ln(2x)$

   (c) $y = \sin^2 x$

   (d) $y = e^{2x}\cos 3x$

2. Find the derivatives of:

   (a) $y = \dfrac{x}{\cos x}$

   (b) $y = e^x \sin x$

   (c) Obtain the derivative of $y = xe^x\tan x$ using the results of parts (a) and (b).

Answers

1. (a) $\dfrac{dy}{dx} = \tan x + x\sec^2 x$

   (b) $\dfrac{dy}{dx} = 4x^3\ln(2x) + \dfrac{x^4}{x} = x^3(4\ln(2x) + 1)$

   (c) $y = \sin x . \sin x$

   $\therefore \quad \dfrac{dy}{dx} = \cos x \sin x + \sin x \cos x = 2\sin x \cos x = \sin 2x$

   (d) $\dfrac{dy}{dx} = (2e^{2x})\cos 3x + e^{2x}(-3\sin 3x) = e^{2x}(2\cos 3x - 3\sin 3x)$

2. (a) $y = x\sec x \qquad \therefore \quad \dfrac{dy}{dx} = \sec x + x\sec x\tan x$

   (b) $\dfrac{dy}{dx} = e^x\sin x + e^x\cos x = e^x(\sin x + \cos x)$

   (c) The derivative of $y = xe^x\tan x = (x\sec x)(e^x\sin x)$ is found by applying the product rule to the results of (a) and (b):

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}(x\sec x).e^x\sin x + (x\sec x)\frac{d}{dx}(e^x\sin x) \\
&= (\sec x + x\sec x\tan x)e^x\sin x + x\sec x(e^x)(\sin x + \cos x) \\
&= e^x(x + \tan x + x\tan x + x\tan^2 x)
\end{aligned}$$

2. Differentiating a quotient

In this Section we consider functions of the form $y = \dfrac{f(x)}{g(x)}$. To find the derivative of such a function we make use of the following Key Point:

Key Point 10

Quotient Rule

If $\quad y = \dfrac{f(x)}{g(x)} \quad$ then $\quad \dfrac{dy}{dx} = \dfrac{g(x)\dfrac{df}{dx} - \dfrac{dg}{dx}f(x)}{[g(x)]^2} \qquad$ (or $\ y' = \dfrac{gf' - g'f}{g^2}$)

If $\quad y = \dfrac{u}{v} \quad$ then $\quad \dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - \dfrac{dv}{dx}u}{v^2} \qquad$ (or $\ y' = \dfrac{vu' - v'u}{v^2}$)

These are two equivalent versions of the **quotient rule**.

Example 10

Find the derivative of $y = \dfrac{\ln x}{x}$

Solution

Here $f(x) = \ln x$ and $g(x) = x$

$$\therefore \quad \frac{df}{dx} = \frac{1}{x} \quad \text{and} \quad \frac{dg}{dx} = 1$$

Hence $\dfrac{dy}{dx} = \dfrac{x\left(\dfrac{1}{x}\right) - 1(\ln x)}{[x]^2} = \dfrac{1 - \ln x}{x^2}$

Task

Obtain the derivative of $y = \dfrac{\sin x}{x^2}$ (a) using the formula for differentiating a product and (b) using the formula for differentiating a quotient.

(a) Write $y = x^{-2}\sin x$ then use the product rule to find $\dfrac{dy}{dx}$:

Your solution

Answer

$y = x^{-2}\sin x \qquad \therefore \qquad \dfrac{dy}{dx} = (-2x^{-3})\sin x + x^{-2}\cos x \qquad \therefore \qquad \dfrac{dy}{dx} = \dfrac{-2\sin x + x\cos x}{x^3}$

(b) Now use the quotient rule instead to find $\dfrac{dy}{dx}$:

Your solution

Answer

$y = \dfrac{\sin x}{x^2} \qquad \therefore \qquad \dfrac{dy}{dx} = \dfrac{x^2(\cos x) - (2x)\sin x}{(x^2])^2} = \dfrac{x\cos x - 2\sin x}{x^3}$

Exercise

Find the derivatives of the following:

(a) $(2x^3 - 4x^2)(3x^5 + x^2)$

(b) $\dfrac{2x^3 + 4}{x^2 - 4x + 1}$

(c) $\dfrac{x^2 + 2x + 1}{x^2 - 2x + 1}$

(d) $(x^2 + 3)(2x - 5)(3x + 2)$

(e) $\dfrac{(2x + 1)(3x - 1)}{x + 5}$

(f) $(\ln x) \sin x$

(g) $(\ln x)/ \sin x$

(h) e^x/x^2

(i) $\dfrac{e^x \sin x}{\cos 2x}$

Answer

(a) $48x^7 - 84x^6 + 10x^4 - 16x^3$

(b) $\dfrac{2x^4 - 16x^3 + 6x^2 - 8x + 16}{(x^2 - 4x + 1)^2}$

(c) $-\dfrac{4(x + 1)}{(x - 1)^3}$

(d) $24x^3 - 33x^2 + 16x - 33$

(e) $\dfrac{6(x^2 + 10x + 1)}{(x + 5)^2}$

(f) $\dfrac{1}{x} \sin x + (\ln x) \cos x$

(g) $\dfrac{\sin x \left(\dfrac{1}{x}\right) - (\ln x) \cos x}{\sin^2 x} = \text{cosec} x(\dfrac{1}{x} - \cot x \ln x)$

(h) $\dfrac{x^2 e^x - 2xe^x}{x^4} = (x^{-2} - 2x^{-3})e^x$

(i) $\dfrac{\cos 2x(e^x \sin x + e^x \cos x) + 2 \sin 2x e^x \sin x}{\cos^2 2x}$

$$= e^x[(\sin x + \cos x) \sec 2x + 2 \sin x \sin 2x \sec^2 2x]$$

The Chain Rule

11.5

Introduction

In this Section we will see how to obtain the derivative of a composite function (often referred to as a 'function of a function'). To do this we use the **chain rule**. This rule can be used to obtain the derivatives of functions such as e^{x^2+3x} (the exponential function of a polynomial); $\sin(\ln x)$ (the sine function of the natural logarithm function); $\sqrt{x^3+4}$ (the square root function of a polynomial).

Prerequisites

Before starting this Section you should ...

- be able to differentiate standard functions
- be able to use the product and quotient rule for finding derivatives

Learning Outcomes

On completion you should be able to ...

- differentiate a function of a function using the chain rule
- differentiate a power function

1. The meaning of a function of a function

When we use a function like $\sin 2x$ or $e^{\ln x}$ or $\sqrt{x^2+1}$ we are in fact dealing with a composite function or **function of a function**.

$\sin 2x$ is the sine function of $2x$. This is, in fact, how we 'read' it:

$\sin 2x$ is read 'sine of $2x$'

Similarly $e^{\ln x}$ is the exponential function of the logarithm of x:

$e^{\ln x}$ is read 'e to the power of $\ln x$'

Finally $\sqrt{x^2+1}$ is also a composite function. It is the square root function of the polynomial x^2+1:

$\sqrt{x^2+1}$ is read as the 'square root of (x^2+1)'

When we talk about a function of a function in a general setting we will use the notation $f(g(x))$ where both f and g are functions.

Example 11

Specify the functions $f,\ g$ for the composite functions

(a) $\sin 2x$ (b) $\sqrt{x^2+1}$ (c) $e^{\ln x}$

Solution

(a) Here f is the sine function and g is the polynomial $2x$. We often write:

$f(g) = \sin g$ and $g(x) = 2x$

(b) Here $f(g) = \sqrt{g}$ and $g(x) = x^2+1$

(c) Here $f(g) = e^g$ and $g(x) = \ln x$

In each case the original function of x is obtained when $g(x)$ is substituted into $f(g)$.

Specify the functions $f,\ g$ for the composite functions
(a) $\cos(3x^2 - 1)$ (b) $\sinh(e^x)$ (c) $(x^2 + 3x - 1)^{1/3}$

Your solution
(a)

Answer
$f(g) = \cos g \qquad g(x) = 3x^2 - 1$

Your solution
(b)

Answer
$f(g) = \sinh g \qquad g(x) = e^x$

Your solution
(c)

Answer
$f(g) = g^{1/3} \qquad g(x) = x^2 + 3x - 1$

2. The derivative of a function of a function

To differentiate a function of a function we use the following Key Point:

Key Point 11

The Chain Rule

If $y = f(g(x))$, that is, a function of a function, then

$$\frac{dy}{dx} = \frac{df}{dg} \times \frac{dg}{dx}$$

This is called the **chain rule**.

Example 12

Find the derivatives of the following composite functions using the chain rule and check the result using other methods

(a) $(2x^2-1)^2$ (b) $\ln e^x$

Solution

(a) Here $y = f(g(x))$ where $f(g) = g^2$ and $g(x) = 2x^2 - 1$. Thus

$$\frac{df}{dg} = 2g \qquad \text{and} \qquad \frac{dg}{dx} = 4x \qquad \therefore \qquad \frac{dy}{dx} = 2g.(4x) = 2(2x^2-1)(4x) = 8x(2x^2-1)$$

This result is easily checked by using the rule for differentiating products:

$$y = (2x^2-1)(2x^2-1) \quad \text{so} \quad \frac{dy}{dx} = 4x(2x^2-1)+(2x^2-1)(4x) = 8x(2x^2-1) \quad \text{as obtained above.}$$

(b) Here $y = f(g(x))$ where $f(g) = \ln g$ and $g(x) = e^x$. Thus

$$\frac{df}{dg} = \frac{1}{g} \qquad \text{and} \qquad \frac{dg}{dx} = e^x \qquad \therefore \qquad \frac{dy}{dx} = \frac{1}{g}\cdot e^x = \frac{1}{e^x}\cdot e^x = 1$$

This is easily checked since, of course,

$$y = \ln e^x = x \quad \text{and so, obviously} \quad \frac{dy}{dx} = 1 \text{ as obtained above.}$$

Task

Obtain the derivatives of the following functions

(a) $(2x^2-5x+3)^9$ (b) $\sin(\cos x)$ (c) $\left(\dfrac{2x+1}{2x-1}\right)^3$

(a) Specify f and g for the first function:

Your solution

$f(g) =$ $g(x) =$

Answer

$f(g) = g^9$ $g(x) = 2x^2-5x+3$

Now obtain the derivative using the chain rule:

Your solution

Answer

$9(2x^2-5x+3)^8(4x-5)$. Can you see how to obtain the derivative without going through the intermediate stage of specifying f, g?

(b) Specify f and g for the second function:

Your solution

Answer

$f(g) = \sin g \qquad g(x) = \cos x$

Now use the chain rule to obtain the derivative:

Your solution

Answer

$-[\cos(\cos x)] \sin x$

(c) Apply the chain rule to the third function:

Your solution

Answer

$-\dfrac{12(2x+1)^2}{(2x-1)^4}$

3. Power functions

An example of a function of a function which often occurs is the so-called power function $[g(x)]^k$ where k is any rational number. This is an example of a function of a function in which

$$f(g) = g^k$$

Thus, using the chain rule: if $\quad y = [g(x)]^k \quad$ then $\quad \dfrac{dy}{dx} = \dfrac{df}{dg} \cdot \dfrac{dg}{dx} = k\, g^{k-1} \dfrac{dg}{dx}$.

For example, if $y = (\sin x + \cos x)^{1/3}$ then $\dfrac{dy}{dx} = \dfrac{1}{3}(\sin x + \cos x)^{-2/3}(\cos x - \sin x)$.

Task

Find the derivatives of the following power functions

(a) $y = \sin^3 x$ (b) $y = (x^2+1)^{1/2}$ (c) $y = (e^{3x})^7$

(a) Note that $\sin^3 x$ is the conventional way of writing $(\sin x)^3$. Now find its derivative:

Your solution

Answer

$\dfrac{dy}{dx} = 3(\sin x)^2 \cos x$ which we would normally write as $3\sin^2 x \cos x$

(b) Use the function of a function approach again:

Your solution

Answer

$\dfrac{dy}{dx} = \dfrac{1}{2}(x^2+1)^{-1/2} 2x = \dfrac{x}{\sqrt{x^2+1}}$

(c) Use the function of a function approach first, and then look for a quicker way in this case:

Your solution

Answer

$\dfrac{dy}{dx} = 7(e^{3x})^6(3e^{3x}) = 21(e^{3x})^7 = 21e^{21x}$

Note that $(e^{3x})^7 = e^{21x}$ $\quad\therefore\quad$ $\dfrac{dy}{dx} = 21e^{21x}$ directly - a much quicker way.

Exercise

Obtain the derivatives of the following functions:

(a) $\left(\dfrac{2x+1}{3x-1}\right)^4$ (b) $\tan(3x^2+2x)$ (c) $\sin^2(3x^2-1)$

Answer

(a) $-\dfrac{20(2x+1)^3}{(3x-1)^5}$ (b) $2(3x+1)\sec^2(3x^2+2x)$

(c) $6x\sin(6x^2-2)$ (remember $\sin 2x \equiv 2\sin x \cos x$)

Parametric Differentiation

11.6

Introduction

Sometimes the equation of a curve is not be given in Cartesian form $y = f(x)$ but in parametric form: $x = h(t),\ y = g(t)$. In this Section we see how to calculate the derivative $\dfrac{dy}{dx}$ from a knowledge of the so-called parametric derivatives $\dfrac{dx}{dt}$ and $\dfrac{dy}{dt}$. We then extend this to the determination of the second derivative $\dfrac{d^2y}{dx^2}$.

Parametric functions arise often in particle dynamics in which the parameter t represents the time and $(x(t),\ y(t))$ then represents the position of a particle as it varies with time.

Prerequisites

Before starting this Section you should ...

- be able to differentiate standard functions
- be able to plot a curve given in parametric form

Learning Outcomes

On completion you should be able to ...

- find first and second derivatives when the equation of a curve is given in parametric form

1. Parametric differentiation

In this subsection we consider the parametric approach to describing a curve:

$$\underbrace{x = h(t) \qquad y = g(t)}_{\text{parametric equations}} \qquad\qquad \underbrace{t_0 \leq t \leq t_1}_{\text{parametric range}}$$

As various values of t are chosen within the parameter range the corresponding values of $x,\ y$ are calculated from the parametric equations. When these points are plotted on an xy plane they trace out a curve. The Cartesian equation of this curve is obtained by eliminating the parameter t from the parametric equations. For example, consider the curve:

$$x = 2\cos t \qquad y = 2\sin t \qquad 0 \leq t \leq 2\pi.$$

We can eliminate the t variable in an obvious way - square each parametric equation and then add:

$$x^2 + y^2 = 4\cos^2 t + 4\sin^2 t = 4 \qquad \therefore \qquad x^2 + y^2 = 4$$

which we recognise as the standard equation of a **circle** with centre at $(0, 0)$ with radius 2.
In a similar fashion the parametric equations

$$x = 2t \qquad y = 4t^2 \qquad -\infty < t < \infty$$

describes a **parabola**. This follows since, eliminating the parameter t:

$$t = \frac{x}{2} \qquad \therefore \qquad y = 4\left(\frac{x^2}{4}\right) \qquad \text{so} \quad y = x^2$$

which we recognise as the standard equation of a parabola.
The question we wish to address in this Section is 'how do we obtain the derivative $\dfrac{dy}{dx}$ if a curve is given in parametric form?' To answer this we note the key result in this area:

Key Point 12

Parametric Differentiation

If $x = h(t)$ and $y = g(t)$ then

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$$

We note that this result allows the determination of $\dfrac{dy}{dx}$ without the need to find y as an explicit function of x.

Example 13

Determine the equation of the tangent line to the semicircle with parametric equations

$$x = \cos t \qquad y = \sin t \qquad 0 \leq t \leq \pi$$

at $t = \pi/4$.

Solution

The semicircle is drawn in Figure 9. We have also drawn the tangent line at $t = \pi/4$ (or, equivalently, at $x = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$, $y = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$.)

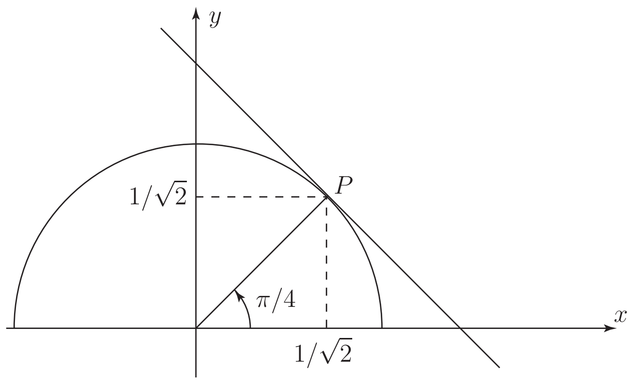


Figure 9

Now

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{\cos t}{-\sin t} = -\cot t.$$

Thus at $t = \frac{\pi}{4}$ we have $\frac{dy}{dx} = -\cot\left(\frac{\pi}{4}\right) = -1.$

The equation of the tangent line is

$$y = mx + c$$

where m is the gradient of the line and c is a constant.

Clearly $m = -1$ (since, at the point P the line and the circle have the same gradient).

To find c we note that the line passes through the point P with coordinates $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. Hence

$$\frac{1}{\sqrt{2}} = (-1)\frac{1}{\sqrt{2}} + c \qquad \therefore \qquad c = \frac{2}{\sqrt{2}}$$

Finally,

$$y = -x + \frac{2}{\sqrt{2}}$$

is the equation of the tangent line at the point in question.

We should note, before proceeding, that a derivative with respect to the parameter t is often denoted by a 'dot'. Thus

$$\frac{dx}{dt} = \dot{x}, \quad \frac{dy}{dt} = \dot{y}, \quad \frac{d^2x}{dt^2} = \ddot{x} \text{ etc.}$$

Task

Find the value of $\dfrac{dy}{dx}$ if $x = 3t, \quad y = t^2 - 4t + 1.$

Check your result by finding $\dfrac{dy}{dx}$ in the normal way.

First find $\dfrac{dx}{dt}, \dfrac{dy}{dt}$:

Your solution

Answer

$\dfrac{dx}{dt} = 3, \ \dfrac{dy}{dt} = 2t - 4$

Now obtain $\dfrac{dy}{dx}$:

Your solution

Answer

$\dfrac{dy}{dx} = \dfrac{dy}{dt} \div \dfrac{dx}{dt} = \dfrac{2t-4}{3} = \dfrac{2}{3}t - \dfrac{4}{3},$

or, using the 'dot' notation $\dfrac{dy}{dx} = \dfrac{\dot{y}}{\dot{x}} = \dfrac{2t-4}{3} = \dfrac{2}{3}t - \dfrac{4}{3}$

Now find y explicitly as a function of x by eliminating t, and so find $\dfrac{dy}{dx}$ directly:

Your solution

Answer

$t = \dfrac{x}{3} \qquad \therefore \qquad y = \dfrac{x^2}{9} - \dfrac{4x}{3} + 1.$ Finally: $\dfrac{dy}{dx} = \dfrac{2x}{9} - \dfrac{4}{3} = \dfrac{2t}{3} - \dfrac{4}{3}.$

Task

Find the value of $\dfrac{dy}{dx}$ at $t = 2$ if $x = 3t - 4\sin \pi t, \quad y = t^2 + t\cos \pi t, \quad 0 \le t \le 4$

First find $\dfrac{dx}{dt}, \dfrac{dy}{dt}$:

Your solution

Answer

$$\frac{dx}{dt} = 3 - 4\pi\cos\pi t \qquad \frac{dy}{dt} = 2t + \cos\pi t - \pi t\sin\pi t$$

Now obtain $\dfrac{dy}{dx}$:

Your solution

Answer

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{2t + \cos\pi t - \pi t\sin\pi t}{3 - 4\pi\cos\pi t}$$

or, using the dot notation, $\dfrac{dy}{dx} = \dfrac{\dot{y}}{\dot{x}} = \dfrac{2t + \cos\pi t - \pi t\sin\pi t}{3 - 4\pi\cos\pi t}$

Finally, substitute $t = 2$ to find $\dfrac{dy}{dx}$ at this value of t.

Your solution

Answer

$$\left.\frac{dy}{dx}\right|_{t=2} = \frac{4+1}{3-4\pi} = \frac{5}{3-4\pi} = -0.523$$

2. Higher derivatives

Having found the first derivative $\dfrac{dy}{dx}$ using parametric differentiation we now ask how we might determine the second derivative $\dfrac{d^2y}{dx^2}$.

By definition:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

But

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} \qquad \text{and so} \qquad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{\dot{y}}{\dot{x}}\right)$$

Now $\dfrac{\dot{y}}{\dot{x}}$ is a function of t so we can change the derivative with respect to x into a derivative with respect to t since

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \left\{\frac{d}{dt}\left(\frac{dy}{dx}\right)\right\}\frac{dt}{dx}$$

from the function of a function rule (Key Point 11 in Section 11.5).

But, differentiating the quotient $\dot{y}/\dot{x}$, we have

$$\frac{d}{dt}\left(\frac{\dot{y}}{\dot{x}}\right) = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^2} \qquad \text{and} \qquad \frac{dt}{dx} = \frac{1}{\left(\dfrac{dx}{dt}\right)} = \frac{1}{\dot{x}}$$

so finally:

$$\frac{d^2y}{dx^2} = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^3}$$

Key Point 13

If $x = h(t), \quad y = g(t)$ then the first and second derivatives of y with respect to x are:

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} \qquad \text{and} \qquad \frac{d^2y}{dx^2} = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^3}$$

Example 14

If the equations of a curve are $x = 2t, \quad y = t^2 - 3$, determine $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$.

Solution

Here $\dot{x} = 2, \;\dot{y} = 2t \quad \therefore \quad \dfrac{dy}{dx} = \dfrac{\dot{y}}{\dot{x}} = \dfrac{2t}{2} = t.$

Also $\ddot{x} = 0, \quad \ddot{y} = 2 \quad \therefore \quad \dfrac{d^2y}{dx^2} = \dfrac{2(2) - 2t(0)}{(2)^3} = \dfrac{1}{2}.$

These results can easily be checked since $t = \dfrac{x}{2}$ and $y = t^2 - 3$ which imply $y = \dfrac{x^2}{4} - 3$. Therefore the derivatives can be obtained directly: $\dfrac{dy}{dx} = \dfrac{2x}{4} = \dfrac{x}{2}$ and $\dfrac{d^2y}{dx^2} = \dfrac{1}{2}$.

Exercises

1. For the following sets of parametric equations find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$

   (a) $x = 3t^2 \quad y = 4t^3$ (b) $x = 4 - t^2 \quad y = t^2 + 4t$ (c) $x = t^2e^t \quad y = t$

2. Find the equation of the tangent line to the curve

$$x = 1 + 3\sin t \qquad y = 2 - 5\cos t \qquad \text{at} \quad t = \frac{\pi}{6}$$

Answers

1. (a) $\dfrac{dy}{dx} = 2t, \quad \dfrac{d^2y}{dx^2} = \dfrac{1}{3t}.$ (b) $\dfrac{dy}{dx} = -1 - \dfrac{2}{t}, \quad \dfrac{d^2y}{dx^2} = -\dfrac{1}{t^3}$

   (c) $\dfrac{dy}{dx} = \dfrac{e^{-t}}{2t + t^2}, \quad \dfrac{d^2y}{dx^2} = -\dfrac{e^{-2t}(t^2 + 4t + 2)}{(t + 2)^3t^3}$

2. $\dot{x} = 3\cos t \qquad \dot{y} = +5\sin t$

$$\therefore \quad \frac{dy}{dx} = \frac{5}{3}\tan t \qquad \therefore \quad \left.\frac{dy}{dx}\right|_{t=\pi/6} = \frac{5}{3}\tan\frac{\pi}{6} = \frac{5}{3}\frac{1}{\sqrt{3}} = \frac{5\sqrt{3}}{9}$$

The equation of the tangent line is $y = mx + c$ where $m = \dfrac{5\sqrt{3}}{9}$.

The line passes through the point $x = 1 + 3\sin\dfrac{\pi}{6} = 1 + \dfrac{3}{2}, \quad y = 2 - 5\dfrac{\sqrt{3}}{2}$ and so

$$2 - 5\frac{\sqrt{3}}{2} = \frac{5\sqrt{3}}{9}(1 + \frac{3}{2}) + c \qquad \therefore \quad c = 2 - \frac{35\sqrt{3}}{9}$$

Implicit Differentiation

11.7

Introduction

This Section introduces implicit differentiation which is used to differentiate functions expressed in implicit form (where the variables are found together). Examples are $x^3 + xy + y^2 = 1$, and $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ which represents an ellipse.

Prerequisites

Before starting this Section you should ...

- be able to differentiate standard functions
- be competent in using the chain rule

Learning Outcomes

On completion you should be able to ...

- differentiate functions expressed implicitly

1. Implicit and explicit functions

Equations such as $y = x^2$, $y = \dfrac{1}{x}$, $y = \sin x$ are said to define y **explicitly** as a function of x because the variable y appears alone on one side of the equation.

The equation

$$yx + y + 1 = x$$

is not of the form $y = f(x)$ but can be put into this form by simple algebra.

Task

Write y as the subject of

$$yx + y + 1 = x$$

Your solution

Answer

We have $y(x+1) = x - 1$ so

$$y = \frac{x-1}{x+1}$$

We say that y is defined **implicitly** as a function of x by means of $yx+y+1 = x$, the actual function being given **explicitly** as

$$y = \frac{x-1}{x+1}$$

We note than an equation relating x and y can implicitly define **more than one** function of x.

For example, if we solve

$$x^2 + y^2 = 1$$

we obtain $y = \pm\sqrt{1-x^2}$ so $x^2 + y^2 = 1$ defines implicitly two functions

$$f_1(x) = \sqrt{1-x^2} \qquad f_2(x) = -\sqrt{1-x^2}$$

Task

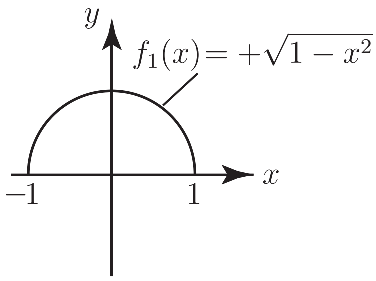
Sketch the graphs of $f_1(x) = \sqrt{1-x^2}$ $\qquad f_2(x) = -\sqrt{1-x^2}$

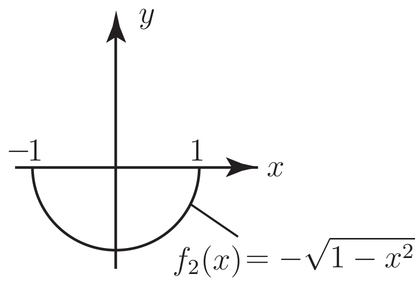
(The equation $x^2 + y^2 = 1$ should give you the clue.)

Your solution

Answer

Since $x^2 + y^2 = 1$ is the well-known equation of the circle with centre at the origin and radius 1, it follows that the graphs of $f_1(x)$ and $f_2(x)$ are the upper and lower halves of this circle.





Sometimes it is difficult or even impossible to solve an equation in x and y to obtain y explicitly in terms of x.

Examples where explicit expressions for y cannot be obtained are

$$\sin(xy) = y \qquad x^2 + \sin y = 2y$$

2. Differentiation of implicit functions

Fortunately it is not necessary to obtain y in terms of x in order to **differentiate** a function defined implicitly.

Consider the simple equation

$$xy = 1$$

Here it **is** clearly possible to obtain y as the subject of this equation and hence obtain $\dfrac{dy}{dx}$.

Task Express y explicitly in terms of x and find $\dfrac{dy}{dx}$ for the case $xy = 1$.

Your solution

Answer
We have immediately $y = \dfrac{1}{x}$ so $\dfrac{dy}{dx} = -\dfrac{1}{x^2}$

We now show an alternative way of obtaining $\dfrac{dy}{dx}$ which does **not** involve writing y explicitly in terms of x at the outset. We simply treat y as an (unspecified) function of x.

Hence if $xy = 1$ we obtain

$$\frac{d}{dx}(xy) = \frac{d}{dx}(1).$$

The right-hand side differentiates to zero as 1 is a constant. On the left-hand side we must use the **product** rule of differentiation:

$$\frac{d}{dx}(xy) = x\frac{dy}{dx} + y\frac{dx}{dx} = x\frac{dy}{dx} + y$$

Hence $xy = 1$ becomes, after differentiation,

$$x\frac{dy}{dx} + y = 0 \quad \text{or} \quad \frac{dy}{dx} = -\frac{y}{x}$$

In this case we can of course substitute $y = \dfrac{1}{x}$ to obtain

$$y = -\frac{1}{x^2}$$

as before.

The method used here is called **implicit differentiation** and, apart from the final step, it can be applied even if y cannot be expressed explicitly in terms of x. Indeed, on occasions, it is **easier** to differentiate implicitly even if an explicit expression is possible.

Example 15

Obtain the derivative $\dfrac{dy}{dx}$ where

$$x^2 + y = 1 + y^3$$

Solution

We begin by differentiating the left-hand side of the equation with respect to x to get:

$$\frac{d}{dx}(x^2 + y) = 2x + \frac{dy}{dx}.$$

We now differentiate the right-hand side of with respect to x. Using the chain (or function of a function) rule to deal with the y^3 term:

$$\frac{d}{dx}(1 + y^3) = \frac{d}{dx}(1) + \frac{d}{dx}(y^3) = 0 + 3y^2\frac{dy}{dx}$$

Now by equating the left-hand side and right-hand side derivatives, we have:

$$2x + \frac{dy}{dx} = 3y^2\frac{dy}{dx}$$

We can make $\dfrac{dy}{dx}$ the subject of this equation:

$$\frac{dy}{dx} - 3y^2\frac{dy}{dx} = -2x \quad \text{which gives} \quad \frac{dy}{dx} = \frac{2x}{3y^2 - 1}$$

We note that $\dfrac{dy}{dx}$ has to be expressed in terms of both x and y. This is quite usual if y cannot be obtained explicitly in terms of x. Now try this Task requiring implicit differentiation.

Task

Find $\dfrac{dy}{dx}$ if $2y = x^2 + \sin y$

Note that your answer will be in terms of both y and x.

Your solution

Answer

We have, on differentiating both sides of the equation with respect to x and using the chain rule on the $\sin y$ term:

$$\frac{d}{dx}(2y) = \frac{d}{dx}(x^2) + \frac{d}{dx}(\sin y) \quad \text{i.e.} \quad 2\frac{dy}{dx} = 2x + \cos y\frac{dy}{dx} \qquad \text{leading to} \qquad \frac{dy}{dx} = \frac{2x}{2 - \cos y}.$$

We sometimes need to obtain the second derivative $\dfrac{d^2y}{dx^2}$ for a function defined implicitly.

Example 16

Obtain $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ at the point $(4,2)$ on the curve defined by the equation

$$x^2 - xy - y^2 - 2y = 0$$

Solution

Firstly we obtain $\dfrac{dy}{dx}$ by differentiating the equation implicitly and then evaluate it at $(4,2)$.

We have $$2x - x\frac{dy}{dx} - y - 2y\frac{dy}{dx} - 2\frac{dy}{dx} = 0 \qquad (1)$$

from which $$\frac{dy}{dx} = \frac{2x-y}{x+2y+2} \qquad (2)$$

so at $(4,2)$ $\dfrac{dy}{dx} = \dfrac{6}{10} = \dfrac{3}{5}$.

To obtain the second derivative $\dfrac{d^2y}{dx^2}$ it is easier to use (1) than (2) because the latter is a quotient. We simplify (1) first:

$$2x - y - (x+2y+2)\frac{dy}{dx} = 0 \qquad (3)$$

We will have to use the product rule to differentiate the third term here.

Hence differentiating (3) with respect to x:

$$2 - \frac{dy}{dx} - (x+2y+2)\frac{d^2y}{dx^2} - (1+2\frac{dy}{dx})\frac{dy}{dx} = 0$$

or

$$2 - 2\frac{dy}{dx} - 2\left(\frac{dy}{dx}\right)^2 - (x+2y+2)\frac{d^2y}{dx^2} = 0 \qquad (4)$$

Note carefully that the third term here, $\left(\dfrac{dy}{dx}\right)^2$, is the square of the first derivative. It should not be confused with the second derivative denoted by $\dfrac{d^2y}{dx^2}$.

Finally, at $(4,2)$ where $\dfrac{dy}{dx} = \dfrac{3}{5}$ we obtain from (4): $\quad 2 - 2(\dfrac{3}{5}) - 2(\dfrac{9}{25}) - (4+4+2)\dfrac{d^2y}{dx^2} = 0$

from which $\quad \dfrac{d^2y}{dx^2} = \dfrac{1}{125}$ at $(4,2)$.

This Task involves finding a formula for the curvature of a bent beam. When a horizontal beam is acted on by forces which bend it, then each small segment of the beam will be slightly curved and can be regarded as an arc of a circle. The radius R of that circle is called the **radius of curvature** of the beam at the point concerned. If the shape of the beam is described by an equation of the form $y = f(x)$ then there is a formula for the radius of curvature R which involves only the first and second derivatives $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$.

Find that equation as follows.

Start with the equation of a circle in the simple implicit form

$$x^2 + y^2 = R^2$$

and perform implicit differentiation twice. Now use the result of the first implicit differentiation to find a simple expression for the quantity $1 + (dy/dx)^2$ in terms of R and y; this can then be used to simplify the result of the second differentiation, and will lead to a formula for $\dfrac{1}{R}$ (called the **curvature**) in terms of $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$.

Your solution

Answer

Differentiating: $x^2 + y^2 = R^2$ gives:

$$2x + 2y\frac{dy}{dx} = 0 \qquad (1)$$

Differentiating again: $$2 + 2\left(\frac{dy}{dx}\right)^2 + 2y\frac{d^2y}{dx^2} = 0 \qquad (2)$$

From (1)

$$\frac{dy}{dx} = -\frac{x}{y} \qquad \therefore \quad 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{x^2}{y^2} = \frac{y^2 + x^2}{y^2} = \left(\frac{R}{y}\right)^2 \qquad (3)$$

So $1 + \left(\dfrac{dy}{dx}\right)^2 = \left(\dfrac{R}{y}\right)^2$.

Thus (2) becomes $$2\left(\frac{R}{y}\right)^2 + 2y\left(\frac{d^2y}{dx^2}\right) = 0 \qquad \therefore \qquad \frac{d^2y}{dx^2} = -\frac{R^2}{y^3} = -\left(\frac{1}{R}\right)\left(\frac{R}{y}\right)^3$$

so $$\frac{d^2y}{dx^2} = -\frac{1}{R}\left(\frac{R}{y}\right)^3 \qquad (4)$$

Rearranging (4) to make $\dfrac{1}{R}$ the subject and substituting for $\left(\dfrac{R}{y}\right)$ from (3) gives the result:

$$\frac{1}{R} = -\frac{\dfrac{d^2y}{dx^2}}{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{3/2}}$$

The equation usually found in textbooks omits the minus sign but the sign indicates whether the circle is above or below the curve, as you will see by sketching a few examples. When the gradient is small (as for a slightly deflected horizontal beam), i.e. $\dfrac{dy}{dx}$ is small, the denominator in the equation for $(1/R)$ is close to 1, and so the second derivative alone is often used to estimate the radius of curvature in the theory of bending beams.