Engineering Maths First Aid Kit

8.1

Introduction to differentiation

Introduction

This leaflet provides a rough and ready introduction to **differentiation**. This is a technique used to calculate the gradient, or slope, of a graph at different points.

1. The gradient function

Given a function, for example, $y = x^2$, it is possible to derive a formula for the gradient of its graph. We can think of this formula as the **gradient function**, precisely because it tells us the gradient of the graph. For example,

$$\text{when } y = x^2 \qquad \text{the gradient function is} \qquad 2x$$

So, the gradient of the graph of $y = x^2$ at any point is twice the x value there. To understand how this formula is actually found you would need to refer to a textbook on calculus. The important point is that using this formula we can calculate the gradient of $y = x^2$ at different points on the graph. For example,

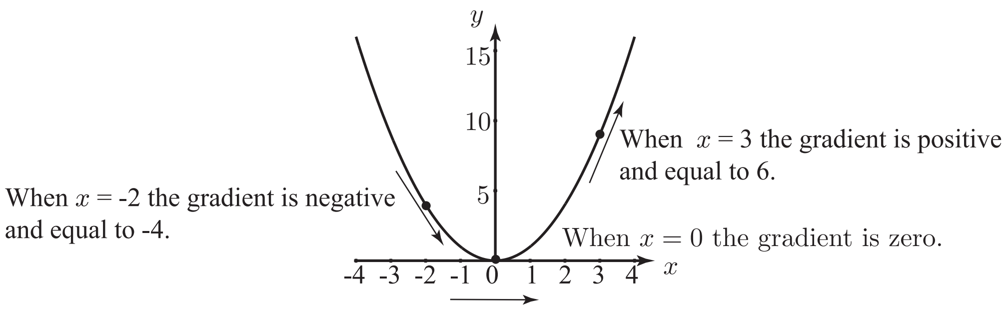
$$\text{when } x = 3, \text{ the gradient is } 2 \times 3 = 6.$$

$$\text{when } x = -2, \text{ the gradient is } 2 \times (-2) = -4.$$

How do we interpret these numbers ? A gradient of 6 means that values of y are increasing at the rate of 6 units for every 1 unit increase in x. A gradient of -4 means that values of y are decreasing at a rate of 4 units for every 1 unit increase in x.

Note that when $x = 0$, the gradient is $2 \times 0 = 0$.

Below is a graph of the function $y = x^2$. Study the graph and you will note that when $x = 3$ the graph has a positive gradient. When $x = -2$ the graph has a negative gradient. When $x = 0$ the gradient of the graph is zero. Note how these properties of the graph can be predicted from knowledge of the gradient function, $2x$.



Example
When $y = x^3$, its gradient function is $3x^2$. Calculate the gradient of the graph of $y = x^3$ when a) $x = 2$, b) $x = -1$, c) $x = 0$.

Solution
a) when $x = 2$ the gradient function is $3(2)^2 = 12$.

b) when $x = -1$ the gradient function is $3(-1)^2 = 3$.

c) when $x = 0$ the gradient function is $3(0)^2 = 0$.

2. Notation for the gradient function

You will need to use a notation for the gradient function which is in widespread use.

> If y is a function of x, that is $y = f(x)$, we write its gradient function as $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

$\dfrac{\mathrm{d}y}{\mathrm{d}x}$, pronounced 'dee y by dee x', is not a fraction even though it might look like one! This notation can be confusing. Think of $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ as the 'symbol' for the gradient function of $y = f(x)$. The process of finding $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ is called **differentiation with respect to** x.

Example
For any value of n, the gradient function of x^n is nx^{n-1}. We write:

$$\text{if} \qquad y = x^n, \qquad \text{then} \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = nx^{n-1}$$

You have seen specific cases of this result earlier on. For example, if $y = x^3$, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 3x^2$.

3. More notation and terminology

When $y = f(x)$ alternative ways of writing the gradient function, $\dfrac{\mathrm{d}y}{\mathrm{d}x}$, are y', pronounced 'y dash', or $\dfrac{\mathrm{d}f}{\mathrm{d}x}$, or f', pronounced 'f dash'. In practice you do not need to remember the formulas for the gradient functions of all the common functions. Engineers usually refer to a table known as a *Table of Derivatives*. A **derivative** is another name for a gradient function. Such a table is available on leaflet 8.2. The derivative is also known as the **rate of change** of a function.

Exercises
1. Given that when $y = x^2$, $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x$, find the gradient of $y = x^2$ when $x = 7$.

2. Given that when $y = x^n$, $\frac{\mathrm{d}y}{\mathrm{d}x} = nx^{n-1}$, find the gradient of $y = x^4$ when a) $x = 2$, b) $x = -1$.

3. Find the rate of change of $y = x^3$ when a) $x = -2$, b) $x = 6$.

4. Given that when $y = 7x^2 + 5x$, $\frac{\mathrm{d}y}{\mathrm{d}x} = 14x + 5$, find the gradient of $y = 7x^2 + 5x$ when $x = 2$.

Answers
1. 14. 2. a) 32, b) -4. 3. a) 12, b) 108. 4. 33.

