Is there an Algebra Problem after all?

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Overview

- Context of the question: Project on algebra-specific diagnosis and intervention

  Is there really an algebra-specific problem of learning?
- Mainstream YES
  - Didactical cut
- But maybe NO
- Not just this question but a little bit on the projects background ideas

Empty set of students

- To investigate algebra problems in more detail I asked collaborating teachers to…
  - Identify students that perform well on arithmetics
  - … but struggle with symbolic algebra
- Only two students were identified by teachers
- Both showed severe difficulties in arithmetics, but NO algebra specific problems
- Hence hypothesis: **Algebra poses no special cognitive demands**
- Rethinking it from scratch – a personal challenge, clarify things for myself

Pro Arguments

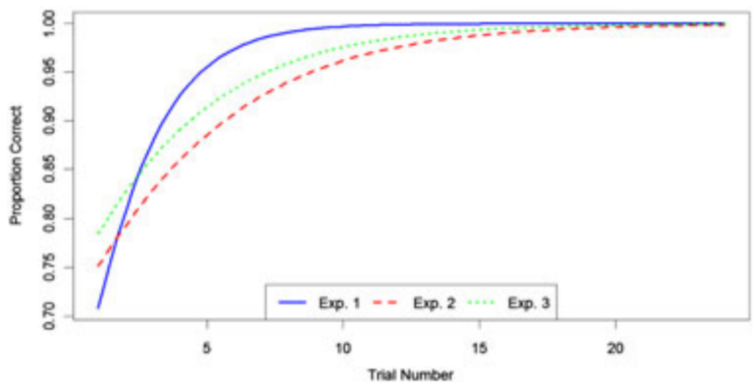
- Numbers and fractions present almost the same generality as variables, e.g. ½ is ½ of *some* unit
- 3 may have elementary meaning, 749839 not more than x
- Coordination of signs (operators, parentheses) is not much different from symbolic algebra
  - Rules of syntax and semantics
  - Ability to carry out algorithms

Pro Arguments

- The role of **substitution** as a central algebraic operation
  - Theoretical: Provides Turing complete computation (lambda calculus)
  - Practical: Provides powerful programming (production systems)
  - Psychological: Subst. of common sub-expressions frees working memory
  - Empirical: In algebra tests: Good in substitution implies well in algebra in general (Oldenburg 2009)
- Substitution is central in arithmetic as well
  - E.g. mental computation: Remainder of 732:4? Substitute 732=700+32
  - Rule "Trade 1 blue chip for 4 red ones". 5 blue→ 20=5*4 red. V.v.: division
- Nota bene:
  - Substitution alone is not everything: Need strategies as well
  - It is difficult
- First Message: Take substitution more seriously in the curriculum!

Pro Arguments

- Study by Christianson et al: Practice makes nearly perfect
  - Considers professor-student type tasks (reversal error)
- Method: Unguided practice!
  - Tasks in pairs: Word order helpful vs. misleading
    - There are six times as many marigolds as there are pansies.
    - The number of marigolds is six times the number of pansies.
  - Students could refine first answer after doing the 2nd
- Algebra problems = lack of practice?



Contra Arguments

- Solving arithmetical and algebraic problems activates different areas of the brain (e.g. Henz&Oldenburg&Schöllhorn 2015)
- Numerical and symbolic items correlate only weakly in tests.
  - Elaborate on this point

Arguments: Tests

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- SMART Tests by Kaye Stacey. Arithmetic

Choose whether each statement is true or false.

$\frac{12\times 17}{6\times 17} = 2$

TRUE
FALSE

$\frac{15\times 100}{5\times 100} = 300$

$5\times 917 = 5^{917}$

$\frac{1}{30} + \frac{1}{40} = \frac{2}{70}$

$\frac{6}{73} + \frac{5}{73} = \frac{11}{73}$

$3\times(2\times 5) = 6\times 15$

$92 - 43 = 43 - 92$

$2\times(300+7) = 600+14$

$\frac{300+6}{3} = 100+2$

$\frac{17000-17}{17} = 1000-17$

Arguments: Tests

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- SMART Tests by Kaye Stacey. AlgebraT

Choose whether each statement is true or false.

$\frac{33x}{11x} = 3$	TRUE / FALSE	$\frac{18x}{6x} = 3x$	
$\frac{1}{3} + \frac{1}{y} = \frac{2}{3+y}$		$5m = m^5$	
$5\times(3x+y) = 15x+5y$		$\frac{s}{t} + \frac{p}{t} = \frac{s+p}{t}$	
$4a(5b-9) = 20ab-9$		$3\times(a\times 2b) = 3a\times 6b$	
$\frac{6x-5}{2} = 3x-5$		$8d-3g = 3g-8d$	

Arguments: Tests

- SMART Tests by Kaye Stacey. AlgebraSem

Lucy bought 6 doughnuts for 12 dollars.
She wanted to work out how much each doughnut cost.
She wrote the equation 6***d*** = 12.
In Lucy's equation, ***d*** stands for:

Choose one answer.

- ○ a. the number of doughnuts
- ○ b. doughnuts
- ○ c. the cost of one doughnut
- ○ d. one doughnut
- ○ e. dollars

David is 10 cm taller than Con.
Con is *h* cm tall.
We write David's height in algebra as:

Choose one answer.

- ○ 10+***h***
- ○ 10***h***
- ○ ***r***
- ○ ***C*** = ***D*** + 10
- ○ ***h***10

The cost of hiring a bicycle is made up of a fixed fee of $25 and a usage charge of $8 per hour.

Which way would your teacher prefer to see the rule for this cost written?

Choose one answer.

- ○ a. 25 ***C*** + 8 ***t*** where the ***C*** is the fixed fee and ***t*** is the time used in hours
- ○ b. 8 ***t*** + 25 where ***t*** is the number of hours used
- ○ c. ***C*** = 25 + 8 ***t*** where ***C*** is the total hiring cost in dollars and ***t*** is the number of hours used
- ○ d. ***C*** = 25 + ***h*** = 8 ***t*** where ***C*** is the total hiring cost in dollars, ***h*** is the hiring charge and ***t*** is the number of hours used

Arguments: Tests

- SMART Tests by Kaye Stacey.
  - Correlation Arithmetics-AlgebraT: 0.41
    - Consider this as high, as cor(arith/2, arith/2) is approx 0.47
  - Correlation Arithmetics-AlgebraSem: 0.19
- Statistical implicative analysis: Measure implications
  - “weak in arithmetics” => “weak in algebra”: $\varphi=0.96$
  - “weak in algebra” => “strong in arithmetics”: $\varphi=0.04$
  - “weak in algebra” => “weak in arithmetics”: $\varphi=0.81$
- Explaining hypothesis 1: As most of school algebra is transformational algebra, students who face problems here are likely to fail in arithmetics as well.

Evaluation of Hypothesis

- But is this really all? Isn't there a didactical cut?
- A closer look using the lense of language levels
- Now: Excursion on language levels
  - Not just to answer the central question, but as background to project

Algebra and Language

- Three aspects
  - **Syntax**: Formal rules. How can symbols be combined?
  - **Semantics**: Reference to objects. What is it?
  - **Pragmatics**: Rules how to use language in context. What do I use and to which end?
- Operating on all levels is important

- Example: Comparing two mobile phone providers
  - 1. Pragmatics: What tool is adequate? Table, expression, equation, inequality, graph? Inequality!
  - 2. Semantics: Fix references: n number of minutes. Comparison: $5+0{,}09n<10+0{,}03n$
  - 3. Syntax: Manipulation: $n<88{,}3$.
  - 4. Semantics: Interpretation: First offer is better when at most 88 Minutes are used.
  - 5. Pragmatics: Act according to result.

Algebra and Language

- Error analysis: „There are 84 balls. 10 more are red than blue“
- Error on pragmatical level: $f(r,b)=r+b+10$
  - Unsuitable math used
- Error on semantical level: $r+(r+10)=84$
  - Reference wrong
- Error on syntactical level: $b+r=84$, $r=10+b$ hence $r=b-84$ and $b-84=10+b$.
  - Structure wrongly operated on

Each level corresponds to specific mental conceptions

Level	Conception	Ex.: Variable
Syntactical	**Structural concept:** How does it look like?	a) A letter b) A structured symbol, x_i
Semantical	**Ontological concept:** What is it?	a) A reference b) A container
Pragmatical	**Role**: What is it used for?	a) Shortcut for known numbers b) Unknowns c) undetermined

Hypothesis: Specific emotions for each level!

Level	**Emotion**	**Examples of positive and negative emotions (p/n)**
Syntax	Aesthetics: well-formed	n: 3(4+)) p: x^3+x^2+x+1
Semantics	Coherence (Logic)	p: Successful check n: Paradoxes
Pragmatics	Utility	p: full decision tree n: Table with missing cases

Algebra and Language

Hypothesis: particular difference between arithmetics and algebra mainly on the pragmatical level

Level	Arithmetics	Algebra
Syntax	Expressions, equations	Just one new thing: variables
Semantics	Operations	As arithmetics + variable values
Pragmatics	Meaning operations	MANY differences! New strategies for problem solving

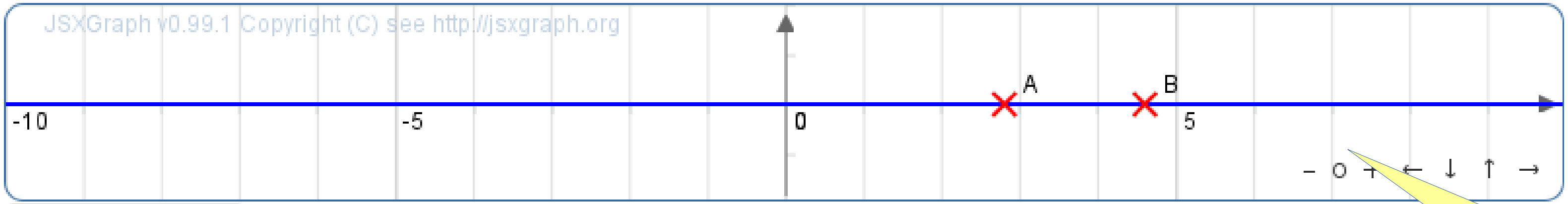
Algebra and Language

- What do we learn from this?
- Arithmetics and algebra are close together with regards to syntax and semantics
  - No particular difficulties here: But make connection transparent
- Substantial differences on the pragmatics level
- Integrate syntax and semantics by a computer tool with some pragmatic ideas

How to link syntax and semantics?

FeliX1D

Intro zum Kennenlernen



Neuer Punkt

Relax

Interpretation

Eq. Semantics

☐ Ganzzahlige Bewegung   Darstellung: ◉ einzeilig ○ mehrzeilig

Variable	Wert	fix	Sichtbar
A	2.8	☐	☑
B	4.6	☐	☑

Gleichung / Term	Defekt/Wert	Gültig
A+2*B=12	0	☑

Neue Gleichung oder Term

Eq. Syntax

V Syntax: Letters

V Reference: Semantics

A pragmatical issue

- A remarkable type of answer to a reversal error test item:
  - The Rhine is 200km longer than the Elbe: **r+200=e-200**
- No explanation for this from ST or other models
- **Hypothesis: Attributes mistaken for operations and v.v.**
- Attributes (e.g. vector arrow) and operations (e.g. complex conjugation) look similar $\vec{a}, \bar{a}$
- Other manifestations of attribute misconception: Agree with the following statements...
  - $-x$ is negative $2y$ is greater than y
- Correlation #reversal errors and #attribute misconceptions : r=0.62 (n=29)
- So reversal error may be caused on the pragmatical level
- Another nice attribute error: Case $a<|1|$ or $a>|1|$

End

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- Summarizing the hypothesis
  - There are parts of algebra (syntactical, semantical) that are close to arithmetics, so no special algebra problem there
  - But: Pragmatics is different
  - Anyway: The integration of syntaix-semantics-pragmatics needs care
- This explains to some extent why no students weak on algebra but strong of arithmetics could be identified: algebra in school means mainly transformational algebra, where the pragmatical differences don't matter that much
- Discussion