Algebra Artefacts

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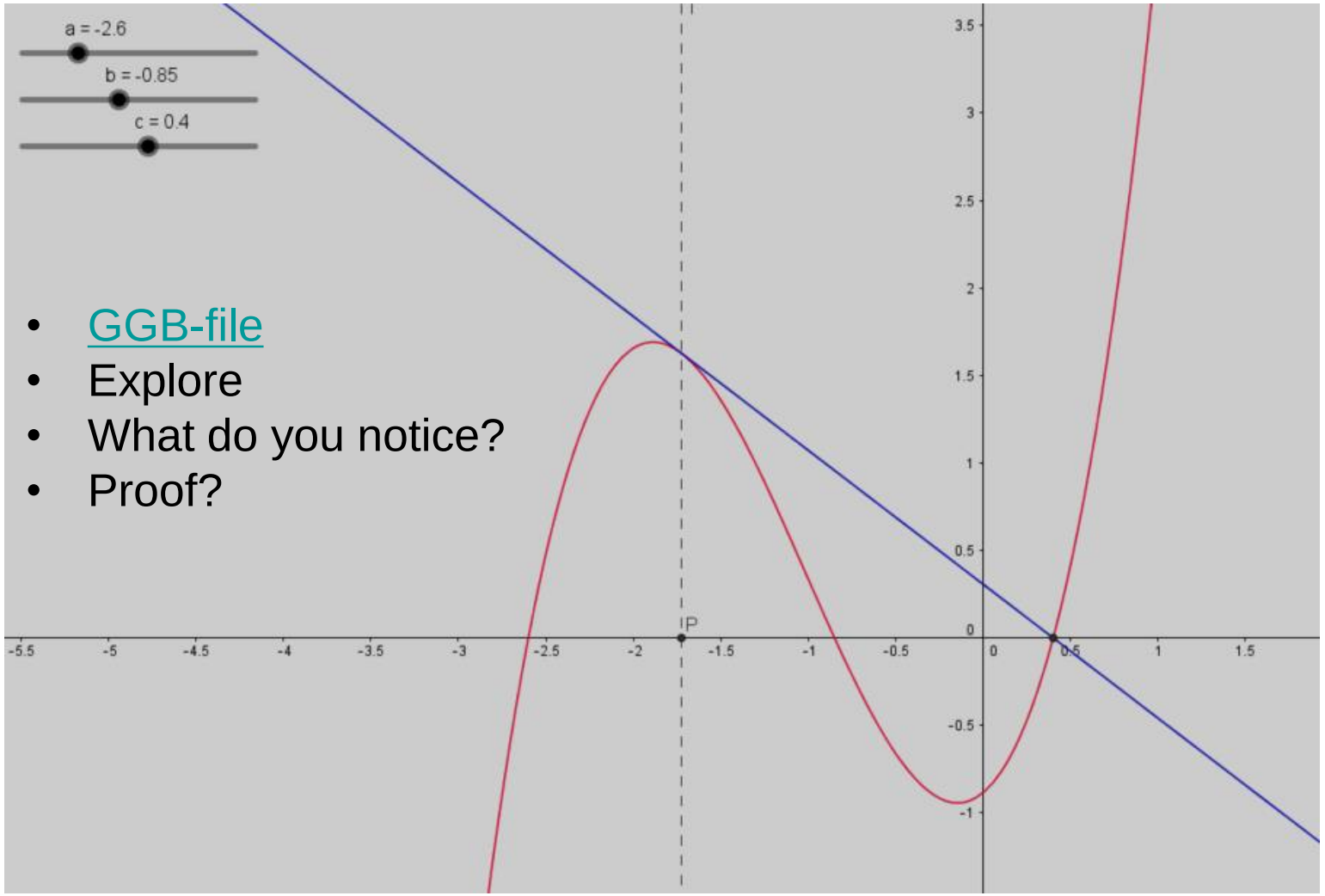
Universiteit Utrecht

[Faculteit Bètawetenschappen
FISME Freudenthal Institute for Science and Mathematics Education]

Outline

1. Appetizing examples
2. Algebra artefacts
3. An instrumentation lens
4. A quantitative lens
5. Assessment with Algebra Artefacts

Example 1: Strange property

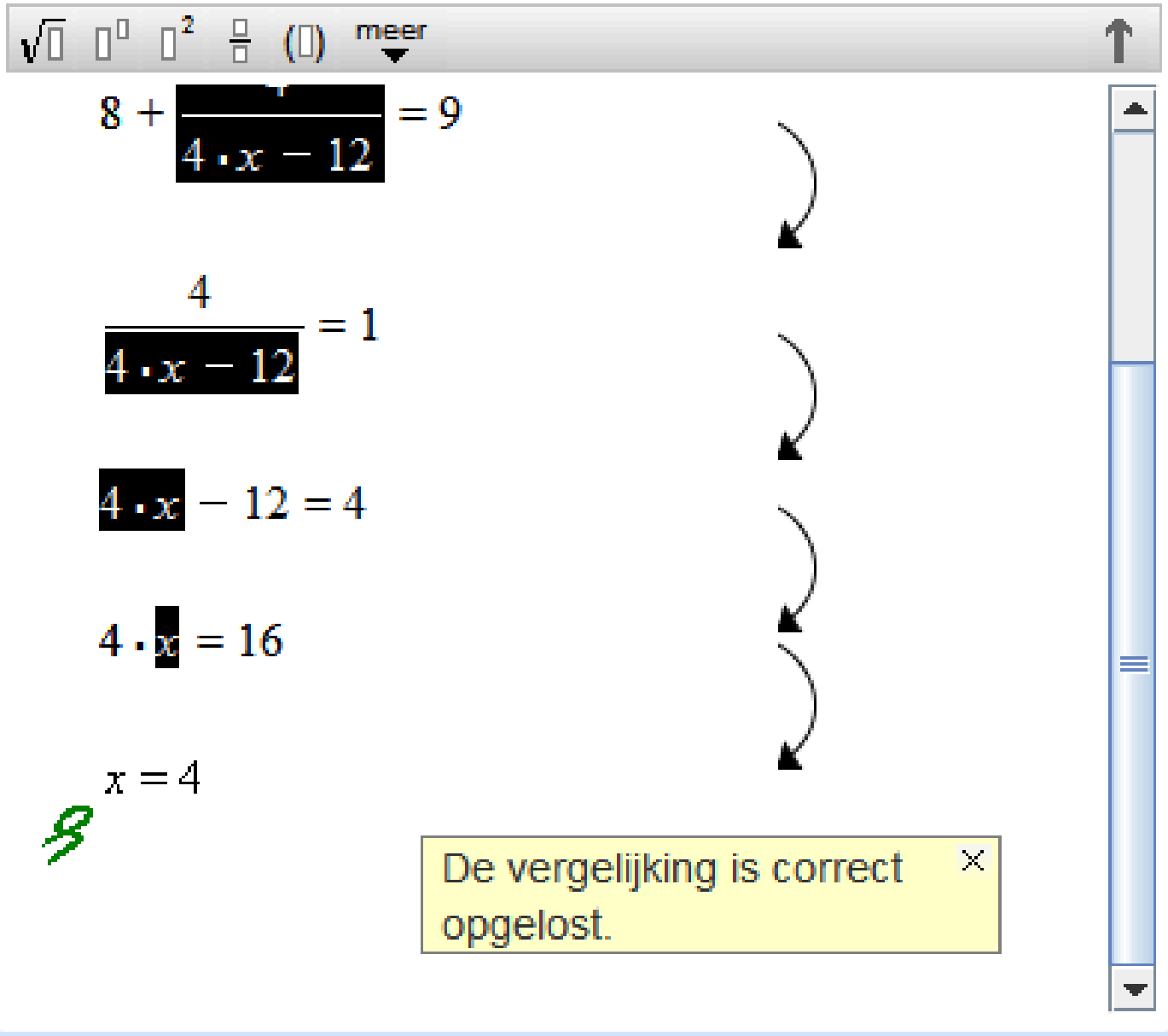


- GGB-file
- Explore
- What do you notice?
- Proof?

Example 1: Strange property

1	v(x):=(x-p)*(x-q)*(x-r) → $v(x) := (-p+x)\ (-q+x)\ (-r+x)$
2	w(x):=Derivative[v(x),x] → $w(x) := (-p+x)\ (-q+x) + (-p+x)\ (-r+x) + (-q+x)\ (-r+x)$
3	Raaklijn(x,a):=w(a)*(x-a)+v(a) → $Raaklijn(x,a) := (-a+x)\ ((a-p)\ (a-q) + (a-p)\ (a-r) + (a-q)\ (a-r)) + (a-p)\ (a-q)\ (a-r)$
4	Raaklijn(r,(p+q)/2) → 0
5	LijndoorR(x):=m*(x-r) → $LijndoorR(x) := m\ (-r+x)$
6	→ $\left\{x = r, x = \frac{p+q+\sqrt{4\,m+p^2-2\,p\,q+q^2}}{2}, x = \frac{p+q-\sqrt{4\,m+p^2-2\,p\,q+q^2}}{2}\right\}$

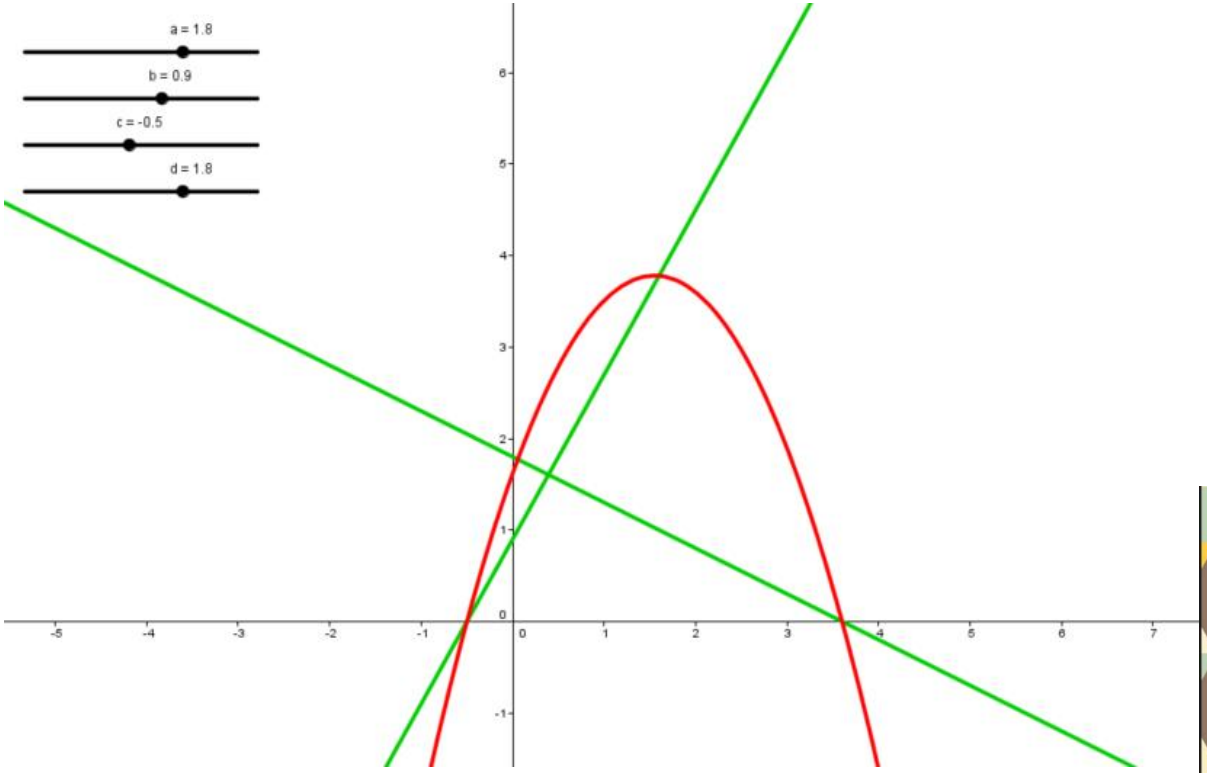
Example 2: Hard equations



http://www.dwo.nl/site/index_en.html

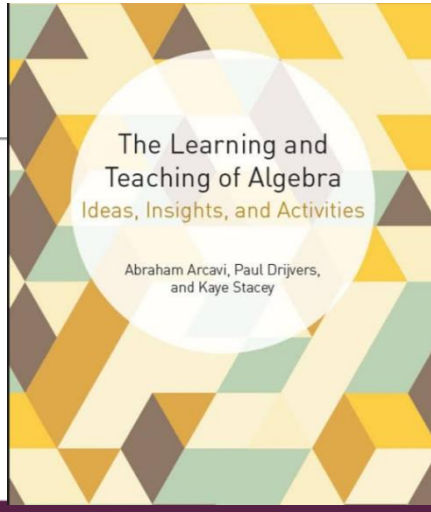
Jupri Drijvers, & Van den Heuvel-Panhuizen, 2016)

Example 3: Line game

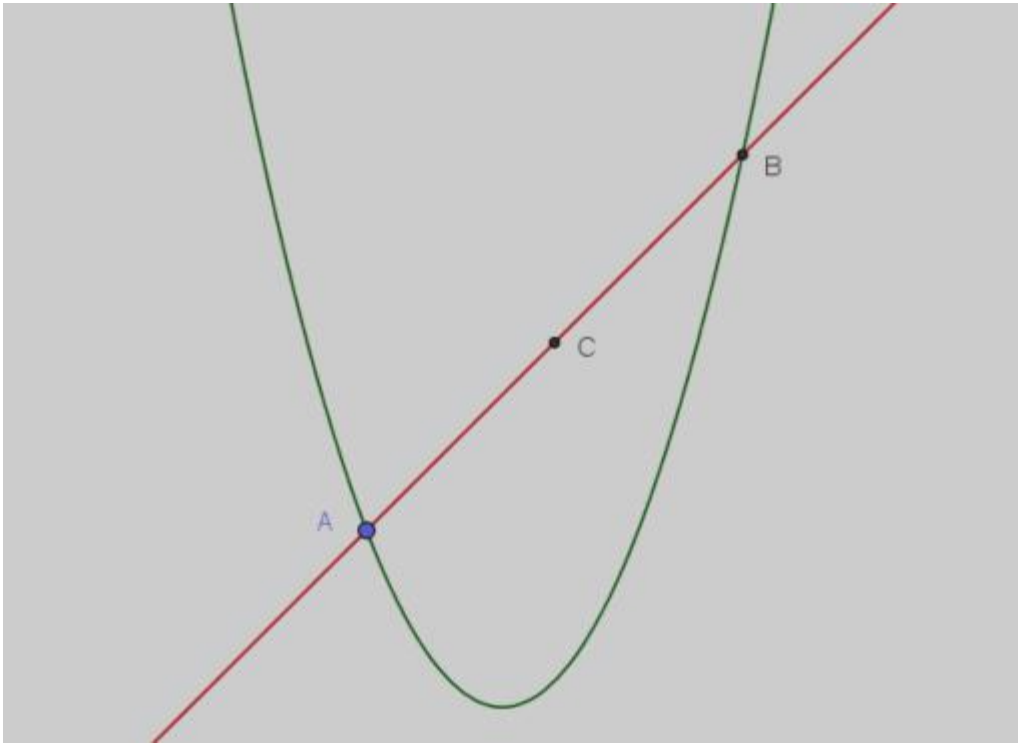


GGB-file

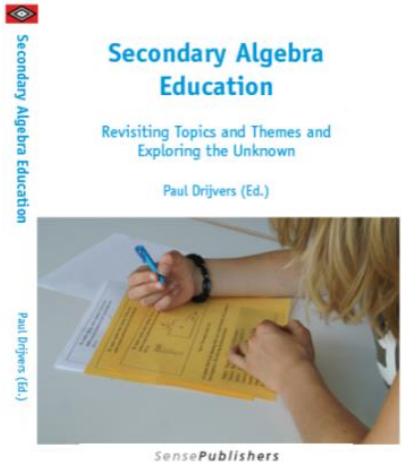
Arcavi, Drijvers, Stacey 2017, p. 111



Example 4: Parabola



Drijvers, Goddijn, & Kindt, 2011 p. 15



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What do we call an Algebra Artefact?

- A “thing”, a “tool” for doing algebra
- Can be material, digital, ….
- Today I focus on digital algebra artefacts
- AAs allows for “outsourcing” procedural work
- The use of AAs may both require and foster the development of algebraic thinking and symbol sense

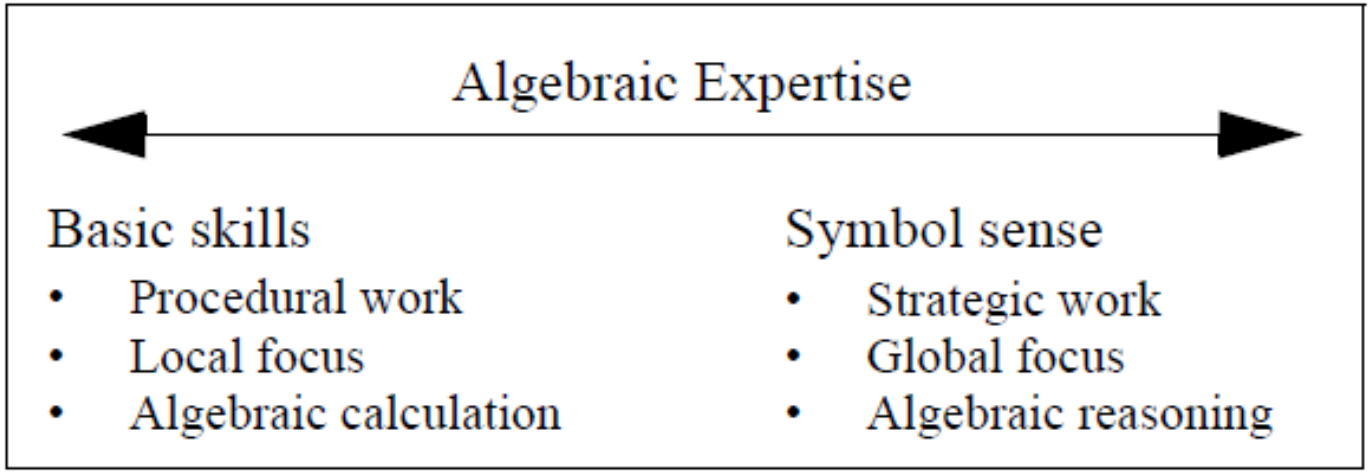
What do we mean by symbol sense?

- Arcavi (1994): A “non-algorithmic” feel for algebraic expressions and formulas

Symbol Sense: Informal Sense-making in Formal Mathematics

ABRAHAM ARCAVI

- Drijvers, Goddijn, Kindt (2011):



An artefact / tool: the hammer

One single tool for everything:

"I suppose it is tempting, if the only tool you have is a hammer, to treat everything as if it were a nail"

- Abraham Maslow (1966)

The Psychology of Science

Tools can be versatile: the multi-purpose Swiss army knife

Tools matter

Tools matter: they stand between the user and the phenomenon to be modelled, and shape activity structures. (Hoyles & Noss, 2003, p. 341)

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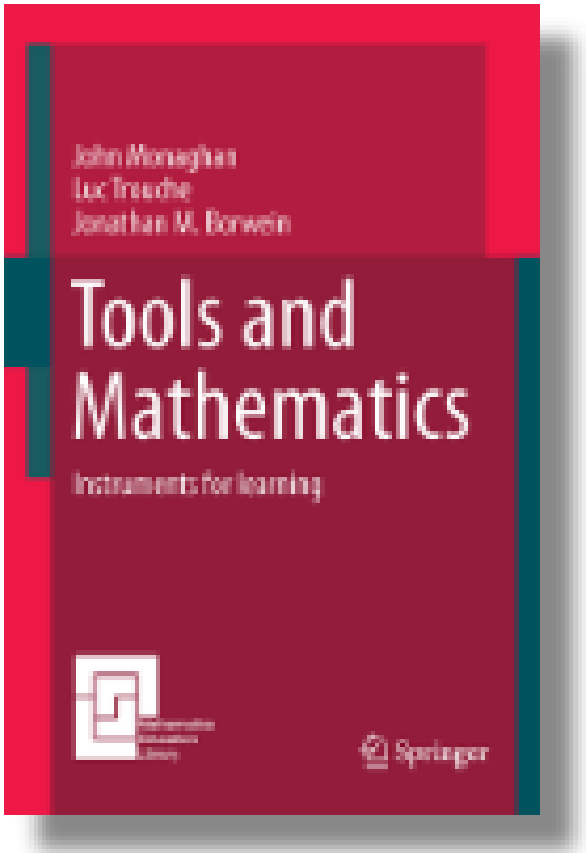
“These boots are made for walking”

Tools matter: they stand between the user and the phenomenon to be modelled, and shape activity structures. (Hoyles & Noss, 2003, p. 341)

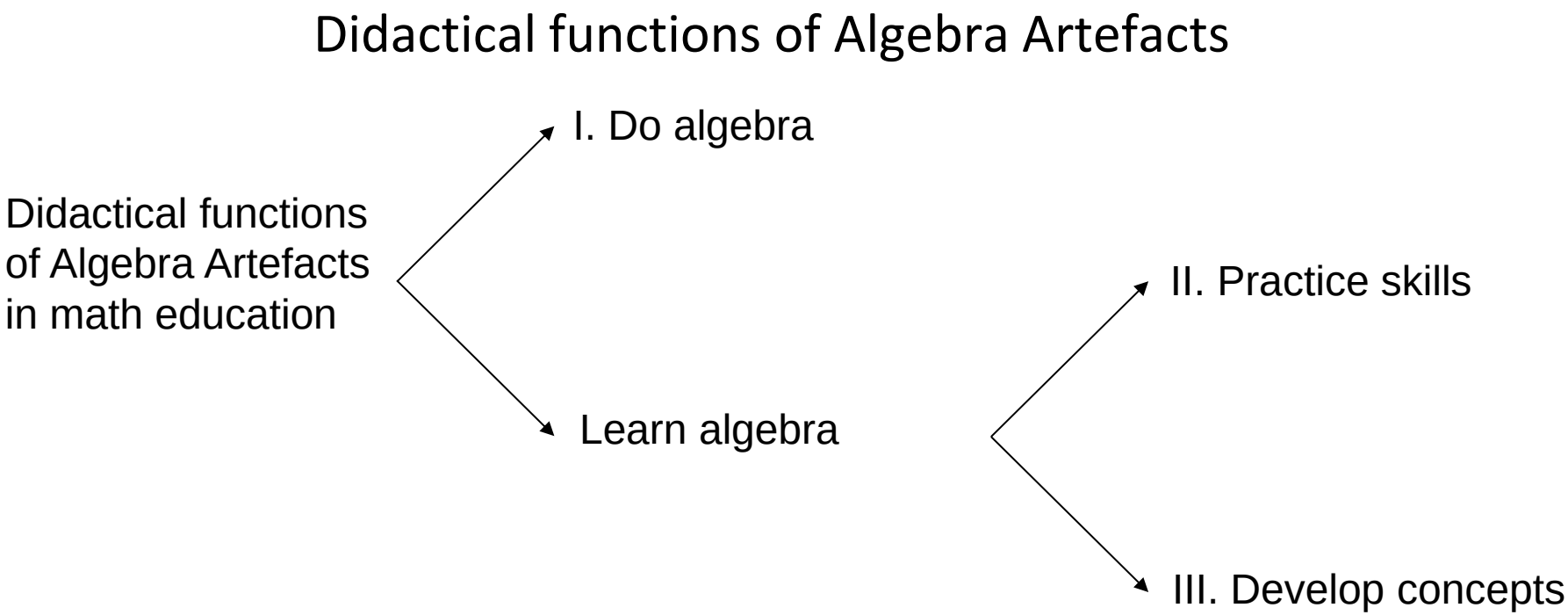
Cognitive tools

There is a subtle interplay between a user's knowledge and her tool use

Read more?
Monaghan, Trouche, & Borwein (2017). *Tools and Mathematics.* Instruments for learning.



Three didactical functions of AA



Drijvers, P., Boon, P., & Van Reeuwijk, M. (2011). Algebra and technology. In P. Drijvers (Ed.), *Secondary algebra education. Revisiting topics and themes and exploring the unknown* (pp. 179-202). Rotterdam: Sense.

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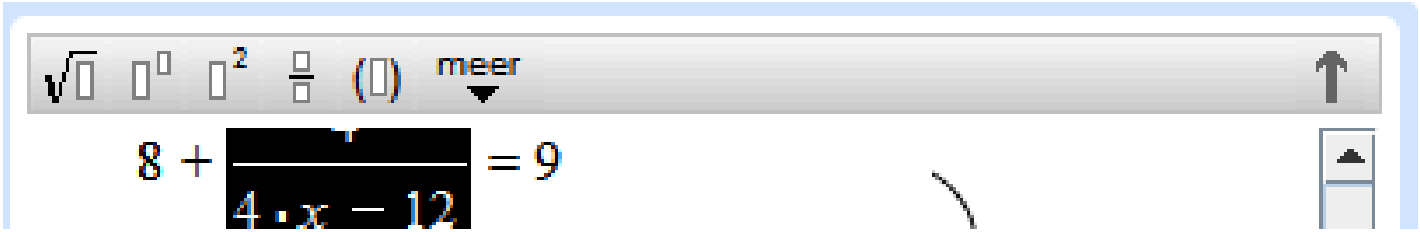
The subtle interplay (1)



Step	Conceptual aspect	Related technical aspect
1.	By scanning the equation, recognizing the equation as being of the form $f(z) = c$, so of the form <expression> = <numerical value>, with the unknown appearing only once on the left hand side. Realizing that the task is to rewrite this equation in the form <unknown> = <value>, which provides its solution. In this case, the expression is $\frac{48}{8(z+1)}$ and the numerical value is 3. As a consequence, the cover-up strategy can be applied.	No specific techniques involved in this step.
2.	By further inspection, recognizing the structure of the expression in left hand side of the equation. In this case, for example, the division of 48 by $8(z + 1)$ should be recognized as the central operator.	No specific techniques involved in this step. The equation has already been given in the solution window, i.e., $\frac{48}{8(z+1)} = 3$.
3.	Identifying a sub-expression to be covered as to start the cover-up strategy. In this case, this could be $8(z + 1)$.	Highlighting the identified sub-expression using the mouse. The applet puts the sub-expression in a new line and adds the equal sign. In this case, the result would be $8(z + 1) = \cdots$.
4.	Assigning a numerical value to the covered sub-expression to	Typing the value after the equal sign, and

The subtle interplay (2)

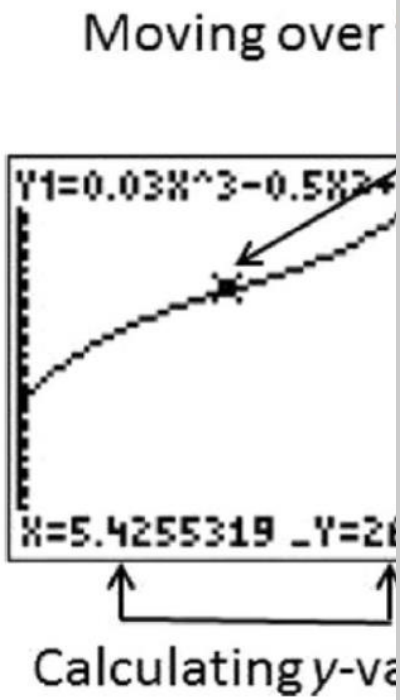


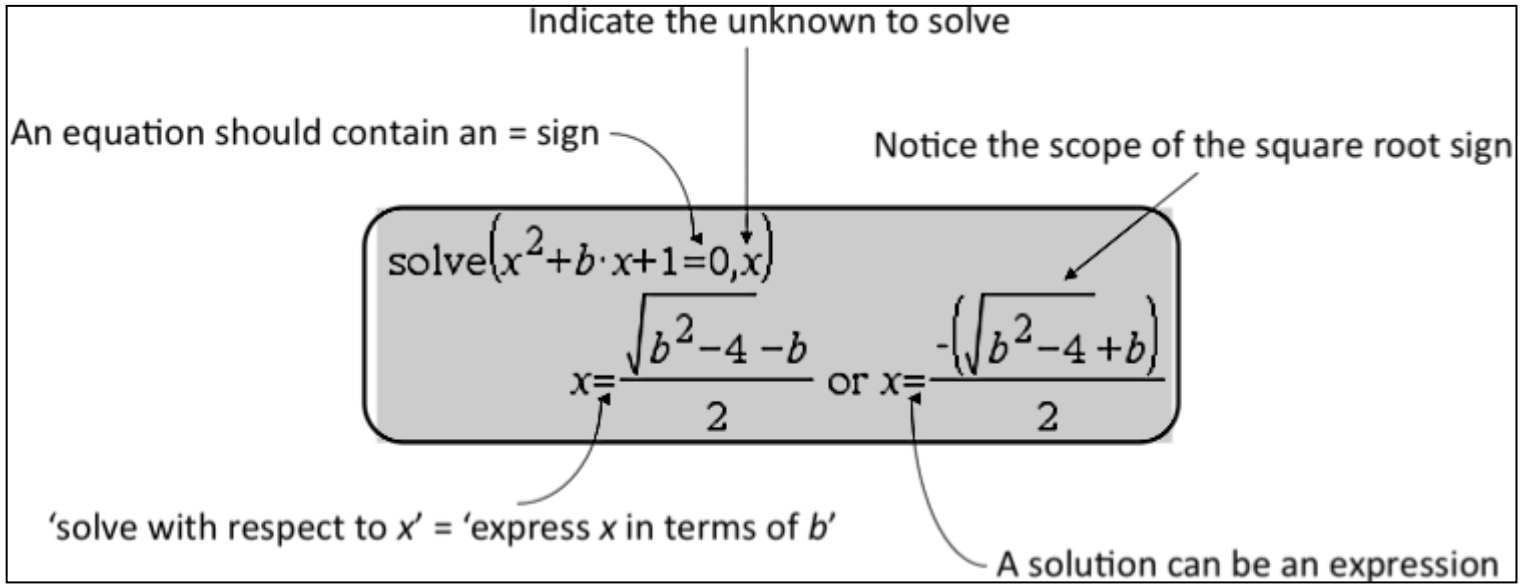
Fig. 4 An example of Andy's trac

Table 2 Overview of the schemes developed by Andy in TBI-1

Instrumentation scheme	Techniques	Conceptual elements	Technical elements
Tangent scheme	Draw a tangent on paper.	Rate of change is related to the steepness of the graph: the steepness of a tangent represents the steepness of the graph at one point.	Calculate the differences of y and x and calculate $\frac{\Delta y}{\Delta x}$.
Trace scheme	Plot the graph, move the cursor over the graph and look at the increase in y.	To find a minimum increase look at the slowest increase of subsequent y-values.	Plot the graph, press trace and scroll over the graph. The trace option gives pairs of x- and y-values. The GC makes equal steps in the x-values.
Trace–value scheme	Calculate the values with the GC and look at the increase over a unit interval.	The increase over a unit interval is an approximation to the instantaneous rate of change.	In the trace option, put in an x-value, press 'enter' and then the GC calculates the corresponding y-value.

Roorda, Vos, Drijvers, & Goedhart, 2016

The subtle interplay (3)



(Drijvers, 2003; Drijvers, Godino, Font & Trouche, 2013)

The subtle interplay (4)

Maria's scheme seems to reveal the following conceptual difficulties:

1. Imagining substitution as "pasting an expression into all instances of a variable": Maria's substitution scheme seems to be limited to numerical substitutions.
2. Remembering the artefact's syntax of the Substitute command and the meaning of the vertical bar symbol in it: Maria is aware of the need of the vertical bar, but is not always able to use it appropriately.
3. Being able to interpret the result, and particularly to accept an equation with two variables as a result: Maria seems to have difficulties with substitution results such as "b=(−x^2–y+1)/x" (unit U5).

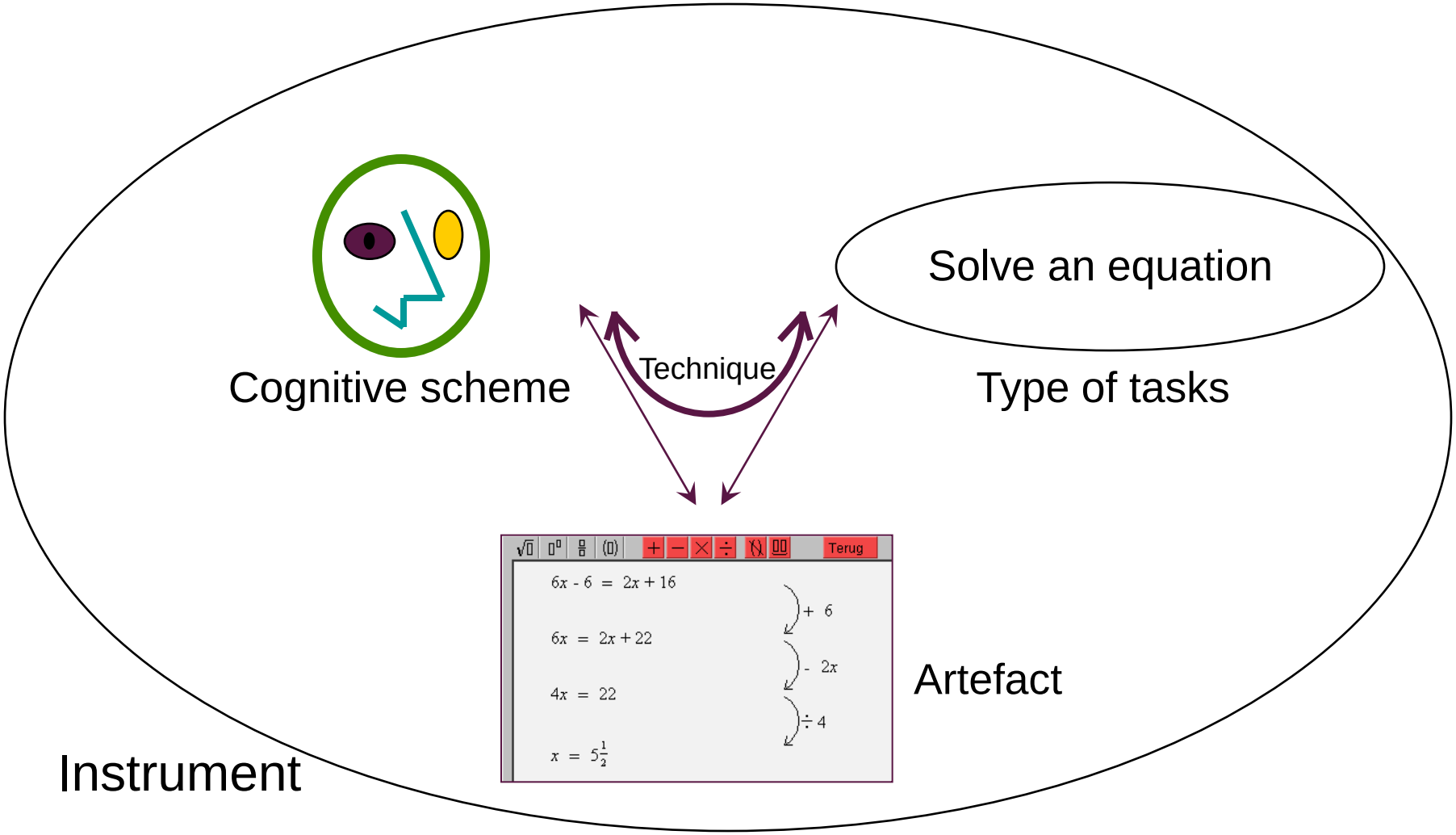
The subtle interplay: the instrumental approach

- Distinction artefact – instrument
- Instrument: mixed entity of artefact and scheme
- Dialectics between technical and conceptual (MTB p. 247)
- Instrumental genesis: the co-emergence of techniques and schemes (Artigue, 2002; Drijvers, 2003; Kieran & Drijvers, 2006; Trouche, 2005)

Artefact and instrument

- Artefact: the object or ‘thing’
- A good artefact does not guarantee nice music!
- Cognitive scheme = Technique + Concepts
- Instrument = Artefact + Schemes

Artefact and Instrument



Scheme and technique

- Scheme: an invariant organization of behaviour for a class of situations (Vergnaud, 1987, 1996)
- Technique: an invariant type of user – artefact interaction, reflecting
- An instrumentation scheme includes conceptual and technical element
- A technique can be considered as the observable part of a scheme.
- Descriptions of (the development of) schemes help us to capture and exploit the potential of Algebra Artefacts

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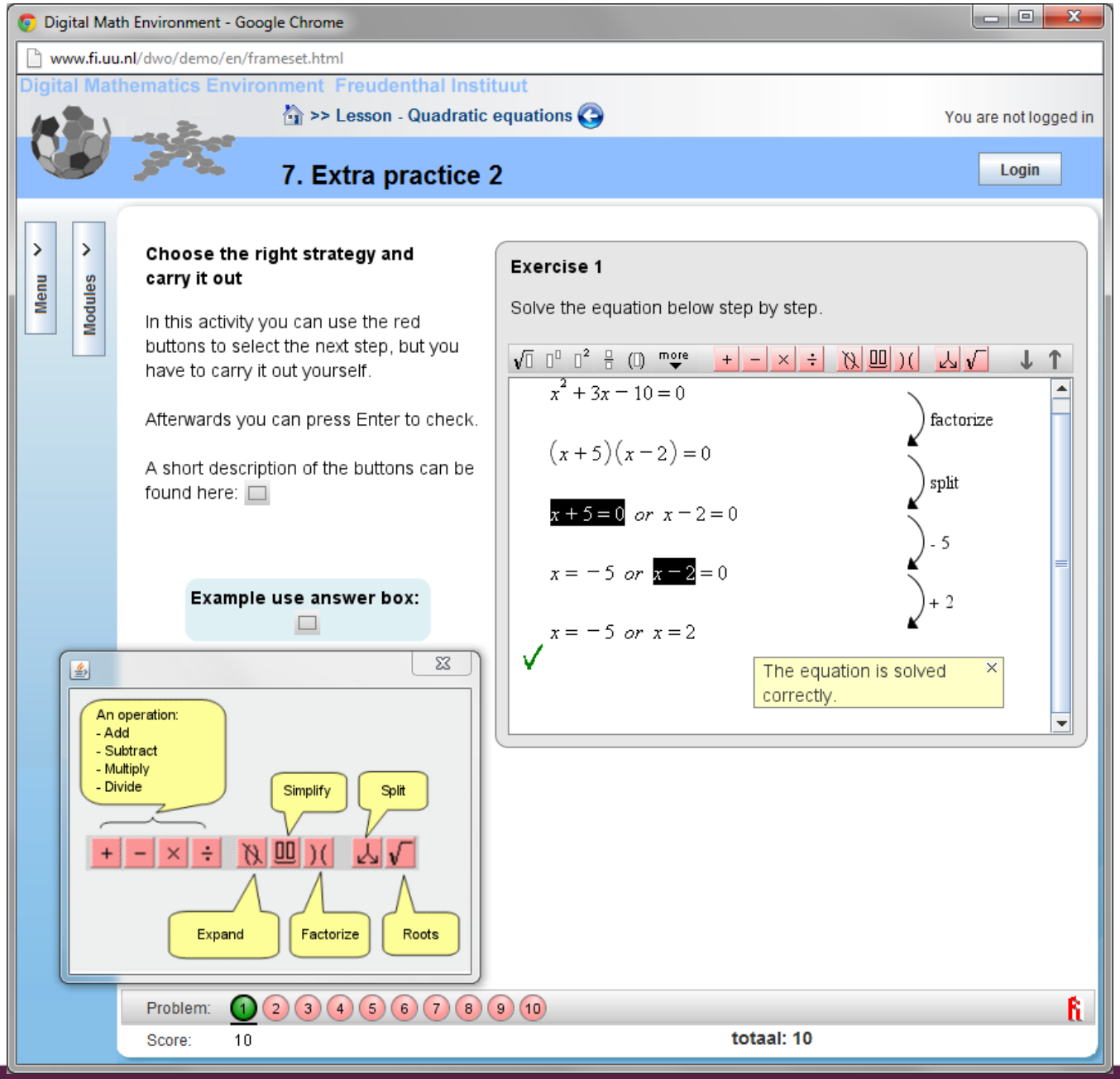
Empirical evidence for the power of Algebra Artefacts? An example

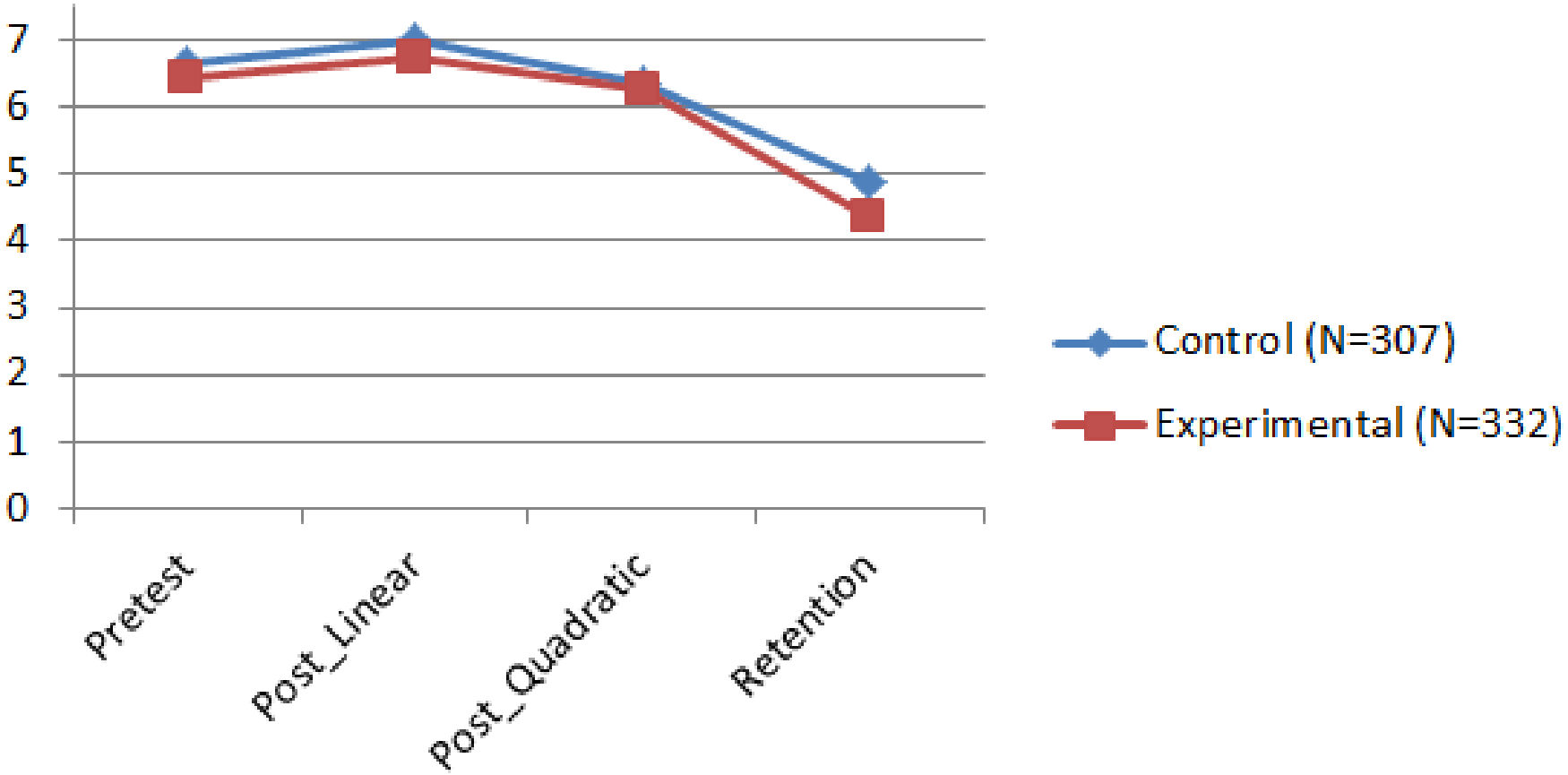
- RQ: What is the effect of learning and practicing of algebraic skills in a digital environment as compared to a traditional learning environment?
- Topic: linear and quadratic equations in grade 8
- ICT: Freudenthal Institute's Digital Mathematics Environment (DME)
- Sample: 842 13-14 year old high-achieving students

Drijvers, P., Doorman, M., Kirschner, P., Hoogveld, B., & Boon, P. (2014). The effect of online tasks for algebra on student achievement in grade 8. *Technology, knowledge and learning, 19*, 1-18.



http://ws.fisme.science.uu.nl/dwo/en/student/#TreeModulePlace:89157

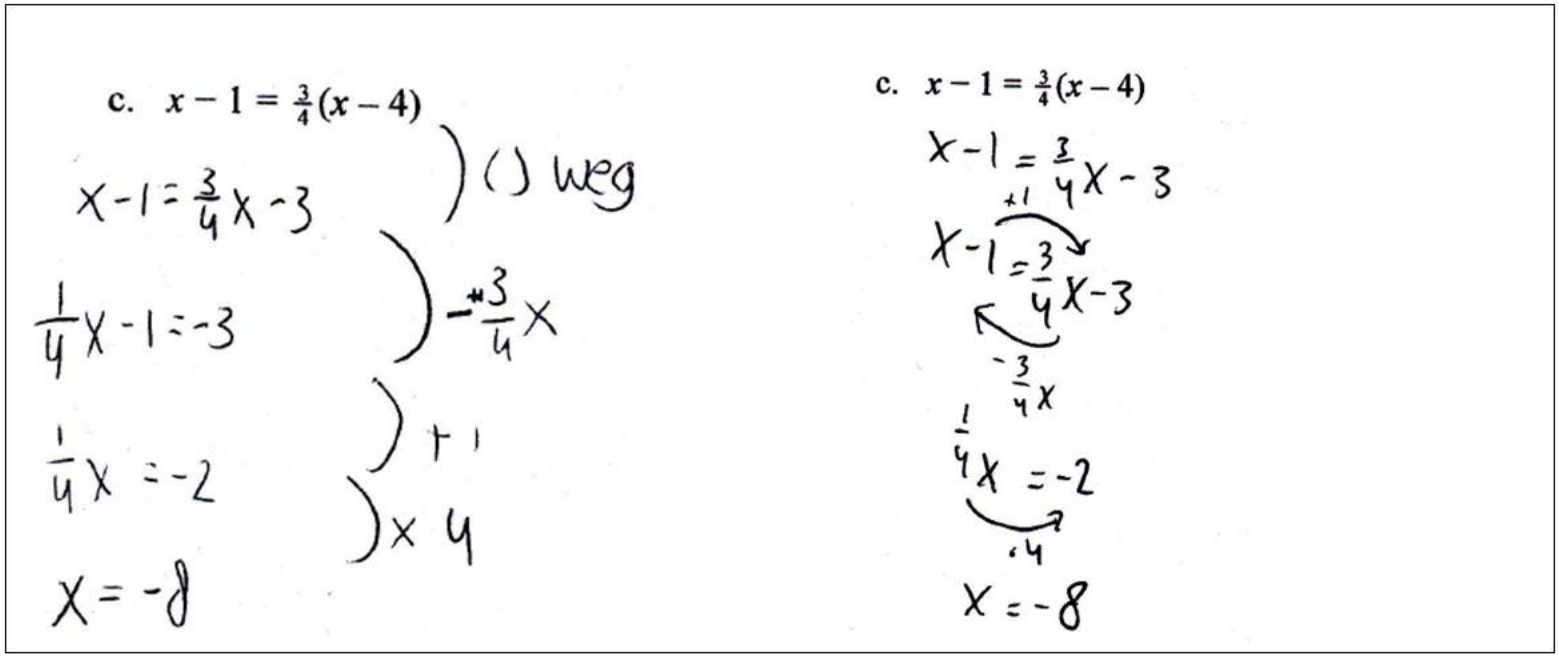
Results



Drijvers, P., Doorman, M., Kirschner, P., Hoogveld, B., & Boon, P. (2014). The effect of online tasks for algebra on student achievement in grade 8. *Technology, knowledge and learning, 19*, 1-18.

Possible explanations

1. Spill-over effect?
2. Too superficial learning?
3. Too limited, mainly strategic, feedback?
4. Lack of integration of computer work and paper-and-pen work?



Review studies: small effects

Study	Number of effect sizes	Average effect size	Global conclusion
Li and Ma 2010	85	$d = 0.28$ (weighted)	Moderate significant positive effects.
Rakes et al. 2010	109	d range 0.151 – 0.165	Small but significant positive effects.
Cheung and Slavin 2013	74	$d = 0.16$	A positive, though modest effect.
Steenbergen-Hu and Cooper 2013	61	g range 0.01 – 0.09	No negative and perhaps a small positive effect.
Sokolowski, Li and Willson 2015	24	$g = 0.60$	A moderate positive effect size.

Table 1: Effect sizes reported in five review studies

Drijvers (in press)

Interpretation of limited evidence:

One interpretation of all this is that building deep, conceptual understanding and higher-order thinking requires intensive teacher-student interactions, and technology sometimes distracts from this valuable human engagement. Another interpretation is that we have not yet become good enough at the kind of pedagogies that make the most of technology; that adding 21st-century technologies to 20th-century teaching practices will just dilute the effectiveness of teaching.

(OECD, 2015, p. 3)

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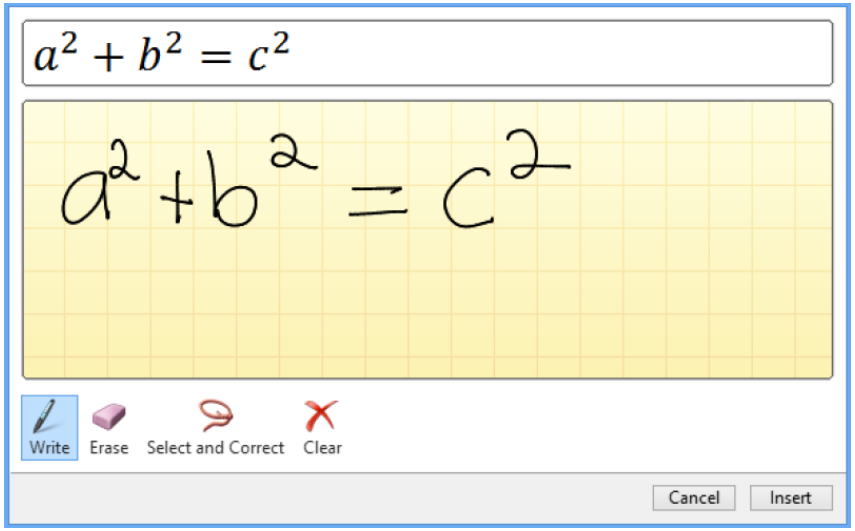
Digital assessment

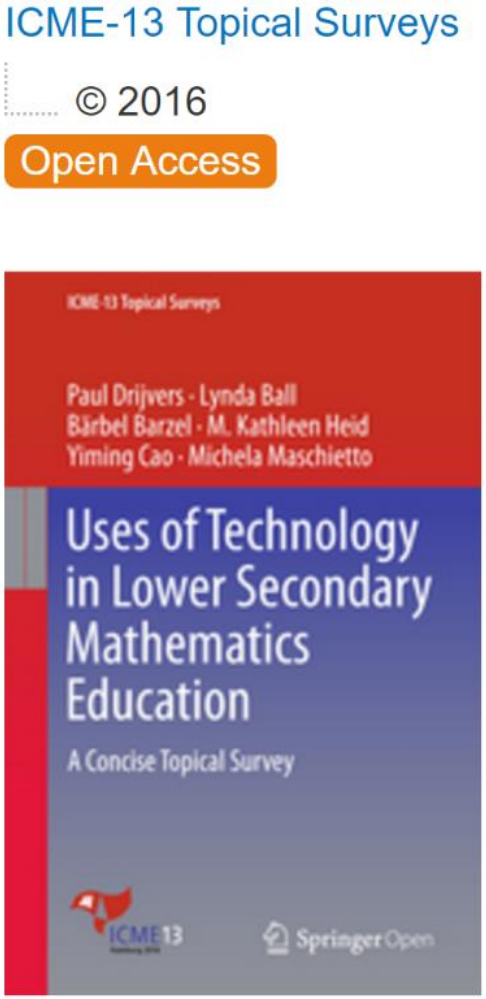
- Challenging with respect to
  - Students' construction room in assessment environments for algebra
  - User-friendliness with respect to entering and manipulating algebraic expressions (-> hand writing recognition?)
  - For automated "intelligent" scoring of student responses.



Criteria for assessment tools for algebra

- A formula editor
- A graphing module
- An expert system (ITS / CAS) to interpret student input in an “intelligent” way, so that scoring and/or feedback can be done in a close-to-human way

Drijvers, P., Ball, L., Barzel, B., Heid, M. K., Cao, Y., & Maschietto, M. (2016). Uses of technology in lower secondary mathematics education; A concise topical survey. New York: Springer

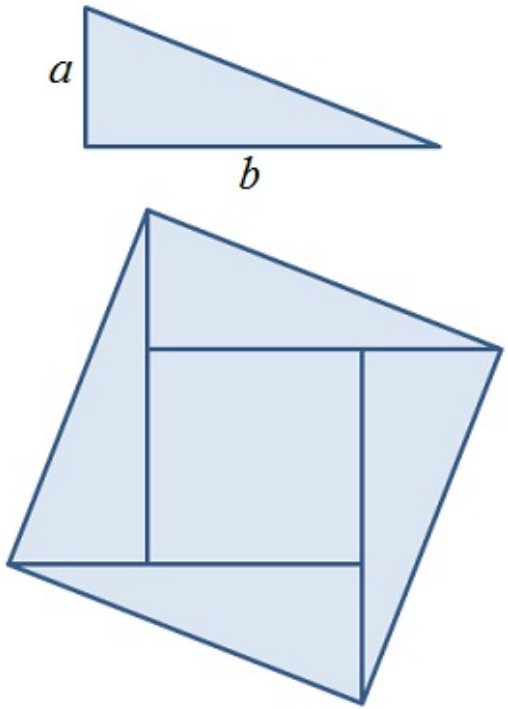


http://www.springer.com/us/book/9783319336657

Example 1: Making a square

Completing the square

Below you see a right-angled triangle. The sides adjacent to the right angle have length a and b, respectively.
Four of these triangles are placed in such a way that they form a square.



Find a formula for the area O of the square, including the inner part.

$O = a^2 + b^2$

$$O = a^2 + b^2$$

$$O = 4 \cdot \frac{1}{2} \cdot a \cdot b + (b - a)^2$$

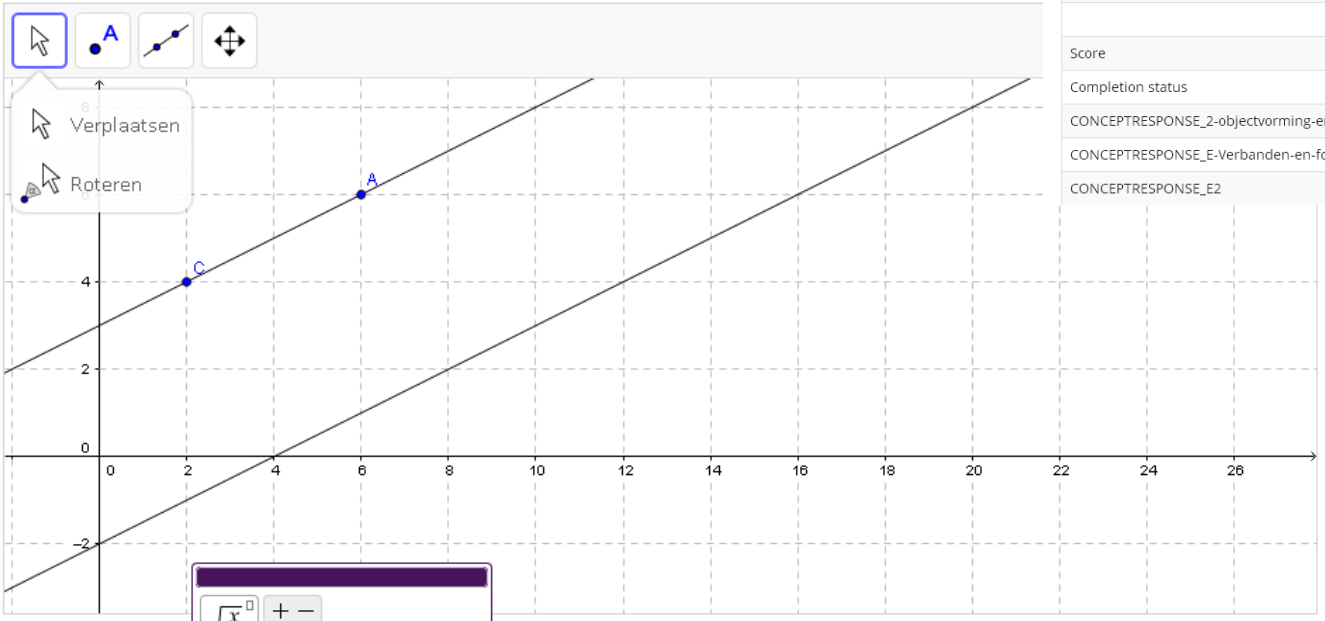
$$O = \left(\sqrt{a^2 + b^2}\right)^2$$

Example 2: Parallel lines

Import User — **Vraag 1 van 1**

dtt-wi-DEMOitem1 - dtt-wi-GGBtestHv286SC

Door punt C gaat een lijn die evenwijdig is aan de zwarte lijn.



Geef de vergelijking

$y = 3 + x \div 2$

Testmonitor

Huidig item | Versieinformatie

Sleutels	$\&[\Leftrightarrow 3 + \frac{x}{2}]$
Sessiedata	
Antwoord	GeoGebra antwoord
Ruwe score	0
Berekende score	0
Outcome	
Score	0
Completion status	completed
CONCEPTRESPONSE_2-objectvorming-en-meervoudige-betekenis	0
CONCEPTRESPONSE_E-Verbanden-en-formules	0
CONCEPTRESPONSE_E2	0

Geef de vergelijking die door punt (Geef de vergelijking die door punt Geef de vergelijking die doo Geef de vergelijking die door punt C gaat.

$y = 3 + \frac{1}{2}x$

$y = \frac{1}{2}x + 3$

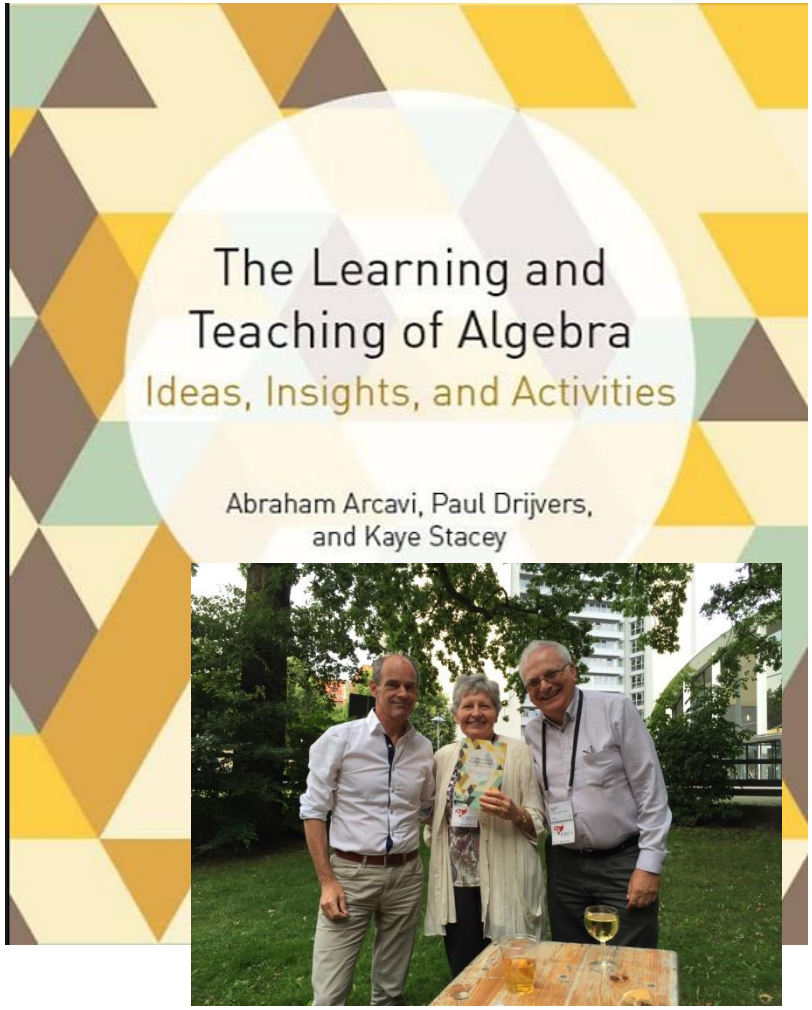
$y = \frac{1}{2}x + 1 + 1 + 1$

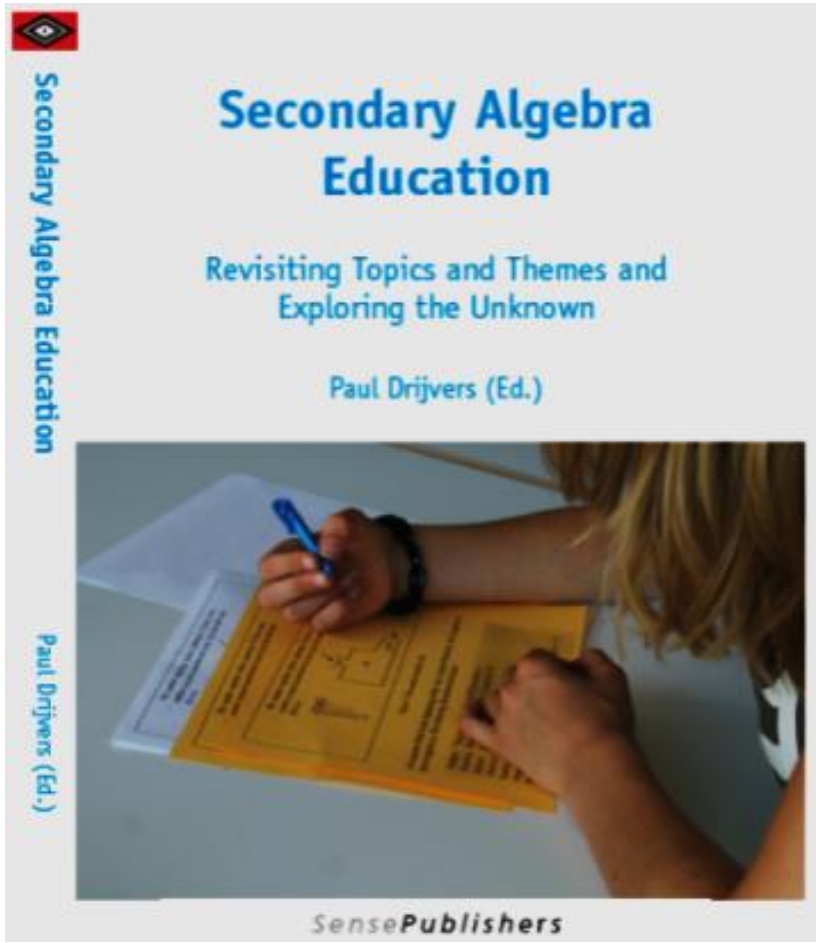
$y = \frac{1}{3}x + 3$

Overall conclusion

- The use of AAs may both require and foster the development of algebraic thinking and Symbol Sense;
- Stronger empirical evidence on how and why this works is needed;
- There is work to be done on assessing algebra with artefacts in a way that resembles the teaching and learning process.

Commercial break:





Thank you for your attention!

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