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SYNTAX AND MEANING AS SENSUOUS, VISUAL, HISTORICAL FORMS OF ALGEBRAIC THINKING

ABSTRACT. Before the advent of symbolism, i.e. before the end of the 16th Century, algebraic calculations were made using natural language. Through a kind of metaphorical process, a few terms from everyday life (e.g. thing, root) acquired a technical mathematical status and constituted the specialized language of algebra. The introduction of letters and other symbols (e.g. "+", "=") made it possible to achieve what is considered one of the greatest cultural accomplishments in human history, namely, the constitution of a symbolic algebraic language and the concomitant rise of symbolic thinking. Because of their profound historical ties with natural language, the emerging syntax and meanings of symbolic algebraic language were marked in a definite way by the syntax and meanings of the former. However, at a certain point, these ties were loosened and algebraic symbolism became a language in its own right. Without alluding to the theory of recapitulation, in this paper, we travel back and forth from history to the present to explore key passages in the constitution of the syntax and meanings of symbolic algebraic language. A contextual semiotic analysis of the use of algebraic terms in 9th century Arabic as well as in contemporary students' mathematical activity, sheds some light on the conceptual challenges posed by the learning of algebra.

KEY WORDS: cultural semiotics, equations, history of algebra, meaning, ontogenesis, phylogenesis, recapitulation, syntax

1. Introduction

Two Grade 7 students were trying to use, for the first time, algebraic techniques to solve a word-problem. After dividing two numbers with a calculator, one of the students got 17.5. Looking at the screen, the other student asked: "How can you have 0.5 of a candy?" With dismay, the first student answered: "I don't know. This is making me mad!"

Situations like this one have prompted, since the early 1970s, an impressive amount of research on the teaching and learning of algebra. The problem, however, is older. It seems to be as old as algebra itself. Thus, in the third century AD the Greek mathematician Diophantus composed a treatise (the *Arithmetica*) containing the germ of some of what we today call key algebraic ideas. In the epistolary section with which the treatise begins, Diophantus says to his friend Dionysius:

... I have tried, beginning from the foundations on which the subject is built, to set forth the nature and power in numbers. Perhaps the subject will appear to you

Educational Studies in Mathematics (2007) 66: 145–164
DOI: 10.1007/s10649-006-9024-6 © Springer 2006

rather difficult, as it is not yet common knowledge, and the minds of beginners are apt to be discouraged by mistakes. . . [1]

Algebra, hence, seems to have always been a source of frustration to many. But what is it that makes algebra so difficult?

Previous research suggests that a great number of important difficulties encountered in the learning of elementary algebra are related to students' understanding of the *meaning* of signs and the *syntax* of the algebraic language.[2] More specifically, students' difficulties are often connected to:

(1) the understanding of the distinctive *manner* in which simple signs (e.g. "x", "n") and compounded signs (e.g. "2 + 5" or "x + 17") stand for the objects that they represent, and
(2) the grasping of the sense of the *operations* carried out on those signs.

To better understand the distinctive manner in which algebraic signs stand for the objects that they represent (i.e. the algebraic way of signifying) it may be helpful to notice that, in school arithmetic, signs designate *known* quantities only; the *unknown* of the problem is the end of the problem-solving process. One does not even need to represent the unknown. In algebra, in contrast, the *unknown* has to be represented (through a word, a drawing, a mark, a rod on an abacus, a letter, etc.), for it is the initial point of the mathematical process. The algebraic text unfolds as calculations from this representation of the unknown are derived.[3]

Because of the algebraic mode of signifying, the sources of meaning of algebra are different from those of arithmetic. Naturally, the question is: where does algebraic meaning come from? Let us travel back to the past again. Many mathematical problems contained in Mesopotamian clay tablets dating back to the first half of the second millennium BC exhibit calculations with unknown quantities. In these problems, the unknown quantities were contextually represented – e.g. as the 'length' and the 'width' of a rectangle. Recent historical studies have shown that the *meaning* of calculations was ensured by *visual geometric transformations* that usually attempted to transform the original figure into a geometric form (e.g. a square) for which a solving procedure was already known (see Høyrup, this volume). However, as time went on, the geometric context that infused the calculations on the unknown with meaning was progressively lost. The geometric context still appeared in al-Khwārizmī's famous 9th Century AD treatise *The concise book of the calculation of al-jabr and al-muqābala* to justify the algebraic procedures, but it faded away as algebraic symbolism emerged in the Renaissance. This important shift led to a focus on *quantities*, which led, in turn, to changes in the mode of endowing signs with meaning. As a result, algebra was seen as completely divorced from geometry and thought of as a mere generalized arithmetic.[4]

It may be wise to state from the outset that we are not pleading here for a curricular organization of algebra that matches its historical development. As it has been argued, recapitulation – i.e. the theory that asserts that, in their intellectual development, our students more or less naturally traverse the same stages that humankind passed through – is highly problematic. The inadequacy of Recapitulation Theory does not mean, however, that life-span developments (*ontogenesis*) are completely independent of past historic-cultural ones (*phylogenesis*). On the contrary, it means that the relationship between ontogenesis and phylogenesis is much more complex than predicted by such a theory.[5]

In this paper we want to address the question of the way in which students make sense of algebraic symbolism. In order to do so, in the next section, we introduce the *Embedment Principle*. This principle offers an explanation of the link between phylogenesis and ontogenesis and, on a methodological level, suggests how insights into our research question can be gained by examining it from a historical dimension and a contemporary one. In the subsequent sections we focus on the way in which equations involving rational coefficients were dealt with in the history of mathematics and in a contemporary Grade 8 classroom. The last section summarizes our results.

2. The phylogenesis/ontogenesis embedment principle

The chief characteristic of an individual's life-span development (ontogenesis) is the fact that it is framed by a world of cultural artifacts and higher evolved forms of cognition that offer the individual an array of lines of conceptual development. Such an array of developmental possibilities was simply absent when the first Homo Sapiens appeared on earth. Homo Sapiens found an environment that was purely circumstantial or natural. In contrast, as Vygotsky (1998, p. 308) wrote, "the environment acts in the development of the child, in the sense of development of the personality and its specific, human properties, in the role of a source of development".

Biological or natural developments unavoidably become *affected* by and *entangled* with the historical-cultural one as individuals use signs and other cultural artifacts, such as language. In fact, the merging of natural and historical developments constitutes the actual line of growth of the individual. Given that it is impossible to untie the merging of the cultural and the natural lines of development, the conceptual growth of each individual cannot *reproduce* a historical-social conceptual formation process. In short, phylogenesis cannot recapitulate ontogenesis.

In light of the previous discussion, we want to suggest that the relationship between phylogenesis and ontogenesis is better cast by the following *Embedment Principle*: our cognitive mechanisms (e.g. perceiving, abstracting, symbolizing) are related, in a crucial manner, to a historical conceptual dimension ineluctably *embedded* in our social practices and in the signs and artifacts that mediate them. Indeed, the contexts in which we think are anchored on an ubiquitous stratum of historically constituted cognitive activity from which we draw in a fundamental way – even if not consciously (Lektorsky, 1984; Pea, 1993).

In terms of our research question, what the *Embedment Principle* asserts is that the algebraic signs and meanings that the students meet in school are bearers of the cognitive activity of previous generations. This historical cognitive activity deposited in signs, the semiotic system that they form, and the social practices that they mediate offer our students certain lines of conceptual development, malleable vectors of cognitive growth that the students can pursue and transform in accordance to the activities they engage with.

In this context, syntax and meaning, we want to suggest, are historically constituted sensuous features of algebraic thinking that empower our students to reflect, interpret, create and imagine in a certain mathematical way – much in the same manner that syntax and meaning do in prose and poetry.

Generally speaking, the historical cognitive activity deposited in signs, their meanings, artifacts and the social practices that they mediate is not something transparent for the students. As we shall see in Section 5, to become aware of such a historical intelligence requires that the students engage in an active process of sense-making; more precisely, a process of *objectification* or of making something apparent (Radford, 2002a).

3. Rational equations: an historical example

In Section 5 we will discuss some of the difficulties encountered by Grade 8 students in dealing with the equation $x + \frac{x+17}{2} = x + 17$. In this section, we shall discuss the symbols associated with fractions in algebra from an historical viewpoint. The question is: How were fractions such as the previous one symbolized and what was their meaning? How were calculations on equations involving fractions of the unknown justified? The first point to notice is that the invention of signs to represent fractions is not part of the elaboration of algebraic language but of the language of arithmetic. It was only later that signs for fractions were adapted to algebra. The *Liber Abacci*, written by Leonardo of Pisa (Fibonacci) in the 13th Century gives us a clear example. In this treatise, we find five different representations of fractions, each one related to a different conceptualization of fractional numbers.[6] In addition to the standard form (e.g. $\frac{8}{9}$), Leonardo also writes expressions such as $\frac{2\ \ 4\ \ 6\ \ 8}{3\ \ 5\ \ 7\ \ 9}$o which represents "eight ninths and six sevenths of eight ninths and four fifths of six sevenths of eight ninths and two thirds of four fifths of six sevenths of eight ninths", that is: $\frac{8}{9} + \frac{6}{7} \times \frac{8}{9} + \frac{4}{5} \times \frac{6}{7} \times \frac{8}{9} + \frac{2}{3} \times \frac{4}{5} \times \frac{6}{7} \times \frac{8}{9}$. In other cases, in the same treatise, the sign $\frac{1\ \ 5\ \ 7}{2\ \ 6\ \ 10}$ (with no circle at the end, as opposed to the previous sign) is used to represent the fraction that, we – with our modern symbolism – can represent as: $\frac{7}{10} + \frac{5}{6} \times \frac{1}{10} + \frac{1}{2} \times \frac{1}{6} \times \frac{1}{10}$.

These ways of representating fractions belong to the Arabic mathematical tradition with which Leonardo became acquainted during his stay in Bugia. Authors in the Muslim world did not write or read those complex signs as we do. These signs were written and read following the right-to-left order of the written Arabic language. This is a token of the influence of the oral language on the specialized symbolism of mathematics and its syntax. (Notice that our left-to-right writing orientation has reversed the order in which we symbolise fractions, thus affecting our symbolic syntax.) In the Arabic mathematical tradition, the oral expressions stress the original meaning and the successive process of partitions: first the tenths, of which seven parts are taken, then a fraction of a tenth, etc. According to Djebbar (1992), the first Arabic treatise to deal with fractions is al-Khwārizmī's *The Indian Calculation*. However in his treatise on algebra (*The concise book of the calculation of al-jabr and al-muqābala*) al-Khwārizmī writes *everything* in vernacular language, with no words other than those of the Arabic language – i.e. he does not use "mathematical" symbols.[7] The Arabic language has the peculiarity that it has specific words to refer to the following unitary fractions: $1/n$, $2 \leq n \leq 10$. These are "expressible" fractions; in contrast, it does not have specific words to express unitary fractions having a denominator bigger than 10; they are called "inexpressibles" or "surd". Thus 1/13 is expressed as a process: "one part of thirteen parts". Similarly, 3/13 is expressed as "three parts of thirteen parts".

How, then, relying on the aforementioned Arabic linguistic form to refer to fractions does al-Khwārizmī conceptualize and solve equations that require one to take fractional parts into consideration? When al-Khwārizmī solves a problem involving fractions in an algebraic context, he does not make recourse to the Arabic expression. Hence, instead of writing the equivalent fractional Arabic jargon to our algebraic fractional equation

$$\frac{x}{x+2} = \frac{1}{2}$$

he writes: “By the division of thing by thing and two dirhams, half a dirham appears as quotient.” As we can see, the calculations do not refer to the meaning of a fraction but to the meaning of a *division*.[8] The statement of the problem is made in such a way that the operations are progressively described until the result “appears”. Before we go further, let us pause on al-Khwārizmī’s algebraic terms. To refer to the unknown, al-Khwārizmī uses the term *shay’* (usually translated as *thing*).[9] *Shay’* or *thing* was incorporated into the language of mathematics as a technical term -along with a few other basic terms.[10] Other quantities are expressed through a combination of the basic terms, like “thing and two dirhams” as in the problem.[11]

Now, how does al-Khwārizmī solve an equation like the previous one? He relies on the numerical meaning of the division. He says: “You have already observed, that by multiplying the quotient by the divisor, the amount (of money) which you divided is restored. This amount (of money) in the present case, is thing. Multiply, therefore, thing and two dirhams by half a dirham [. . .].” (Rosen, 1831, p. 36 of the Arabic).

The same strategy is used later by the Italian abacists. Thus, in Antonio de’ Mazzinghi’s *Trattato di Fioretti* (14th Century) the solution of some problems often involved algebraic fractions. As Franci (1992) remarked, the frequency with which algebraic fractions appeared in Mazzinghi’s *Trattato,* made algebraic fractions, for the first time, virtually into a genuine mathematical domain. One element of interest for our discussion here is that the *Trattato* is not a book as such. Mazzinghi belongs to a historical period prior to the invention of the printing press. Like other manuscripts, the *Trattato* was intended as a set of notes for teaching and was written in vernacular language (Italian). Thus, the equation that we would write as follows:

$$\frac{4000}{x+6000} - \frac{3000}{x+5000} = \frac{1}{15}$$

Mazzinghi writes as “when subtracting $\frac{3000}{1co\ 5000}$ from $\frac{4000}{1co\ 6000}$ it remains $\frac{1}{15}$” (Mazzinghi, 1967, p. 33).

It is interesting to note that the calculations made by Mazzinghi in order to find the *thing* (in Italian *cosa*, that he abbreviated as “*co*”) are based on an arithmetic rule of fractions: Mazzinghi says: “Multiply in cross", etc. The calculation with algebraic fractions can be accomplished once the original meanings of the algebraic expressions have been put back.

The recourse to a complex spatially-based fraction-sign such as the one used by Mazzinghi stresses a visual aspect that cannot be found in al-Khwārizmī's treatise on algebra. The question, of course, is not to determine whether or not one is "better" than the other. The point is that al-Khwārizmī's symbolism relies on the spectrum of resources of natural language, while in Mazzinghi, visuality is solicited. The difference is that the former fully relies on natural language. In the latter, natural language is not dismissed completely, as the linguistic description of actions (e.g. "subtracting") shows. But the human voice starts fading away... With this move, a new kind of conception of symbols and a new form of production of meaning start to become elaborated. The emergence of complex technical symbols removed from the realm of everyday language indeed originates a specialized space of signs that, although always susceptible to be pronounced in natural language, opens a crack or a fissure in the space of mathematical significations. We enter here into a new conceptual space. But how is it possible? We sketch an answer to this question in the next section.

4. Zones of proximal development of the culture

It is our contention that new mathematical ideas are answers worked out in the historically situated *zones of proximal development of their culture*. As with all spaces of action and reflection, such zones embed the historically constituted cognitive activity of previous generations. As zones of knowledge expansion, they point to the future.

Zones of proximal development of the culture are in continuous change, offering different potential answers to practical and theoretical questions. They are dynamic because cultures and their subcultures are in constant motion. This dynamic nature of cultures results not only from changes in the material conditions of life, but also from the fact that thinking, as a sensual-intellectual reflective sign-mediated social praxis, is intrinsically *interpretative* and *imaginative*. Since signs cannot exhaust their objects (Castoriadis, 1975), since there is no full adequacy between sign and object, there is always room for seeing a sign and what it represents in a different way. In other words, the relationship between sign and object is a cultural construct that the continuous effect of interpretation and imagination makes always open to change and transformation, i.e. to new ideas that eventually may become socially accepted.

Let us return to the new conceptual space in which Mazzinghi moves when he deals with complex fractional equations. This conceptual space is still barely defined, floating fuzzily in the Zone of Proximal Development of late Medieval culture. At this point in history, cognition, while located in a situated present, simultaneously points to the past and to the future. It draws from the deposited cognitive activity of previous generations –as implied by the *Embedment Principle*– and, at the same time, creates new cultural forms of signification. It is a space where perception and silence begin to reign supreme and that, historically speaking, will receive a more definite shape as social practices move forward towards a systematization of human actions and the invention of the printing press (Radford, 2004). This will be the reign of pure symbolism that, at this point in human cultural history, starts rising like a star in the sky.

5. Fractional equations: an example from a grade 8 class

The objectification of the deposited intelligence in social practices and the artifacts and semiotic systems that mediate them requires, as mentioned previously, the students' active participation. The results of historical developments are not simply *absorbed* by individuals. Sociocultural theorists have always opposed the idea of learning as a mere absorption of knowledge and the traditional methods of teaching by transmission. Leont'ev –one of Vygotsky's collaborators- tenaciously insisted that the individual's conceptual development (personality, forms of thinking, consciousness, and so on) is also *a product of his/her own activity*:

> In this activity, mediated by contact with other people, is realized the process of the individual's acquisition (*Aneignung*) of the spiritual riches accumulated by the human race (*Menschengattung*) and embodied in an objective, sensible form. (Leont'ev, 1978, p. 19)

However, we are not implying that, for a contemporary student, to learn algebra (or any other subject for that matter) means to *construct* (in a Piagetian sense) the concepts of unknown, equation, variable, etc. As intimated by our discussion of Mesopotamian and Islamic mathematics, contemporary algebra, with all its concepts, is the product of a lengthy historical-cultural process that the students *encounter* in the highly complex social institution that we call the school. To learn algebra is *not* to construct the objects of knowledge (for they have already been constructed) but to *make sense* of them (Radford, 2005).[12]

Bearing these remarks in mind, let us now turn to a Grade 8 classroom teaching sequence and see how contemporary students make sense of the modern signs and meanings of algebra.[13]

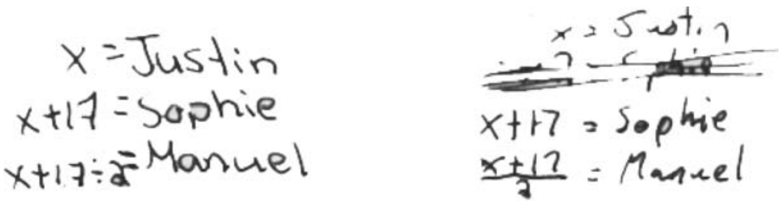

Figure 1. To the left, Judy's answer to Question 1. To the right, Rick's answer.

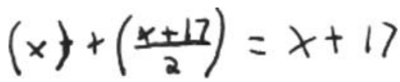

Figure 2. Rick's equation.

The problem that we will discuss (which was the third of a math regular lesson) is the following:

Sophie has 17$ more than Justin. Sophie has twice more than Manuel. Manuel and Justin together have the same amount as Sophie.

Question 1: Let **x** be the number of dollars that Justin has. Find an algebraic expression for the number of dollars that Sophie has and an algebraic expression for the number of dollars that Manuel has.
Question 2: Write an equation for this problem.
Question 3: Solve the equation and clearly explain your steps.

5.1. *From the story-problem to the equation*

The general purpose of teaching algebra was to provide the students with a sophisticated way of thinking mediated by words and alphanumeric signs. Here, we discuss the work done by a group of two students, Judy and Rick. Their answer to Question 1 is shown in Figure 1.

As we can see, the comparative linguistic expressions $C_1(A, B, Z) =$ "A has Z more than B…", and $C_2(A, B, U) =$ "A has U times more than B" were transformed into alphanumeric expressions. However, there seems to be an important difference in the students' understanding of signs. As suggested by the expression corresponding to Manuel's amount of candies, the syntax in Judy's case is based on a procedural understanding (Sfard, 1991) of signs and operations. Rick, in contrast, made recourse to a different sign to express the comparative multiplicative expression $C_2(A, B, U)$. He used a rational expression.

In answering Question 2, Rick produced the equation shown in Figure 2.

One of the difficulties in translating a word-problem into an equation is that it demands that one contract the original story-problem.[14] Although the equation still "narrates" the original story, it does so in a considerably different way. Not only have the *comparative* expressions $C_1(A, B, Z)$ and $C_2(A, B, U)$ been transformed into *calculation* formulas, they have also been reorganized in a *visual* manner.

5.2. *Transforming the equation*

To go further, the story-problem has to be "transformed". This transformation amounts to a *collapse* of the original meanings. In (Radford, 2002b) it was shown how some students, when they have to start transforming the equation, are reluctant to get rid of the brackets. Removing the brackets, (like e.g. in "x + (x + 1)" to obtain "x + x + 1"), leads to a situation where one can no longer identify with certainty the meaning of the terms of the translated story. Thus, to transform the equation in order to simplify it, signs have to change the manner in which they stand for the represented objects.[15] As mentioned in Section 1, this was not the case in the Mesopotamian geometric procedures that infused the transformation of signs with meaning. In those cases, the signs kept their original meaning (or eventually borrowed a geometrical one; see e.g. the oil problem in Høyrup, 2002, pp. 206–07). In the case that we are discussing here, operations and signs must undergo changes in their way of signifying. The students, it may be said, have to position themselves in the space of signification that Mazzinghi started opening in the late Middle Ages.

So, with pen in hand, Rick watched the formula intensively for 20 seconds, thinking about how to simplify it until the teacher came to see the students' work:

1. Teacher: What's wrong with the problem?
2. Rick: Look, well, if we go... I understand... if it's x plus 17 divided by 2, what do you do next ... ? Do we go minus 17?
3. Teacher: Hum ... hum... ?
4. Rick: So you do zero divided by 2?

Rick wanted to subtract 17 from the rational expression ($\frac{x+17}{2}$), and thought that such a subtraction would render the numerator equal to zero. As the next excerpt implies, the teacher called the students' attention to the fact that the whole expression "x + 17" is divided by 2 and suggested another interpretation for the rational expression. However, Rick was not very enthusiastic about pursuing this line of thought for he foresaw the encounter with negative numbers:

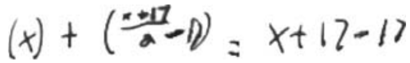

Figure 3. In an attempt to simplify the equation, Rick subtracts 17 from both sides of the equation.

5. Teacher: It's x plus 17 divided by 2, right? well, it's the same thing as saying x divided by 2 plus 17 divided by 2.
6. Rick: But you have to do minus 17 .... so (*looking puzzled at the teacher*) it [$\frac{17}{2} - 17$] will be like a negative?

At this point, the teacher was called to assist another group of students. She encouraged the students to consider making recourse to fractional expressions and left. Rick and Judy continued their work. Judy adopted Rick's equation and wrote it on her activity sheet. Following the idea of subtracting 17, Rick performed the operations as indicated in Figure 3.

In the next line, the indicated numerical operations were carried out (see Figure 4).

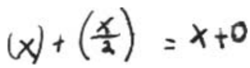

Figure 4. Rick carries out the subtractions.

Immediately after that, Rick multiplied by 2 the expression ($\frac{x}{2}$) on the left, as well as 0 on the right side of the equation (top line of Figure 5) and obtained a new equation (bottom line of Figure 5).

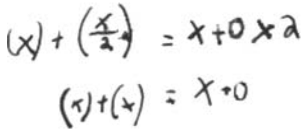

Figure 5. Rick indicates the multiplication by 2 and carries out the calculations.

Then he continued: "So ...x plus x equals x ... [plus] zero, then (see Figure 6) minus x ... minus x" and obtained the answer.

Although the students' understanding of the syntax and meaning of algebraic symbolism has not been fully achieved, Rick's attempts suggest the crucial role played therein by interpretation and imagination. As we shall see, there is a tension between two kinds of meanings –a verbal and a perceptual one.

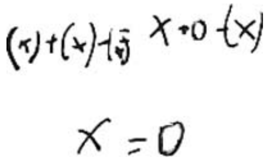

Figure 6. Rick's solution of the equation.

5.3. *The difference between writing and solving an equation: Verbal versus perceptual meaning*

We can say that up to the writing of the equation, the signs had a *verbal meaning*, i.e. a meaning based on the linguistic significations expressed by the story-problem. But what kind of meaning did they have afterwards? To provide some clues to this question, let us notice that, throughout Rick's equation solving process, "x" still represents Manuel's amount of candies. The sign "x" is still the representation of an unknown. But the *manner* in which it is representing the unknown has changed. The original signification has been *suspended* and the letter x is now a *mark*. During the transformation of the equation, the main "personages" of the formal algebraic text are no longer Manuel, Justin and Sophie, but the *operations*. Not exactly the original ones, however, for it is not the initial *verbal meaning* of operations that has primacy in the simplification of the equation. It is a more generalized one, a *perceptual meaning*, i.e. one that attends to the *shape* of the symbolic expressions.

An algebraic equation is in fact like a *diagram* in that an equation exhibits, through algebraic signs, the *relations* between the involved quantities.[16]

To simplify an equation in order to solve it requires, hence, a particular kind of reasoning –a spatial-sensual one.[17] It requires *diagrammatic thinking*.[18] In perceiving the signs as parts of a diagram or "a skeleton-like sketch" stressing "relations between its parts" (Stjernfelt, 2000, p. 363), a definite shift of attention occurs: attention moves from the verbal meaning of signs to the *shapes* of the expressions that they constitute. This shift leads one to see the equation as an iconic, *spatial object.*

5.4. *The equation as a spatial object*

After the math lesson, during the usual debriefing with the teacher and following the examination of the students' written sheets, we decided that, the next day, the students would be allowed some time to complete the activity and then the teacher would have a general discussion with the class

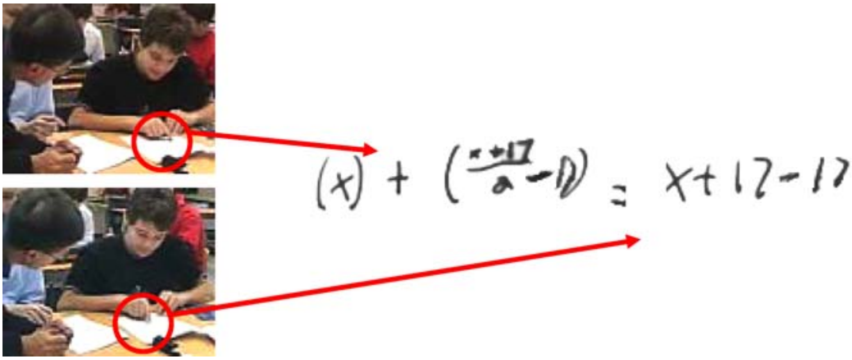

Figure 7. Left, Rick explaining to L.R. some of the transformations of the original equation (see Line 8). Right, the expressions pointed to by Rick's indexical gestures. Judy (on Rick's right side) cannot be seen in the picture.

on the different solutions. It was decided that one of us (L.R.) would discuss their equations and problem-solving procedures with Rick and Judy:

7. LR: Could you please explain to me how you solved this problem?
8. Rick: Here … (*he points to the left side of the equation; see Figure 7, top picture*) plus x (*he refers to the sign " x" in "(x + 17)/2"*), well to make it equal to x we subtract 17 by 17, which only gives x, so (*he indicates the right side of the equation*) [we] have to do the same thing on this side (*Figure 7; bottom picture*) 17 minus 17, so (*still indicating the right side of the equation*) x plus 0, then you multiply by 2 to get x only… […]
9. LR: Wait, wait, wait, so here (*he points to the rational expression on the left side of the equation*) you have 17, right? […] you took away 17, you say. Where did you take 17 away from?
10. Rick : Umf! here, you take away 17 so that it will only be x divided by 2 […] you take away … 17 from Manuel, so [you] have to take away 17 from the total (*that is to say, from the right side of the equation*)
11. LR: but, the 2 that divides there, at x plus 17, does it also divide at 17?
12. Rick: yes… or maybe… (inaudible) […]
13. LR: Maybe you have to divide beforehand because there, the 17 is divided by 2, so, if you divide 17 by 2, in fact, you don't have 17, you have less than 17… right?

As we can see, in Line 8, Rick talks about subtracting 17. There is no mention of the qualities of the objects. The sign 17 passed from signifying an adjectival number of candies to a general object.[19] The equation is seen as a kind of *situated spatial object* having two sides. The perception of shapes becomes interdependent with the actions that are carried out. Rick has a general sense of the transformations: equal actions have to be performed on both sides of the equation. What escapes Rick's attention is that operations have to be consistent not only with the meaning of the equation but also with the *internal* meaning of the compounded signs (here $\frac{x+17}{2}$). Our efforts to convince Rick that the rational expression could also be seen as the sum of two fractions did not pay off. We see that the equation as a diagram is a complex cultural artifact. It possesses various layers of meaning whose coordination was a source of difficulty for Rick and other students as well.

After the previous discussion, the students were invited to make the corrections (Judy's participation was not only limited to copying in her paper what Rick said or did. She continuously asked Rick for explanations and, sometimes, she suggested some ideas). The students produced a second solution that, just like the previous one, proved to be unsatisfactory. The students solved the equation again, starting from the beginning on a new sheet (see Figure 8).

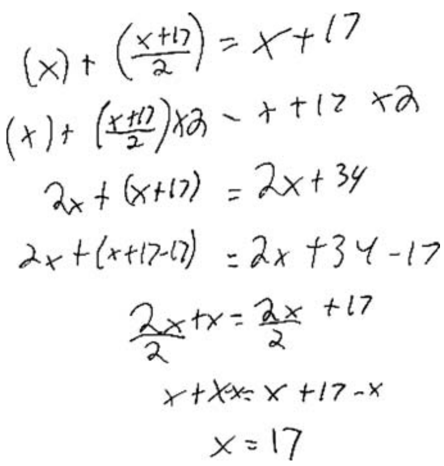

Figure 8. Judith and Rick's third solution.

The final text suggests that although there was a refinement in the students' solving process, the understanding of the syntax and meaning of algebraic symbolism was not secured yet. The division by two towards the end of the problem-solving procedure shows a partial understanding of the meaning of algebraic transformations. The general discussion, conducted right after by the teacher, helped the students to fill some gaps. Because of space constraints, we will not go into the details of the classroom discussion and the opportunities that the teacher created for the students to continue their mathematics objectifying processes. By way of conclusion, in the next section, we will instead provide a synthesis of our theoretical approach and stress the way in which the history of mathematics informs it.

6. Synthesis and concluding remarks

One of the central ideas that framed our research question is that social practices and the semiotic systems and artifacts that mediate them are bearers of a cultural-historical cognitive activity. By becoming reflectively engaged in them, i.e., by becoming deeply involved in creative and imaginative processes of objectification and sense-making, sign- and artifact-mediated practices offer the students a malleable array of vectors of cognitive growth.

The array of vectors of cognitive growth should not be understood as something that culture imposes upon its individuals. Culture is not a strait-jacket. As Baxandall suggests,

> Cultures do not impose uniform cognitive and reflective equipment on individuals [. . .] Living in a culture, growing up and learning to survive in it, involves us in a special perceptual training. It endows us with habits and skills of discrimination that affect the way we deal with the new data that sensation offers the mind. (Baxandall, 1985, p. 107)

Of course, to move successfully in the direction intimated by the vectors of cognitive growth offered by culture in general and the school in particular, some challenges have to be overcome. As we have repeatedly said in the course of this article, the objectification of the deposited intelligence in the mediated social practices is not a matter of passive reception. There would not be a better way of misunderstanding the *Embedment Principle* stated in this paper than by seeing it as the conveyor of a model of passive transmission of knowledge.[20]

As suggested by our analyses, solving equations through algebraic symbolism rests on a mode of diagrammatic thinking. The historical roots of such a kind of thinking were put into evidence in Sections 3 and 4. They help us realize that the advent of algebraic symbolism fits perfectly into the extremely visual culture of the Renaissance (Radford, in press). This is why –all proportions kept– it may be said that a diagram like the one shown in Figure 8 is much like a Renaissance painting.

However, unveiling the historical roots of diagrammatic thinking has not to be seen as a matter of academic curiosity or pedantic interest. For us, it is part of an essential endeavor to help us understand the historical intelligence deposited in our contemporary practices of school algebra and to better appreciate the students' attempts to make sense of it. For one thing, as a form of diagrammatic thinking, symbolic algebraic thinking requires the cognitive ability to switch between verbal and perceptual meanings and to become conscious that the latter is governed by the shape of expressions whose syntactic complexity may lead to multilayered perceptual meanings (like in the expression $(\frac{x+17}{2})$). The short classroom episodes presented in this paper illustrate some of the difficulties encountered by students when trying to make sense of these matters. Certainly, from an ontogenetic viewpoint, the interpretation or production of an algebraic text is encompassed by previous experiences with other kinds of texts. Arithmetic texts are prominent here, as Judy's efforts intimate (see Figure 1). To produce an algebraic text requires the students' creation of experiential associations with the cultural ways in which (simple and compounded) signs in algebra stand for the objects that they represent and with the operations carried out on those signs. What students interpret are their associations with the text rather than the text itself (Smagorinsky, 2001). Brackets are very revealing in this respect. Brackets (see e.g. Figure 8) indicate the parts of the text on which the students' attention is being put. For the students, at this point in their cognitive development, brackets are not merely disembodied syntactic signs. On the contrary, they "signal" those parts of the text that are being attended to by the students in their unfolding spatial-temporal mathematical experience. Producing an algebraic text is a sense-making labour. It is a culturally mediated subjective sensual-intellectual endeavour that includes outer and inner speech, the kinaesthetic motion of writing, gesturing, perceiving, etc. (see Figure 7).

While syntax and meaning in poetry are deeply dependent on rhythm and phonetic match, syntax and meaning in algebra are deeply dependent on visualization. Syntax and meaning in algebra, as well as in poetry, are historically constituted sensuous features of thought. In each case, they empower our students to reflect, interpret, create and imagine in a certain way. The difference is that syntax and meaning in algebra stress aspects of human reflection where the focus is on quantities. The difference between syntax and meaning in poetry and algebra does not prevent that, in one case as in the other, their objectification arises in the contact between ontogenesis and phylogenesis, in this space where subjectivity encounters culture.

ACKNOWLEDGMENT

This article is a result of a research program funded by the Social Sciences and Humanities Research Council of Canada (SSHRC/CRSH).

NOTES

1. Quoted from Thomas' (1939, p. 519) translation of Tannery's (1893–95) Greek editon of Diophantus' *Arithmetica*.
2. See e.g. Bednarz, Kieran and Lee, 1996; Wagner and Kieran, 1989; Kaput and Sims-Knight, 1983, Filloy and Rojano, 1989; Sutherland *et al.*, 2001.
3. This point was recognized by Vieta as one of the chief characteristics of algebra. For Vieta, algebra was indeed an *analytic art*.
4. This conception of algebra was promoted by mathematicians and philosophers of the 18th Century, like Newton and Kant. Thus, for Kant, the essential difference between algebra and geometry is their recourse to two different types of object construction. While geometry uses ostensive representations (e.g. a triangle is represented as a three-side figure), algebra can only represent its objects *symbolically*. In the *Critique of Pure Reason*, Kant says: "Even the method of algebra with its equations [...] is not indeed geometrical in nature [...]." (Kant, 1781, 1787/1929; A734/B762, p. 590).
5. Furinghetti and Radford, 2002; Radford, 2000a. We shall return to this point in the next section.
6. See Boncompagni, 1857, p. 24; Sigler, 2002, pp. 49–50.
7. For a recent edition of the 9th Century AD *The concise book of the calculation of al-jabr and al-muqābala* and *The Indian Calculation*, see al-Khwārizmī, 1939, 1992, respectively.
8. In the next section we will see that a same conceptualizing strategy appears in one of our Grade 8 students.
9. According to Rashed, *shay'* is a Koranic term used also in the language of philosophy. In this context it means "all what can be imagined without nonetheless been realized in an object" (Rashed, 1984; pp. 120–123); consequently, *shay'* has a kind of empty character, capable of acquiring any content.
10. See an analysis of al-Khwārizmī's algebraic language in Puig (2004) and Puig and Rojano (2004).
11. It is interesting to note that al-Khwārizmī does not say "thing and two" but qualifies the number "two" as "two dirhams" the designation for a monetary unit in the Muslim world at the time, something that reveals the practical and situated origin of algebra, much in the same vein as Mesopotamian scribes, who employed the words 'length' and 'width' as we indicated it in Section 1.
12. Naturally, to make sense of something does not presuppose a mere conformity to what is given. Indeed, because of their reflexive, interpretative and imaginative nature, processes of meaning-making also mean transformation, a going-beyond, an outdoing of what is given. The subjective dimension of meaning-making, as something accomplished by historically-situated and unique individuals, makes possible the overcoming of the actual and the expansion and modification of knowledge and culture.
13. The data comes from a 6-year longitudinal classroom based research program and was collected during classroom activities that are a part of the regular school mathematics program. In these activities, the students work together in small groups. At some points, the teacher conducts general discussions allowing the students to expose, confront and discuss their different solutions. Details of the methodology can be found in Radford, 2000b. It is important to bear in mind that our students were introduced to symbolic algebra in Grade 7, and solved word-problems that led them to tackle equations like $\frac{x}{2} + (\frac{x}{2} + 5) + x = 145$.

14. Thus, the original story is now expressed through substantially fewer signs.

15. This transformation is underpinned by a process of abstraction that goes from the reference made to the concrete quantities in the story-problem to the representation of previous and new relationships and equivalences generated at the level of symbolic expression. We shall come back to this point later.

16. As Peirce suggested "every algebraical equation is an icon" (CP. 2.282), and he conceived of diagrams as kinds of icons "A *diagram* is an *icon* or schematic image" (Peirce, quoted in Stjernfelt, 2000, p. 368). See also CP 2.255, 2.258, 2.279, 4.233.

17. See also Kirshner and Awtry, 2004.

18. For Peirce, the diagram represents, in an intuitive way, the relations that are abstractly expressed in the problem. A visualization of these relations leads one to ponder certain ideas or hypotheses. "In order to test this, various experiments are made upon the diagram, which is changed in various ways". As a result of this experimentation, "the conclusion is compelled to be true by the conditions of the construction of the diagram." This, Peirce said, is called "diagrammatic, or schematic, reasoning." (CP 2.778)

19. The original context is mentioned in Line 10, when reference is made to one of the personages of the original story –Manuel– but this is only done briefly.

20. For instance, we saw how Rick was not very enthusiastic about pursuing some ideas that we suggested, for he foresaw the encounter with negative numbers; we also saw how vain our efforts were to convince Rick that the rational expression could also be seen as the sum of two fractions. The adults cannot transmit to the students their own concepts, simply because concept attainment is a dynamic experiential process. "Concepts", wrote Vygotsky, "cannot be assimilated by the child in a ready-made form, but have to undergo a certain development." (Vygotsky, 1986, p. 146).

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