

Loughborough University
Department of Mathematical Sciences
MATHEMATICAL CHALLENGE
CHRISTMAS - 2021

Problem 1. Find all the pairs (a, b) such that each of the quadratic equations $x^2 + ax + b = 0$ and $x^2 + bx + a = 0$ have two integer roots.

Problem 2. Let $f(x) = x^2 + c$ be a quadratic function with real c and consider its n -th iteration: $f^{(n)}(x) := f(f(\dots(f(x))\dots))$, $n \in \mathbb{N}$.

Show that for $c < 0$ the equation $f^{(n)}(x) = 0$ has real roots for all n and find the limit of the largest of them when $n \rightarrow \infty$.

Can the equation $f^{(n)}(x) = 0$ have exactly two distinct real roots for all n ?

Can it have exactly 2^n distinct real roots for all n ? Justify your answer.

Problem 3. A mysterious flu hit the hobbits in the Shire. The disease is severe but lasts only for one day, after which the hobbit is immune against it, although only for the next day. A vaccine exists but it is also effective only for one day. The disease is spread through the visits of the hobbits to their sick friends, which the healthy hobbits always do. Each hobbit has a number of friends, but friendship is not necessarily transitive (a friend of a friend need not be a friend). After visiting, the healthy hobbits become sick the next day unless they are immune.

Prove that if there is no vaccination, then the flu will end itself, and provide an upper bound on its duration if the population of hobbits is 1000.

If, however, there were vaccinated hobbits with immunity on the very first day of the flu, then show that in certain cases the epidemic could continue forever. Justify your answer.

Remarks.

1. There will be a first prize of £50 to the person handing in what will be considered to be the best effort to these problems. There may also be special prizes for the most original solutions.
2. Any student registered on one of the undergraduate programmes in the Department of Mathematical Sciences may submit solutions to any or all of these problems.
3. Solutions should be scanned and e-mailed on or before January 31, 2022 to either Prof. A.P. Veselov (a.p.veselov@lboro.ac.uk) or Dr. B. Winn (b.winn@lboro.ac.uk), who will be the judges for the Challenge.