Integrable systems and applications

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Abstracts

Deniz Bilman, University of Cincinnati, US
On Universal Wave Patterns in Rogue Wave Formation

It is known from our recent work that both fundamental rogue wave solutions (with Peter Miller and Liming Ling) and multi-pole soliton solutions (with Robert Buckingham) of the nonlinear Schrödinger (NLS) equation exhibit the same universal asymptotic behavior in the limit of large order in a shrinking region near the peak amplitude point, despite the quite different boundary conditions these solutions satisfy at infinity. We review these results and show that this profile arises universally from arbitrary background fields. We then show how rogue waves and solitons of arbitrary orders can be placed within a common analytical framework in which the ”order” becomes a continuous parameter, allowing one to tune continuously between types of solutions satisfying different boundary conditions. In this scheme, solitons and rogue waves of increasing integer orders alternate as the continuous order parameter increases. We show that in a bounded region of the space-time of size proportional to the order, these solutions all appear to be the same when the order is large. However, in the unbounded complementary region one sees qualitatively different asymptotic behavior along different sequences. This is joint work with Peter Miller.

Gino Biondini, University at Buffalo, US
On one-dimensional reductions and integrability of the Whitham equations for the Kadomtsev-Petviashvili equation

In the last few years, the Whitham modulation equations for the Kadomtsev-Petviashvili (KP) equation were derived and successfully used to characterize various initial value problems for the KP equation. In this talk I will present some recent results about various one-dimensional reductions of the KP-Whitham system and their integrability properties.

Jerry Bona, University of Illinois Chicago, US
KdV in a quarter plane: asymptotic periodicity and mass transport

Laboratory studies have suggested two properties of solutions of an initial-boundary value problem that arose in attempting to model water wave experiments in a channel. One has to do with asymptotic periodicity of solutions while the other is concerned with mass transport. Various efforts using classical methods have yielded results indicating that at least part of the conjectures provided by study of experiments are valid, at least for certain model equations.

Using the more subtle tools of inverse-scattering theory as it applies to initial-boundary-value probems via Fokas’ methodoloy, Jonatan Lenells and I initiated a detailed study of these issues in the context of the Korteweg-de Vries model. The lecture will report on the outcome of that study.

Sonia Boscolo, Aston University, UK
Control of complex nonlinear wave dynamics in dissipative systems by machine learning

Ultrafast mode-locked fibre lasers exploiting nonlinearity in the pulse formation process are well-known to display a rich landscape of “dissipative soliton” dynamics, which results from the interplay of the nonlinearity with dispersion and dissipation. Reaching a desired operating regime in a fibre laser generally depends on precisely adjusting multiple parameters in a high-dimensional space, in connection with the wide range of accessible pulse dynamics, which is usually performed through a trial-and-error experimental procedure, due to the lack of analytic relationship between the cavity parameters and the pulse features. The practical difficulties associated with such a procedure can be circumvented by machine-learning strategies and the use of evolutionary and genetic algorithms (GAs) [1], which are well-suited to the global optimisation problem of complex functions. In this talk, we will provide a snapshot of our recent progress in the control of non-stationary nonlinear dynamics in fibre lasers by using GAs.

Breathing solitons exhibiting periodic oscillatory behaviour form an important part of many different classes of nonlinear wave systems. Recently, thanks to the development of real-time detection techniques, they have also emerged as a ubiquitous mode-locked regime of ultrafast fibre lasers [2, 3]. The excitation of breather oscillations in a laser naturally triggers a second characteristic frequency in the system, which therefore shows competition between the cavity repetition frequency and the breathing frequency. Nonlinear systems with two competing frequencies show frequency locking, in which the system locks into a resonant periodic response featuring a rational frequency ratio, and quasi-periodicity following the hierarchy of the Farey tree and the structure of the devil's staircase [4]. Whilst frequency-locking phenomena have been extensively studied theoretically and experimentally in many physical systems, all the investigations so far relate to systems where an external, accurately controllable modulation adds a new characteristic frequency to the system. In [5], we introduced an approach based on a GA for the generation of breather dynamics in fibre lasers with controlled characteristics, which relies on specific features of the radio-frequency spectrum of the breather laser output to optimise the intra-cavity nonlinear transfer function through computer-controlled polarisation control. In this talk, benefiting from this approach and further developing it to directly pinpoint frequency-locked breathers, we demonstrate that a breather mode-locked fibre laser is a passive system showing frequency locking at Farey fractions [6]. The frequency-locked states occur in the sequence they appear in the Farey tree and within a pump-power interval given by the width of the corresponding step in the devil's staircase. The breather laser may therefore serve as a simple model system to explore universal synchronisation dynamics of nonlinear systems.

First introduced in the context of oceanic waves, the concept of extreme events or rogue waves (RWs), i.e., statistically-rare giant-amplitude waves, has been transferred to other natural environments such as the atmosphere, as well as to the solid grounds of research laboratories [7]. As RWs appear from nowhere and disappear without a trace, their emergence is unpredictable and non-repetitive, which make them particularly challenging to control. Here, we extend the use of GAs to the active control of extreme events in a fibre laser cavity [8]. Feeding real-time spectral measurements into an GA controlling the electronics to optimise the cavity parameters, we are able to trigger wave events in the cavity that have the typical statistics of RWs in the frequency domain and on-demand intensity. This accurate control enables the generation of the strongest optical RWs observed so far with a spectral peak 32.8 times higher than the significant intensity threshold. The extreme spectral events observed correlate with extreme variations of the pulse energy, thus qualifying as temporal RWs as well. Importantly, significant frequency up- or down-shifting of the optical spectrum is also associated with the emergence of these waves, which suggests a new physical scenario for RW formation. Given the generality of our control strategy, which relies on the statistical defining characteristics of RWs independent of the particular physical model, it is reasonable to expect the machine-learning method used in this work to be applicable to the control of RWs in many different systems.

Based on joint work with Xiuqi Wu, Yu Zhang, Junsong Peng, Christophe Finot, and Heping Zeng.

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Roberto Camassa, University of North Carolina, US

Some issues in the mathematical modeling of internal wave propagation

While modeling internal wave dynamics in stratified fluids have met with some success and advances in recent as well as more distant past, many fundamental questions remain, at both mathematical and physical levels. This talk will illustrate some of these questions and the challenges they present with a mix of theory, numerics and experiments.

Henry Carr, Northumbria University, UK

Riemann problem for a dense soliton gas for the KdV equation: a numerical study

We numerically realise the generalised Riemann problem for dense soliton gas of the Korteweg-de Vries (KdV) equation whereby soliton gas is prepared in different macroscopically homogeneous states on the left- and right-hand side of space. The emerging non-equilibrium dynamics are compared with the analytical predictions of the spectral kinetic theory of soliton gas. The numerical scheme is based on the construction of an appropriate exact N-soliton solution of the KdV equation with N large and random distribution of initial phases.

Marta Dell'Atti, University of Kent, UK and **Pierandrea Vergallo**, University of Salento, Italy

Classification of degenerate non-homogeneous hydrodynamic type operators

We investigate non-homogeneous Hamiltonian operators composed of a first order Dubrovin-Novikov operator and an ultralocal one. The study of such operators turns out to be fundamental for a class of Hamiltonian scalar equations once inverted into quasilinear systems of first order PDEs. Often, the involved operators are degenerate in the first order term. For this reason a complete classification of the operators with degenerate leading coefficient in 2 and 3 components is presented with applications.

Maciej Dunajski, University of Cambridge, UK

Dispersionless integrable systems in null Kähler geometry

The radial solutions to the sinh–Gordon equation and the elliptic Tzitzeica equation can be interpreted as abelian vortices on certain surfaces of revolution. These surfaces have a conical excess angle at infinity (in a way which makes them similar to Elizabethan ruff collars, or certain green algae). They can not be embedded in the Euclidean 3-space but can be globally embedded in the hyperbolic space.

Gennady El, Northumbria University, UK

Generalised Riemann problem for soliton gas

Soliton gas represents an infinite random ensemble of interacting solitons displaying nontrivial collective, hydrodynamic behaviours, ultimately determined by the properties of elementary two-soliton nonlinear interactions. The macroscopic dynamics of soliton gases in integrable systems are described by the nonlinear integro-differential kinetic equation for the density of states in the spectral (Lax) phase space. This equation has been recently shown to possess various families of integrable hydrodynamic reductions. Some of these reductions are linearly degenerate while others are genuinely nonlinear. We will consider Riemann problems for both types of reductions and construct exact solutions describing generalised rarefaction and dispersive shock waves in a soliton gas for the Korteweg-de Vries (KdV) equation. Analytical solutions are compared with direct numerical simulations of the KdV soliton gases.

Sergey Gavrilyuk, Aix-Marseille University, France

Soliton limit for the Whitham modulation equations for the BBM equation

The soliton limit for the Whitham modulation equations for the BBM equation is obtained in explicit form for any wave amplitude by passing to a singular limit in the wave action conservation law. The limit system is written in terms of the Riemann invariants. The problem of interaction of solitary waves with rarefaction waves is analytically solved. A very good agreement with the corresponding numerical solutions to the exact BBM equation is found.
This is a joint work with K.-M. Shyue.

Georgi Grahovski, University of Essex, UK

Real Hamiltonian forms of affine Toda field theories and exceptional Lie algebras

We will present real Hamiltonian forms of 2-dimensional Toda field theories related to exceptional simple Lie algebras, and the spectral theory of the associated Lax operators. Real Hamiltonian forms are a special type of 'reductions' of Hamiltonian systems, similar to real forms of semi-simple Lie algebras. Examples of real Hamiltonian forms of affine Toda field theories related to exceptional complex untwisted affine Kac-Moody algebras will be presented. Along with the associated Lax representations, we will also outline the relevant Riemann-Hilbert problems and the minimal sets of scattering data that determine uniquely the scattering matrices and the potentials of the Lax operators. This is a joint work with Vladimir Gerdjikov and Alexander Stefanov.

Mark Hoefer, University of Colorado, US

Modulated Multiphase Waves in Soliton-Mean Flow Interaction

The Kortweg-de Vries equation subject to step-like initial conditions and a soliton is considered using multiphase Whitham modulation theory. The step-like initial condition results in either a rarefaction wave (RW) or a dispersive shock wave (DSW) that interact with the soliton. Soliton-RW interaction is described using a degenerate limit of the 1-phase (cnoidal wave) modulation equations. Soliton-DSW interaction is described using degenerate limits of the 2-phase modulation equations. The interaction is shown to result in a modulated breather solution whose trajectory is a characteristic of the 2-phase modulation equations. In the case of the transmission of a soliton through the DSW, a modulated bright breather is created during the interaction. For soliton-DSW trapping, a modulated dark breather describes the penetration of a quasi-soliton into the DSW, never to exit. The long-time asymptotics of the initial value problem using inverse scattering theory agrees with the modulation description outside of the soliton-RW/DSW interaction region. Extensions of bright and dark breather solutions to non-integrable equations and corresponding fluid dynamics experiments will also be presented.

Rossen Ivanov, Technological University of Dublin, Ireland
Integrable systems on symmetric spaces

The soliton hierarchies are characterised by a pair of Lax operators (L and M), where L is usually polynomial in the spectral parameter. The equations arising from the negative flows (when M involves negative powers of the spectral parameter) are usually equations in non-evolutionary form. These include important examples such as the Camassa-Holm, Qiao and the Degasperis-Procesi equations.
When L is quadratic (the so-called quadratic pencil or quadratic bundle) there are negative flows as well, the best known example being the Fokas-Lenells (FL) equation. We formulate possible extensions of the FL and other integrable equations related to quadratic pencil on Hermitian symmetric spaces. Our approach originates from the classical papers such as [1,2], see for example [3,4].
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Robert Jenkins, University of Central Florida, US
Connecting different models of soliton gases

Soliton (and breather) gasses are solutions of integrable systems which can be thought of as many particle limits of multi-soliton (or multi-breather) solutions. Different models for analytically modeling these solutions have been proposed. One popular technique models a soliton gas using the 'primitive potential' approach introduced by Dyachenko, Zakharov, and Zakharov (2016). Another powerful method is to treat the gas as the thermodynamic limit of finite gap solutions of the underlying integrable system (KdV, NLS, etc.) as described in El and Tovbis (2020). In this talk we will show that the two methods are equivalent, at least in certain cases. In particular, we will show that given spectral data for the finite gap model, there exist explicit spectral data for the finite soliton model such that in their "many particle" limit each model approaches the same limiting soliton gas. Time permitting we will discuss some simple examples.

Christian Klein, University of Burgundy, France
Numerical study of Davey-Stewartson systems

In this work we will look at the family of Davey-Stewartson equations from two different angles, using advanced numerical tools. The integrable DS II equation possesses in the focusing case solutions that develop a singularity in finite time. We numerically study the long time behaviour and blow-up of solutions to focusing Davey-Stewartson equations for various initial data and propose conjectures describing the blow up mechanisms. For the integrable case we also study the inverse scattering problem. Both the forward and inverse scattering transformation in this case are reduced to a Dirac system which plays the role that Riemann-Hilbert problems play in one dimensional problems. We will present a hybrid scheme where numerical solutions are obtained for low values of the spectral parameter, complemented by a semi-classical result giving the behaviour at high frequencies.

Yuji Kodama, Ohio State University, US
KP solitons and algebraic curves

It is well-known that soliton solutions of the KdV hierarchy are obtained by singular limits of hyper-elliptic curves. However, there is no general results for soliton solutions of the KP hierarchy, KP solitons. In this talk, I will show that some of the KP solitons are related to the singular space curves associated with certain class of numerical semigroups. This is a joint work with Y. Xie.

Boris Konopelchenko, University of Salento, Italy
On the fine structure and hierarchy of gradient catastrophes for the multi-dimensional homogeneous Euler equation

Blow-ups of derivatives and gradient catastrophes for the n-dimensional homogeneous Euler equation are discussed. It is shown that in the case of generic initial data the blow-ups exhibit a fine structure in accordance to the admissible ranks of certain matrix generated by the initial data. Blow-ups form a hierarchy composed by $n+1$ levels with the strongest singularity of derivatives given by $(\epsilon)^{-\frac{n+1}{n+2}}$. It is demonstrated that in the multi-dimensional case there are certain subspaces with bounded superpositions of blow-up derivatives. Particular case of the potential motion is considered too. Main novelties of the multidimensional case are discussed. Hodograph equations are the basic tool of the analysis.

Boris Kruglikov, UiT the Arctic University of Norway, Norway
On geometry of higher dimensional dispersionless integrable equations

Dispersionless systems in 3D and 4D allow to express their integrability via dispersionless Lax pair in terms of geometry on solutions á la twistor theory. This was initiated in my joint work with E.Ferapontov and then extended in my joint work with D.Calderbank. Integrable dispersionless systems in higher dimensions are necessary degenerate. I will discuss restrictions on geometry of the symbol for integrable equations in higher dimensions.

Antonio Moro, Northumbria University, UK
Statistical ensembles, nonlinear differential equations and integrability: from mean field models to hydrodynamic chains

A number of statistical mechanical models, from classical mean field spin models to random matrix ensembles, appear to be remarkably connected to integrable hierarchies of nonlinear differential equations. Such hierarchies provide a natural unifying mathematical framework, where the correlation functions of a statistical ensemble can be calculated as solutions to a suitable integrable hierarchy of nonlinear differential equations. The initial condition on the hierarchy uniquely characterises the model within the class under consideration.
The thermodynamic regime and universality properties are described via a multiscale asymptotic limit of the underlying hierarchy of differential equations in regime or low viscosity or dispersion. We discuss in detail the case of symmetric random matrix ensemble where the thermodynamic limit leads to a new example of an integrable hydrodynamic chain.

Miguel Onorato, University of Turin, Italy

Thermalization vs integrability in the Fermi-Pasta-Ulam-Tsingou chain

The FPUT system is an Hamiltonian system that describes a one dimensional lattice characterized by an harmonic and anharmonic potential, usually cubic or quartic. It has been introduced by Fermi, Pasta, Ulam and Tsingou to study the phenomenon of thermalization and, more specifically, to understand the role of nonlinearity in approaching the equilibrium. Their numerical simulations show very little tendency toward equipartition. Since then, a lot of papers have been written, trying to explain the unexpected numerical result observed. In this talk, after a quick review of the literature, I will approach the problem using the Wave Turbulence theory and discuss a number of numerical results, supported by theoretical predictions, concerning the large time behaviour of the system, for finite number of masses and in the large box limit.

Alfred R. Osborne, Nonlinear Waves Research Corporation, Alexandria, VA, U. S. A.

Two Fundamental Properties of Nonlinear Integrable Wave Equations with Hamiltonian Perturbations

I discuss two fundamental properties of *integrable wave equations with Hamilton perturbations*:

(1) The solutions of these nonlinear wave equations can be expressed as *quasiperiodic Fourier series* with *finite gap loop integrals* implicit in their formulation [4, 5]. This means that nonlinear integrable equations have general spectral solutions which are a *linear superposition of sine waves*. The results require a theorem by Baker [1] and Mumford [3], and another theorem by Kuksin [2]. First consider the nonlinear integrable wave equations in the Table below.

The Baker-Mumford Theorem states: The most general, single valued, multiperiodic, meromorphic functions are formed from theta functions. Examples are given in the right-hand column of the Table below. Consequently, the expressions relating theta to the solutions of the nonlinear equations (right hand column of Table) generally have a quasiperiodic Fourier series solution:

$$u(x,t) = \sum_{\mathbf{n}\in\mathbf{Z}^N} u_{\mathbf{n}} e^{i\mathbf{n}\cdot\mathbf{k}\,x - i\mathbf{n}\cdot\boldsymbol{\omega}\,t + i\mathbf{n}\cdot\boldsymbol{\phi}} \tag{1}$$

where $u_{\mathbf{n}}$ are coefficients related to the Riemann spectrum of finite gap theory, $\mathbf{n}$ is an integer vector of length N (the genus), $\mathbf{k}$ is a vector of wavenumbers, $\boldsymbol{\omega}$ is a vector of frequencies and $\boldsymbol{\phi}$ is a vector of phases. The "dot" notation indicts a scalar product. Therefore, the *solutions of nonlinear equations are linear superpositions of sine waves*, a surprising result. This outcome contrasts to the often expressed opinion that nonlinear pdes cannot be so generally written. Given a Hamiltonian perturbation of an integrable equation, the solution (1) remains on tori, with a small perturbation of the dispersion relation! This is a marvelous result of Kuksin [2].

(2) The *probability structure of nonlinear equations* and their *Hamiltonian perturbations* can be analytically computed from ideas implicit in the Baker-Mumford theorem. This means that one can always find the probability structure of nonlinear integrable equations. One first determines the probability density of the theta function $\theta(x,t)$ (a work-able problem) and then uses the expressions in the right-hand column of the above Table to determine the probability density of the solutions of the associated wave equations $u(x,t)$. *This establishes a one-to-oneness between the solutions of nonlinear integrable equations* $u(x,t)$ *(right hand column of Table) and their probability structure* $p(u)$ [6].

Name of Equation	The Equation	Solution in terms of theta functions
Korteweg-deVries	$u_t + 6uu_x + u_{xxx} = 0$	$u(x,t) = 2\partial_{xx} \ln \theta(x,t)$
Kadomtsev-Petviashvili	$(u_t + 6uu_x + u_{xxx})_x + u_{yy} = 0$	$u(x,y,t) = 2\partial_{xx} \ln \theta(x,y,t)$
Nonlinear Schroedinger	$iu_t + u_{xx} + 2\|\psi\|^2\psi = 0$	$u(x,t) = \theta(x,t\,\|\tau,\phi^-)/\theta(x,t\,\|\tau,\phi^+)$
Sine-Gordon	$u_{xx} - u_{tt} = \sin u$	$u(x,t) = 2i \ln[\theta^*(x,t)/\theta(x,t)]$
Gardner Eq.	$u_t + 6uu_x + u_{xxx} + u^2 u_x = 0$	$u(x,t) = 2\partial_x \ln[\theta(x,t\,\|\mathbf{B},\phi^-)/\ (x,t\,\|\mathbf{B},\phi^+)]$

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Lev Ostrovsky, University of Colorado Boulder, US

Joint Effects of Rotation and Topography on Internal Solitary Waves

We present the results of the recent study of dynamics of nonlinear oceanic solitary waves under the influence of the combined effects of nonlinearity, Earth's rotation, and depth inhomogeneity. Our consideration is based on the extended model of the Korteweg-de Vries (KdV) equation that in general accounts for the quadratic and cubic nonlinearity (the Gardner equation) with the additional terms incorporating the effects of rotation and slowly varying depth. After a brief historical outline, using the asymptotic (adiabatic) theory, we describe a complex interplay between these factors. As an application, the case of a two-layer fluid with the variable-depth lower layer is considered using the approximate theory, as well as through numerical solutions of the governing equation that includes all the above factors under realistic oceanic conditions. In particular, different scenarios of the soliton propagating toward the 'internal beach' (e.g., zero lower-layer depth) are studied in which the terminal damping can be caused by radiation or disappearing quadratic nonlinearity (when the layers' depths become equal). We also consider interaction of a soliton with a long wave providing the energy 'pump' compensating the radiation losses due to rotation so that the soliton can exist infinitely. The limitations of the adiabatic approach due to the radiation and other factors are also demonstrated.

Mahendra Panthee, University of Campinas, Brasil

Global analytic solution to the generalised Benjamin equation

In this talk we consider the Cauchy problem for the generalised Benjamin equation

$$\begin{cases} \partial_t u - l\mathcal{H}\partial_x^2 u - \partial_x^3 u + \partial_x(u^p) = 0, \quad x \in \mathbb{R},\ t \geq 0,\ p > 1 \\ u(x,0) = u_0(x), \end{cases} \tag{2}$$

where $u = u(x,t)$ is a real valued function, $0 < l < 1$ and $\mathcal{H}$ is the Hilbert transform. This model was introduced by Benjamin (J. Fluid Mech. 245 1992) and describes unidirectional propagation of long waves in a two-fluid system where the lower fluid with greater density is infinitely deep and the interface is subject to capillarity.

We prove that the local solution to the Cauchy problem (2) for given data in the spaces of functions analytic on a strip around the real axis continue to be analytic without shrinking the width of the strip in time. We also study the evolution in time of the radius of spatial analyticity and show that it decreases as the time advances. Finally, we present an algebraic lower bound on the possible rate of decrease in time of the uniform radius of spatial analyticity.

This is a joint work with Renata O. Figueira.

Emilian Parau, University of East Anglia, UK

A dissipative Nonlinear Schrödinger model for wave propagation in the marginal ice zone

Sea ice attenuates waves propagating from the open ocean. Here we model the evolution of energetic unidirectional random waves in the marginal ice zone with a nonlinear Schrödinger equation, with a frequency dependent dissipative term consistent with current model paradigms and recent field observations.

Barbara Prinari, University at Buffalo, US

Inverse Scattering Transform, solitons and soliton interactions for the complex coupled short-pulse equation

We present the inverse scattering transform (IST) for the complex coupled short pulse equation (ccSPE) on the line. Our work extends to the complex, coupled case the Riemann-Hilbert approach to the IST for the real, scalar short-pulse equation proposed by A. Boutet de Monvel and collaborators in 2016. One-soliton solutions are also investigated within the framework of the IST. The simplest soliton solutions, fundamental solitons, are found to be the natural vector generalization of scalar one-soliton solutions of the complex short-pulse equation. But in the coupled case one can also have more complicated, composite soliton solutions, corresponding to two fundamental solitons having the same amplitude and velocity but different carrier frequencies, as well as solutions that, while still corresponding to a minimal set of discrete eigenvalues, cannot be reduced to simple superposition of fundamental solitons. Moreover, it is found that the same constraint on the discrete eigenvalues which leads to regular, smooth one-soliton solutions in the complex SPE, also holds in the coupled case, for both a single fundamental soliton and a single fundamental breather, but not, in general, in the case of a composite breather. If time permits, the interactions of fundamental solitons will be discussed.

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Moahammad Reza Rahmati, Universidad De La Salle Bajio, Mexico and
Gerardo Flores, Centro de Investigaciones en Optica, Mexico
Extended trace formula for vertex operators

We present an extension of the trace of a vertex operator and explain a representation-theoretic interpretation of the trace. Specifically, we consider a twist of the vertex operator with infinitely many Casimir operators and compute its trace as a character formula. To do this, we define the Fock space of infinite level $\mathfrak{F}^{\infty}$. Then, we prove a duality between $\mathfrak{gl}_{\infty}$ and $\mathfrak{a}_{\infty} = \widehat{\mathfrak{gl}}_{\infty}$ of Howe type, which provides a decomposition of $\mathfrak{F}^{\infty}$ into irreducible representations with joint highest weight vector for $\mathfrak{gl}_{\infty}$ and $\mathfrak{a}_{\infty}$. The decomposition of the Fock space $\mathfrak{F}^{\infty}$ into highest weight representations provides a method to calculate and interpret the extended trace.
The Fock spaces provide an alternative to model many particle statistical systems. It plays a crucial role in quantum mechanics, in computation of Feynman integrals. The trace formula calculates the energy or the expectation value of N interacting particles in a dynamical system. Specially it is a candidate to compute the quantum mean field expectation of N gas or liquid solitons for the KdV hierarchy.

Michael Shearer, North Carolina State University, US
The Dispersive Riemann Problem for the BBM Equation

The BBM (Benjamin-Bona-Mahoney) equation,

$$u_t + uu_x - u_{xxt} = 0, \tag{3}$$

is a non-integrable variation of the Korteweg-deVries equation. In this talk, I describe solutions of the Riemann problem for (3), with initial data a smoothed jump:

$$u(x,0) = u^{\epsilon}(x) = \frac{1}{2}(u_L + u_R) + \frac{1}{2}(u_R - u_L)\tanh\left(\frac{x}{\epsilon}\right) \approx \begin{cases} u_L, & x < 0, \\ u_R, & x > 0. \end{cases}$$

A beginning observation is that if $u_L = -u_R$, then a smooth jump symmetric about $u = 0$ persists. Non-symmetric initial jumps yield a rich variety of solution patterns, which are explored with numerical simulations and analysis.

Matteo Sommacal, Northumbria University, UK
Integrability, instabilities, and the onset of rogue waves

Recently, a direct construction of the eigenmodes of the linearization of 1+1, multicomponent, nonlinear, partial differential equations of integrable type has been introduced. This construction employs only the associated Lax pair, with no reference to spectral data and boundary conditions. In particular, this technique allows to study the instabilities of continuous wave solutions in the parameter space of their amplitudes and wave numbers, leading to the construction of the so-called stability spectra, which, for multi-component systems with more than two components, in general differs from the continuous spectra of the spatial Lax operator. In the context of modulation instability, it provides also a necessary condition in the parameters for the onset of rational solitons. The theory will be illustrated using the example of the plane wave solutions for a system of two coupled nonlinear Schrödinger equations in the defocusing, focusing and mixed regimes. The derivation of the stability spectra is completely algorithmic, and, in the case of plane waves, their study makes use of some basic ideas from algebraic geometry. Indeed, it turns out that, for a Lax Pair that is polynomial in the spectral parameter, the problem of classifying the stability spectra is transformed into a problem of classification of certain complex curves. The method is general enough to be applicable to a large class of integrable systems and in principle to all typologies of their solutions: additionally to the system of two coupled nonlinear Schrödinger equations, it has already been successfully applied to the study of the plane wave stability for the system modelling the resonant interaction of three waves, and for a novel long wave-short wave system, which contains both the Yajima-Oikawa and Newell models as special cases. Moreover, when this method is applied to continuous wave solutions, the corresponding spectra can be used to predict the values of the spectral parameter leading to rational soliton (rogue wave) solutions and the instability regimes allowing for their formation. This is a joint work with Marcos Caso-Huerta (Northumbria University), Antonio Degasperis (Roma “La Sapienza”), Priscila Leal da Silva (Loughborough University) and Sara Lombardo (Loughborough University).

Alexander Tovbis, University of Central Florida, US
Recent developments in spectral theory of soliton gases for integrable equations

Considering the focusing NLS as a model example, we introduce soliton gas through the thermodynamic limit of multi-phase (finite gap) solutions. This limit can be characterized by the growing genus 2N of the corresponding hyperelliptic Riemann surface combined with simultaneous (exponentially fast in N) shrinking of the bands. We derive the average densities and fluxes of solution gases, study the thermodynamic limit of quasimomentum and quasienergy differentials and introduce and discuss periodic soliton gases.

Mats Vermeeren, Loughborough University, UK
A Lagrangian perspective on integrability

Many integrable systems can be understood as a hierarchy, consisting of the integrable equation and its symmetries, with a (bi-)Hamiltonian structure. This poster presents a lesser-known Lagrangian perspective on integrability. The ”Lagrangian multiform” principle captures a whole hierarchy in a single variational principle. It can be used to describe many types of integrable systems (fully discrete, differential-difference, and continuous equations, of various dimensions). In some cases it can be used to show surprising implications or equivalences between sets of integrable equations.

Raffaele Vitolo, University of Salento, Italy
Projective geometry of homogeneous second-order Hamiltonian operators

In this talk we will show that second order homogeneous Hamiltonian operators are invariant under reciprocal transformations of projective type, thus allowing for a projective classification of the operators. Then, we will describe how the above operators generate known and new integrable systems, and discuss the invariance properties of the systems under projective transformations. Finally, we will compare the properties of second-order homogeneous Hamiltonian operators with those of third-order homogeneous Hamiltonian operators, and give possible directions for more general results. Joint work with P. Vergallo: https://arxiv.org/abs/2203.04237

Derchyi Wu, Academia Sinica, Taiwan
Stability of the Kadomtsev-Petviashvili multi-line solitons

Using the inverse scattering theory, we will provide an L^∞-stability of Kadomtsev-Petviashvili multi-line solitons.