Dynamic Inverse Problem for a hyperbolic equation and continuation of boundary data.

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Abstract. We consider an inverse problem for a second order hyperbolic initial boundary value problem on a compact Riemannian manifold $M$ with boundary. Assume that we know $\partial M$ and the Cauchy data on $\partial M \times [0,T]$ of the solutions with vanishing initial data. In the paper we consider two problems. Firstly, when $T$ is sufficiently large and the Riemannian manifold satisfies an additional geometrical condition, we show that we can continue the data on $\partial M \times \mathbb{R}_+$ without solving the inverse problem. Secondly, we show that it is possible to determine manifold $M$ and the wave operator to within the group of the generalized gauge transformations.

1. Introduction and main result.

In the paper we study an inverse problem for the hyperbolic initial boundary value problem

$$u_{tt} + bu_t + a(x,D)u = 0 \text{ in } M \times \mathbb{R}_+ \quad (1.1)$$

$$u|_{\partial M \times \mathbb{R}_+} = f; \quad u|_{t=0} = u_t|_{t=0} = 0; \quad f \in H^1_0(\partial M \times \mathbb{R}_+) \quad (1.2)$$

on a compact connected $C^\infty$-Riemannian manifold $M, \text{dim } M = m \geq 1$, with metric $g = (g^{jl})_{j,l=1}^m$ and non-empty boundary $\partial M$. The operator $a(x,D)$ is a first order perturbation of the Laplace Beltrami operator $-\Delta_g$,

$$a(x,D) = -\Delta_g + P + q. \quad (1.3)$$

Here in local coordinates $P = P^l \partial_l$ is a complex valued $C^\infty$-vector field and $q$ and $b$ are complex valued $C^\infty$-functions on $M$. The symbol $a(x,D)$ is, in general, not formally symmetric. Later in the paper we refer to the case (1.1), (1.2) with $b(x) \neq 0$ and $a(x,D)$ of form (1.3) as to a ”generic case”.

Remark. Any uniformly elliptic symbol on a differentiable manifold can be written in form (1.3).

In the paper we also study separately the ”selfadjoint case”,

$$a(x,D) = -\Delta_g + q, \quad b(x) = 0, \quad (1.4)$$

where $q$ is a real-valued function.

By $H^s(A)$ we denote the Sobolev space of the functions on $A$ and by $H^s_0(\partial M \times [0,t])$ the space of $u \in H^s(\partial M \times \mathbb{R})$ for which $\text{supp } u \subset \partial M \times [0,t]$. We denote by $\nu$ the unit normal vector to $\partial M$ with respect to $g$. We define the boundary operator $Bu = \partial_\nu u - P_\nu u|_{\partial M \times [0,T]}$, where $\partial_\nu$ and $P_\nu = (\nu, P)_g$ are the normal derivative and the normal component of $P$, correspondingly.
Definition 1. Let $T > 0$. We define the response operator $R_T : H^1_0(\partial M \times [0, T]) \to L^2(\partial M \times [0, T])$ by

$$R_T(f) = \partial_v u^f - P_v u^f|_{\partial M \times [0, T]} = Bu^f|_{\partial M \times [0, T]},$$

where $u^f$ is the solution of the problem (1.1), (1.2).

In the paper we consider two problems:

Problem I. Let $\partial M$ and $R_T$ be given with some $T > 0$. Can we reconstruct from these data the operator $R_t$ with any $t > T$ without solving the inverse problem?

Problem II. Let $\partial M$ and $R_T$ be given with some $T > 0$. Do these data determine $(M, a(x, D), b)$ uniquely?

We give answers to Problems I-II assuming in the generic case that some geometric conditions are posed upon $(M, g)$.

In the following we call the pair $\{\partial M, R_T\}$ the dynamical boundary data and abbreviate it by DBD.

To answer positively to problem I we have to assume that the waves sent at time $t = 0$ from boundary can reach all points in $M$ and return back before time $t = T$. Hence in the selfadjoint case we assume that $T > 2r$ where $r = \max\{\text{dist}(x, \partial M) : x \in M\}$ is the geodesic radius of $M$. In the generic case we pose the following geometrical condition (for details see [1]) which generalizes the condition that the rays of the geometrical optics hit the boundary transversally.

Definition 2. $(M, g)$ satisfies the Bardos-Lebeau-Rauch condition if there is $t_* > 0$ and an open conic neighborhood $O$ of the set of the not-nondiffractive points $(x, t, \xi, \omega) \in T^*(M \times [0, t_*])$, $x \in \partial M$ such that any generalized bicharacteristic of the wave operator $\partial_t^2 - \Delta_g$ passes through a point of $(x, t, \xi, \omega) \in T^*(M \times [0, t_*]) \setminus O$, $x \in \partial M$.

Before stating our main results we discuss shortly Problem II. It is well known that Problem II can not have a positive answer since the generalized gauge transformations preserve the boundary data. This means that by replacing $a(x, D)$ by $a_\kappa(x, D)$,

$$a_\kappa(x, D) = \kappa a(x, D) \kappa^{-1},$$

where $\kappa|_{\partial M} = 1$, $\kappa \neq 0$ on $M$ we do not change $R_T$. Thus the best we can hope to recover is the equivalence class of $a(x, D)$ with respect to the generalized gauge transformations, namely the set

$$[a(x, D)] := \{\kappa a(x, D) \kappa^{-1} : \kappa \in C^\infty(M; \mathbb{C}), \kappa|_{\partial M} = 1, \kappa \neq 0 \text{ on } M\}.$$

The above hyperbolic inverse problem and its analogs were considered in several papers. Paper [14] considered the inverse problem in $M \subset \mathbb{R}^m$ with Euclidean metric $g^{ij} = \delta^{ij}$. The corresponding inverse boundary spectral problem was studied in [11]. A local variant of the dynamic inverse problem with data prescribed only on a part of the boundary was considered in [5] where it was assumed that $g^{ij} = \delta^{ij}$. In [13] the uniqueness of the reconstruction of the conformally Euclidean metric in $M \subset \mathbb{R}^m$ and the lower order terms (with some restrictions upon these terms)
was proven for the geodesically regular domains \( M \). The present work is based on paper [9] of the authors where an analogous problem was studied for the Gel’fand inverse boundary spectral problem.

In an anisotropic case an analogous inverse problem was considered in [6], [7] for the self-adjoint case and in [8] for the non-selfadjoint case, \( a^*(x, D) \neq a(x, D) \) where is was, however, assumed that \( b = 0 \).

This paper is based on the Boundary Control method introduced in [2] (see also [3]). Particularly, we use here the geometrical formulation of the Boundary Control method (see [7]) and exact controllability results [1].

The main results of the paper are:

**Theorem 1.1.** Assume that

i. In the generic case the Riemannian manifold \((M, g)\) satisfies the Bardos-Lebeau-Rauch condition with \( t_* \) and \( R_T \) is known for \( T > 2t_* \);

ii. In the self-adjoint case \( R_T \) is known for \( T > 2r \).

Then these data determine uniquely \( R_t \) for any \( t > 0 \).

In the selfadjoint case (1.4) it is known (see e.g. [7]) that BSD determines \((M, g)\) and \( q \) uniquely. We give a dynamic version of this result which is valid in generic case:

**Theorem 1.2.** In generic case let the Riemannian manifold \((M, g)\) satisfies the Bardos-Lebeau-Rauch condition with \( t_* \). Let \( \partial M \) and \( R_T \) be given and \( T > 2t_* \). Then these data determine \( M, b \) and the equivalence class \([a(x, D)]\) uniquely.

Before stating the proofs, we explain what we mean by the reconstruction of a Riemannian manifold \((M, g)\). Since a manifold is an ‘abstract’ collection of coordinate patches we construct a representative of an equivalence class of the isometric Riemannian manifolds or a metric space \( X \) which is isometric to \((M, g)\). After constructing \( X \) one can take any coordinatisation and construct the vector field \( P \) and the potential \( q \) in local coordinates.

2. Continuation of data in the selfadjoint case.

In this section we consider Problem I for the initial boundary value problem

\[
\begin{align*}
\partial_t^2 u^f - \Delta_g u^f + qu^f &= 0 \text{ in } M \times \mathbb{R}^+, \\
\partial^2 u^f |_{\partial M \times \mathbb{R}^+} &= f; \quad u^f |_{t=0} = u_t^f |_{t=0} = 0,
\end{align*}
\]

where \( q \) is a real valued function. We point out that we do not assume that the Bardos-Lebeau-Rauch condition is valid.

By \( \lambda_j \) and \( \phi_j \) we denote the Dirichlet eigenvalues and the normalized eigenfunctions of the operator \(-\Delta_g + q\). In this section all spaces \( L^2(M) \) etc are spaces of real valued functions.

We start with a well-known result of approximable controllability.

**Lemma 2.1.** The pairs \((u^f(2r), u_t^f(2r))\), \( f \in C_0^\infty(\partial M \times [0, 2r]) \) are dense in \( H^1_0(M) \times L^2(M) \).

**Proof.** Assume that a pair

\[
(\psi, -\phi) \in (H^1_0(M) \times L^2(M))' = H^{-1}(M) \times L^2(M)
\]
satisfy the duality
\[(u^f(2r), \psi)_{H_0^1, H^{-1}} + (u^f_t(2r), -\phi)_{L^2} = 0\]
for all \(f \in C_0^\infty(\partial M \times [0, 2r])\). Let
\[e_{tt} - \Delta_\gamma e + qe = 0 \text{ in } M \times [0, 2r],\]
\[e|_{\partial M} = 0; \quad e|_{t=2r} = \phi, \quad e_t|_{t=2r} = \psi.\]
By part integration
\[0 = \int_{M \times [0, 2r]} [(e_{tt} - \Delta_\gamma e + qe)u^f - (u^f_{tt} - \Delta_\gamma u^f + qu^f)e] \, dx \, dt = \]
\[= \int_M (u^f_t(2r) \phi - u^f(2r) \psi) \, dx + \int_{\partial M} \int_0^{2r} f \, \partial_r e \, dS_x \, dt = \int_{\partial M} \int_0^{2r} f \, \partial_r e \, dS_x \, dt\]
for all \(f \in C_0^\infty(\partial M \times [0, 2r])\). This yields that
\[e|_{\partial M \times [0, 2r]} = \partial_v e|_{\partial M \times [0, 2r]} = 0.\]
Since by (2.1) \(e \in \mathcal{D}'(\mathbb{R}, H_0^1(M))\) Tataru's Holmgren-John uniqueness theorem [15] is applicable and we obtain \(e(r) = e_t(r) = 0\). Hence \(e = 0\) identically on \(M \times [0, 2r]\) and thus \(\phi = \psi = 0\).

Consider a bilinear form
\[E(u^f, u^g, t) = \int_M \left[ (\nabla u^f(t), \nabla u^g(t))_g + u^f_t(t)u^g_t(t) + qu^f(t)u^g(t) \right] \, dx\]
and denote \(E(u^f, t) = E(u^f, u^f, t)\).

**Lemma 2.2.** Operator \(R_t^f\) determines \(E(u^f, u^g, t)\) for \(f, g \in C_0^\infty(\partial M \times [0, t])\).

**Proof.** By part integration
\[\frac{\partial}{\partial t} E(u^f, t) = 2 \int_M \left[ (\nabla u^f_t(t), \nabla u^f(t))_g + u^f_t(t)u^g_t(t) + qu^f(t)u^g(t) \right] \, dx = \]
\[= 2 \int_M -\Delta_\gamma u^f(t) + u^f_{tt}(t) + qu^f(t) \right] u^f_t(t) \, dx + 2 \int_{\partial M} u^f_t(t) \partial_r u^f(t) \, dS_x \]
\[= 2 \int_{\partial M} f_t(t) R^f f(t) \, dS_x.\]
Since \(E(u^f, 0) = 0\) we can determine \(E(u^f, t)\). Since \(4E(u^f, u^g, t) = E(u^{f+g}, t) - E(u^{f-g}, t)\), this proves the assertion.

Next we show that we can continue data without solving the inverse problem.

**Proof.** (of Theorem 1.1. in the selfadjoint case) It is sufficient to show that \(R^T\) determines \(R^T f\) for any \(f \in C_0^\infty(\partial M \times [0, 2r])\).
Let \( \varepsilon = (T - 2r)/2 \) and \( t_0 = 2r + \varepsilon \). By Lemma 2.1 there are \( f_n \in C^\infty_0(\partial M \times [0, 2r]) \) such that

\[
\lim_{n \to \infty} (u^n(t_0), u^n_0(t_0)) = (u(t_0), u_0(t_0)) \quad (2.2)
\]

in \( H_0^1(M) \times L^2(M) \)-topology. We want to show that (2.2) is valid if and only if for every \( h \in C^\infty_0(\partial M \times [0, 2r]) \)

\[
\lim_{n \to \infty} E(u^{g_n}, t_0) = 0, \quad (2.3)
\]

\[
\lim_{n \to \infty} E(u^{g_n}, u^h, t_0) = 0, \quad (2.4)
\]

\[
\lim_{n \to \infty} ||R^{t_0+\varepsilon} g_n||_{L^2(\partial M \times [t_0, t_0+\varepsilon])} = 0, \quad (2.5)
\]

where \( g_n(t) = f(t) - f_n(t - \varepsilon) \). Since the direct problem depends continuously on initial data [10], we see that (2.2) obviously yields (2.3)-(2.5). Thus assume that (2.3)-(2.5) are valid. We use the eigenfunction expansion \( u^{g_n}(t_0) = \sum_j a^n_j \phi_j \) and \( u^h(t_0) = \sum_j b_j \phi_j \). Then by (2.3)

\[
\lim_{n \to \infty} \left( \sum_{j=0}^\infty \lambda_j(a^n_j)^2 + ||u^n_t||_{L^2}^2 \right) = 0. \quad (2.6)
\]

Let \( j_0 \) be such that \( \lambda_j > 0 \) for \( j > j_0 \) and \( \lambda_j \leq 0 \) for \( j \leq j_0 \) and let \( P \) be the orthogonal projection in \( H_0^1(M) \) onto the space of the eigenfunctions corresponding \( \lambda_j = 0 \). Using these notations we rewrite (2.6) in the following form

\[
\sum_{j \leq j_0} -\lambda_j(a^n_j)^2 = \sum_{j > j_0} \lambda_j(a^n_j)^2 + ||u^n_{t_0}(t_0)||_{L^2(M)}^2 + o(1) \quad (2.7)
\]

where \( o(1) \) goes to zero when \( n \to \infty \).

First we show that \( a^n_j \to 0 \) for \( j \) satisfying \( \lambda_j < 0 \). Indeed, assume that there is \( k \) with \( \lambda_k < 0 \) such that \( a^n_k \neq 0 \). By choosing a subsequence, the sign of \( a^n_j \) depends only upon \( j \). Moreover, without loss of generality we can assume that \( a^n_j \geq 0 \).

Since \( (u^g(t_0), u^n_t(t_0)) \) are dense in \( H_0^1(M) \times L^2(M) \) we can choose \( h \) such that its Fourier coefficients \( (b_j) \) satisfy \( b_j = \delta_{j \leq j_0} + c_j \) where \( ||(\lambda_j c_j)||_{\ell^2} < \varepsilon \) and \( ||u^n_{t_0}(t_0)||_{L^2(M)} < \varepsilon, \varepsilon \in (0, \frac{1}{2}] \). Then (2.4) yields that

\[
\sum_{j \leq j_0} -\lambda_j a^n_j(1 + c_j) = \sum_{j > j_0} \lambda_j a^n_j c_j + (u^n_{t_0}(t_0), u^n_{t_0}(t_0))_{L^2(M)} + o(1).
\]

Hence by (2.7)

\[
\sum_{j \leq j_0} -\lambda_j a^n_j(1 + c_j) \leq \left( \sum_{j > j_0} \lambda_j(a^n_j)^2 \right)^\frac{1}{2} \left( \sum_{j > j_0} \lambda_j(c_j)^2 \right)^\frac{1}{2} + ||u^n_{t_0}(t_0)||_{L^2} ||u^n_{t_0}(t_0)||_{L^2} + o(1)
\]

\[
\leq \varepsilon \left( \sum_{j > j_0} \lambda_j(a^n_j)^2 \right)^\frac{1}{2} + \varepsilon ||u^n_{t_0}(t_0)||_{L^2} + o(1) = \varepsilon \left( \sum_{j \leq j_0} -\lambda_j(a^n_j)^2 \right)^{1/2} + o(1). \quad (2.8)
\]
On the other hand, there is $C > 0$ independent of $\varepsilon$ such that
\[ \sum_{j \leq j_0} -\lambda_j a_j^n (1 + c_j) \geq C \left( \sum_{j \leq j_0} -\lambda_j (a_j^n)^2 \right)^{\frac{1}{2}}. \]
But for some $k$ with $\lambda_k < 0$ $a_k^n \neq 0$. This leads to a contradiction with (2.8).

Thus we have proven that $a_j^n \to 0$ for all $j$ satisfying $\lambda_j < 0$. By (2.6), this implies that
\[ \sum_{j=0}^{\infty} |\lambda_j|(a_j^n)^2 < C \]
uniformly for all $n$. Thus the pairs $((1 - P)u^{g_n}, u^{g_n}_t)$ are uniformly bounded in $H^1_0(M) \times L^2(M)$. By (2.4), this implies that $E(u^{g_n}, a, t_0) \to 0$ for any $a \in H^1_0(M) \times L^2(M)$. Hence $(1 - P)u^{g_n}(t_0) \to 0$ in $H^1_0(M)$ and $u^{g_n}_t(t_0) \to 0$ in $L^2(M)$.

It remains to show that $|a^n_j| \to 0$ when $\lambda_j = 0$ and $g_n$ satisfy (2.3)-(2.5). Since the solution of the direct problem depends continuously on the data,
\[ \lim_{n \to \infty} \|R^T g_n - \sum_{\lambda_j=0}^{\infty} a^n_j \partial_r \phi_j |_{\partial M} \|_{L^2(\partial M \times [t_0, T])} = 0. \]
Since $\partial_r \phi_j |_{\partial M}$ are linearly independent, (2.5) can be valid only if $a^n_j \to 0$.

Thus (2.2) and (2.3)-(2.5) are equivalent.

We can use Lemma 2.2 to construct $f_n$ which satisfy conditions (2.2). The functions $y_n(t) = u^{f_n}(t)$ for $t \in [2r, T]$ are the solutions of the initial value problem
\[
\begin{align*}
y^n_{tt} - \Delta y^n &= 0 \text{ in } M \times [2r, T] \\
y^n &= 0 \text{ on } \partial M \times [2r, T]; \quad y^n|_{t=2r} = u^{f_n}(2r), \quad y^n_t|_{t=2r} = u^{f_n}_t(2r).
\end{align*}
\]
However, $y(t) = u^f(\cdot + \varepsilon)$ satisfies the same equation with initial data $y|_{t=2r} = u^f(t_0)$, $y_t|_{t=2r} = u^f_t(t_0)$. Then it follows from (2.2) and continuous dependence of solutions on the initial data (see e.g [10]) that
\[ \lim_{n \to \infty} \partial_r y^n|_{\partial M \times [2r, T]} = \partial_r y|_{\partial M \times [2r, T]} \]
in $L^2$-topology. Since we know $y_n(t)|_{\partial M \times [2r, T]} = (R^T f_n)(t)$, $t \in [2r, T]$ we can determine $R^T + \varepsilon f$.

By iterating the above consideration, we get the assertion. \hfill \Box

3. Continuation of data and uniqueness results in the non-selfadjoint case.

In this section we study the inverse problem for the initial-boundary value problem in generic case
\[
\begin{align*}
u^f_{tt} + bu^f_t + a(x, D)u^f &= 0 \text{ in } M \times \mathbb{R}_+ \quad (3.1) \\
u^f|_{\partial M \times \mathbb{R}_+} &= f; \quad u^f|_{t=0} = u_t|_{t=0} = 0; \quad f \in H^1_0(\partial M \times \mathbb{R}_+), \quad (3.2)
\end{align*}
\]
where $a(x, D)$ is of form (1.3) and $(M, g)$ satisfies the Bardos-Lebeau-Rauch condition. We use the notations
\[ U^f(t) := \begin{pmatrix} u^f_1(x, t) \\ u^f_2(x, t) \end{pmatrix} \in L^2(M)^2, \quad J\left( \begin{pmatrix} u^1 \\ u^2 \end{pmatrix} \right) = \begin{pmatrix} u^2 + bu^1 \\ u^1 \end{pmatrix}. \quad (3.3)\]
and denote the inner product in $L^2(M)^2$ by $(\cdot, \cdot)$.
3.1 Adjoint equation.

Let \( v^g(x, s) \) be the solution to the adjoint initial-boundary value problem,

\[
v^g_{tt} + b v^g_t + a^*(x, D)v^g = 0 \quad \text{in} \quad M \times \mathbb{R}_+ \tag{3.4}
\]

\[
v^g|_{\partial M \times \mathbb{R}_+} = g, \quad v^g|_{t=0} = v^g_t|_{t=0} = 0. \tag{3.5}
\]

We denote

\[
V^g(t) = \left( \begin{array}{c} v^g(x, t) \\ v^g_t(x, t) \end{array} \right). \tag{3.6}
\]

For the adjoint equation we define the response operator \( R^T_\ast : H^1_0(\partial M \times [0, T]) \to L^2(\partial M \times [0, T]) \),

\[
R^T_\ast(g) = B^* v^g, \quad B^* v := \partial_
u v|_{\partial M \times [0, T]}. \tag{3.7}
\]

Lemma 3.1. For any \( t_0 > 0 \) \( R^{t_0} \) determines \( R^{t_0}_\ast \).

Proof. Let \( f, h \in H^1_0(\partial M \times [0, t_0]) \) and let \( e^h \) be the solution of the backward wave equation

\[
e^h_{tt} - b e^h_t + a^*(x, D)e^h = 0 \quad \text{in} \quad M \times [0, t_0], \tag{3.8}
\]

\[
e^h|_{\partial M \times [0, t_0]} = h; \quad e^h|_{t=t_0} = e^h_t|_{t=t_0} = 0. \tag{3.9}
\]

Notice that for \( h(t) = g(t_0 - t) \) we have \( e^h(t) = v^g(t_0 - t) \). Part integration together with initial and final conditions (3.2), (3.9) yield that

\[
0 = \int_{t_0}^0 \int_M ((u^f_{tt} + bu^f_t + a(x, D)u^f) \bar{e}^h - u^f (e^h_{tt} - b e^h_t + a^*(x, D)e^h)) \, dx \, dt
\]

\[
= \int_{t_0}^0 \int_{\partial M} (Bu^f \bar{e}^h - u^f \overline{B^* e^h}) \, dS_x \, dt = \int_{t_0}^0 \int_{\partial M} (R^{t_0} f \bar{h} - f \overline{B^* e^h}) \, dS_x \, dt.
\]

Since \( f \) is arbitrary and \( R^{t_0} f \) is known, we can determine \( B^* e^h|_{\partial M \times [0, t_0]} \) for each \( h \in H^1_0(\partial M \times [0, t_0]) \), i.e. to find \( R^{t_0}_\ast \).

3.2 Controllability results and continuation of \( R^T \).

In the following we denote by \( L^s, s \in \mathbb{R} \) the subspace of functions in \( H^{s+1}_0(M) \times H^s(M) \) which satisfy the natural boundary compatibility conditions for the hyperbolic problem (3.1), (3.2) for \( t \notin \text{supp} \ f \) (see e.g [12]) and by \( L^s_{\text{ad}} \) the analogous subspace for (3.4), (3.5).

We use the following exact controllability result.

Theorem 3.2. [1] Assume that \( (M, g) \) satisfies the Bardos-Lebeau-Rauch condition. Then

\[
\{ U^f(t_1) : \ f \in H^{s+1}_0(\partial M \times [0, t_0]) \} = L^s, \quad t_1 \geq t_0 > t_*, s \geq 0.
\]

The analogous result is valid for the adjoint equation.
Lemma 3.3. Assume that we know $R^T$. For given $f, g \in H^1_0(\partial M \times [0, T])$, $t+s \leq T$ we can evaluate

$$(JU^I(t), V^g(s)) =$$

$$= \int_M [u^f_t(x, t)v^g(x, s) + u^f(t)v^g_t(x, s) + b(x)u^f(x, t)v^g(x, s)]dx.$$  

Proof. By part integration

$$(\partial_t - \partial_s)(JU^I(t), V^g(s)) =$$

$$= \int_M [(u^f_{tt} + bu^f_t v^g - u^f)v^g_t + bu^f v^g_t]dx =$$

$$= - \int_{\partial M} [(u^f(t)\bar{B}v^g(s) - Bu^f(t)v^g(s))]dS_x = \int_{\partial M} [R^T f(t) g(s) - f(t) R^T_* g(s)]dS_x. \quad (3.9')$$

As $R^T$ and $R^T_*$ are known, all the functions in the last integral are known. Hence (3.9') is a differential equation along the characteristic $t+s = \text{const}$. Furthermore,

$$(JU^I(0), V^g(s)) = (JU^I(t), V^g(0)) = 0$$

due to the initial conditions (3.2), (3.5). Equation (3.9') together with the above initial condition indicates the possibility to find $(JU^I(t), V^g(s))$. 

Next we prove that, in the generic case, $R^t$ can be determined for all $t > 0$.

Proof. (of Theorem 1.1) Let $\varepsilon < T/2 - t_*$, $T_0 = T/2$. We will first prove that when $R^T$ and $R^T_*$ are given, it is possible to find $R^{T+\varepsilon}$ and $R^{T+\varepsilon}_*$.

Clearly it is sufficient to determine $R^{T+\varepsilon} f$ for any $f \in H^1_0(\partial M \times [0, T_0])$. As $T_0 - \varepsilon > t_*$ then by Theorem 3.2 there is $\tilde{f} \in H^1_0(\partial M \times [0, T_0 - \varepsilon])$ for which

$$U^f(T_0) = U^f(T_0 - \varepsilon).$$

Moreover, this function can be found by choosing $\tilde{f}$ which satisfies the following equation

$$(JU^f(T_0), V^g(T_0)) = (JU^{\tilde{f}}(T_0 - \varepsilon), V^g(T_0))$$

for all $g \in H^1_0(\partial M \times [0, T_0])$.

Let now $F \in H^1_0(\partial M \times [0, T])$ be the function

$$F(x, t) = \tilde{f}(x, t) \text{ for } t \in [0, T_0 - \varepsilon], \quad F(x, t) = 0 \text{ for } t \in [T_0 - \varepsilon, T].$$

Let $\phi = u^{f}|_{t=T_0}$ and $\psi = u^{\tilde{f}}|_{t=T_0}$. Since $u^{f}$ solves the equation

$$u_{tt} + bu_t + a(x, D)u = 0 \text{ in } M \times [T_0, T + \varepsilon],$$

$$u^{f}|_{\partial M \times [T_0, T+\varepsilon]} = 0,$$

$$u^{f}|_{t=T_0} = \phi, \quad u^{f}|_{t=T_0} = \psi$$

we have
and $u^F$ solves the equation

$$u^F_{tt} + bu^F_t + a(x,D)u^F = 0 \text{ in } M \times [T_0 - \varepsilon, T]$$

$$u^F|_{\partial M \times [T_0 - \varepsilon, T]} = 0,$$

$$u^F|_{t=T_0 - \varepsilon} = \phi, \quad u^F_t|_{t=T_0 - \varepsilon} = \psi$$

we see that

$$u^f(t + \varepsilon) = u^F(t) \text{ for } t \in [T_0 - \varepsilon, T].$$

Hence we get

$$R^{T+\varepsilon}f(\cdot, t) = R^TF(\cdot, t - \varepsilon) \text{ for } t \in [T_0, T + \varepsilon].$$

Since by assertion $(R^TF)(\cdot, t)$ for $t \leq T$ is known, we reconstruct $R^{T+\varepsilon}$. The claim follows similarly for $R^{T+\varepsilon}_s$.

By iterating the above procedure with fixed $T_0$ we reconstruct $R^{T+n\varepsilon}$, $n = 0, 1, 2, \ldots$. This proves Theorem 1.1.

Analogously to Lemma 3.3, we obtain

**Corollary 3.4.** Assume that DBD is given for $T > 2t_*$. Then for given $f,g \in H^1_0(\partial M \times \mathbb{R}_+)$ and $t, s > 0$ we can evaluate $(JU^f(t), V^g(s))$.

### 3.4 Construction of the boundary distance functions.

Let $r_x(y), x \in M$ be the boundary distance functions

$$r_x(y) = d(x, y), \quad y \in \partial M.$$

We define a mapping $\mathcal{R} : M \to L^\infty(\partial M)$ by setting

$$\mathcal{R}(x) = r_x.$$

We are going to show that we can reconstruct the set $\mathcal{R}(M) = \{r_x : x \in M\}$.

In the standard Boundary Control method one constructs the projections to the spaces of the Fourier coefficients of the functions $L^2(A)$, $A \subset M$. Inspired by this we define the following spaces.

**Definition 3.** Let $H \subset \mathcal{L}^s$ be a lineal, $s \geq 0$. We define the control sets $\mathcal{H}^s(H)$ for $H$ by

$$\mathcal{H}^s(H) = \{ f \in H^{s+1}_0(\partial M \times [0,T/2]) : U^f(T) \in H \},$$

$$\mathcal{H}^s_{ad}(H) = \{ g \in H^{s+1}_0(\partial M \times [0,T/2]) : V^g(T) \in H \}.$$

Let $\Gamma \subset M$ be open, $t_0 \geq 0$. Denote

$$M(\Gamma, t_0) = \{x \in M : d(x, \Gamma) \leq t_0\}. \quad (3.10)$$
**Definition 4.** For \(s \geq 0\) let
\[ \mathcal{L}^s(\Gamma, t_0) = \{ U \in \mathcal{L}^s : \text{supp } U \subset \text{cl}(M(\Gamma, t_0)) \}, \]
\[ [\mathcal{L}^s(\Gamma, t_0)]^c = \{ U \in \mathcal{L}^s : \text{supp } U \subset \text{cl}(M \setminus M(\Gamma, t_0)) \} \]
and analogous sets \( \mathcal{L}^s_{ad}(\Gamma, t_0), [\mathcal{L}^s_{ad}(\Gamma, t_0)]^c \).

Our next goal is to find the control sets for the above subsets of \( \mathcal{L}^s \).

In the following let \( m_g \) be the Riemannian measure on \((M, g))

**Lemma 3.5.** Let \( f \in H^{s+1}_0(\partial M \times [0, T/2]), s \geq 0 \). Then for any \( \Gamma \subset \partial M, t_0 \in [0, T/2] \) DBD determine whether
\[ m_g(\text{supp } U^f(T) \cap M(\Gamma, t_0)) = 0 \]
or not. Analogous statement takes place for the adjoint solutions \( V^g(T) \).

**Proof.** Note that \( f(x, t) = 0 \) for \( t > T/2 \). If
\[ m_g(\text{supp } U^f(T) \cap M(\Gamma, t_0)) = 0 \]
then by the finite velocity of the wave propagation
\[ Bu^f|_{\Gamma \times [T-t_0, T+t_0]} = 0 \] and \( f|_{\Gamma \times [T-t_0, T+t_0]} = 0 \).

On the other hand, by Tataru’s Holmgren-John theorem [15] the converse is also true. By Theorem 1.1 \( Bu^f|_{\partial M \times [0, 3T/2]} = R^{3T/2}f \) is known and hence the statement follows. The claim for adjoint solutions follows from Lemma 3.1.

**Corollary 3.6.** Let \( \Gamma \subset \partial M, t_0 \geq 0 \) and \( s \geq 0 \). Then DBD determine lineals \( \mathcal{H}^s(\mathcal{L}^s(\Gamma, t_0)), \mathcal{H}^s([\mathcal{L}^s(\Gamma, t_0)]^c) \) and \( \mathcal{H}^s_{ad}(\mathcal{L}^s_{ad}(\Gamma, t_0)), \mathcal{H}^s_{ad}([\mathcal{L}^s_{ad}(\Gamma, t_0)]^c) \).

**Proof.** By Theorem 3.2,
\[ \{ U^f(T) : f \in H^{s+1}_0(\partial M \times [0, T/2]) \} = \mathcal{L}^s. \]
Thus by Lemma 3.5 DBD determine \( \mathcal{H}^s([\mathcal{L}^s(\Gamma, t_0)]^c) \) and \( \mathcal{H}^s_{ad}([\mathcal{L}^s_{ad}(\Gamma, t_0)]^c) \).

For \( f \in H^{s+1}_0(\partial M \times [0, T/2]) \) we have \( f \in \mathcal{H}^s(\mathcal{L}^s(\Gamma, t_0)) \) if and only if
\[ (JU^f(T), V^g(T)) = 0 \]
for all \( g \in \mathcal{H}^s_{ad}([\mathcal{L}^s_{ad}(\Gamma, t_0)]^c) \). Hence we can determine \( \mathcal{H}^s(\mathcal{L}^s(\Gamma, t_0)) \).

The case \( \mathcal{H}^s_{ad}([\mathcal{L}^s_{ad}(\Gamma, t_0)]) \) can be considered analogously.

**Corollary 3.7.** Let \( \Gamma_i \subset \partial M, t_i^+ > t_i^- \geq 0; \ i = 1, ..., \ I \). Denote by \( M_I \) the set
\[ M_I = \bigcap_{i=1}^I (M(\Gamma, t_i^+) \setminus M(\Gamma, t_i^-)). \] (3.11)

Then DBD determine whether \( m_g(M_I) = 0 \) or not.

**Proof.** Using intersections of sets described in Corollary 3.6 we find whether \( \mathcal{L}^s \) contains functions supported in the closure of \( M_I \). That kind of functions exists if and only if \( m_g(M_I) \neq 0 \).

Corollary 3.7 is the basic analytic tool in reconstruction of \( \mathcal{R}(M) \).
Theorem 3.8. DBD with $T > 2t_*$ determines $R(M)$ uniquely.

Proof. For every $\varepsilon > 0$ we choose a collection $\Gamma_i \subset \partial M, i = 1, ..., I(\varepsilon)$ such that \( \text{diam}(\Gamma_i) \leq \varepsilon, \cup \Gamma_i = \partial M \). Let

$$p = (p_1, ..., p_{I(\varepsilon)}), \quad p_i \in \mathbb{Z}_+, \quad t_i^+ = (p_i + 1)\varepsilon, \quad t_i^- = (p_i - 1)\varepsilon. \quad (3.12)$$

Denote by $M(\varepsilon, p)$ the set $M_I$ (see (3.11)) with $t_i^\pm$ of form (3.12). For every $p$ we define a piecewise constant function $r_p \in L^\infty(\partial M)$ by setting $r_p(y) = p_i\varepsilon$ when $y \in \Gamma_i$. Using Corollary 3.7 we define whether $m_g(M(\varepsilon, p)) > 0$ or not and introduce a set

$$\mathcal{R}_\varepsilon(M) = \{r_p : p \in \mathbb{Z}_+^{I(\varepsilon)} \text{ such that } m_g(M(\varepsilon, p)) > 0 \} \subset L^\infty(\partial M).$$

As $|r_x - r_p| < 2\varepsilon + \max \text{diam } (\Gamma_i)$ when $x \in M(\varepsilon, p)$, then

$$\text{dist}_H(\mathcal{R}_\varepsilon(M), \mathcal{R}(M)) \leq 3\varepsilon.$$

Here $\text{dist}_H(\Omega, \tilde{\Omega})$ is the Hausdorff distance between subsets $\Omega, \tilde{\Omega} \in L^\infty(\partial M)$. When $\varepsilon \to 0$ we find the set $\mathcal{R}(M) \subset L^\infty(\partial M)$ as the limit of $\mathcal{R}_\varepsilon(M)$. \(\square\)

Let $\mathcal{R}(M) \subset L^\infty(\partial M)$ be given. It is shown in [7] that then it is possible to uniquely define a Riemannian structure on $R(M)$ such that $\mathcal{R} : M \to \mathcal{R}(M)$ is an isometry. For the sake of completeness, we construct $(M, g)$ explicitly. To this end we need the following result (see [7]).

Lemma 3.9. $\mathcal{R}(M) \subset L^\infty(\partial M)$ is homeomorphic to $M$.

Proof. Obviously $\mathcal{R}$ is continuous. Assume that $r_x = r_y, x, y \in M$. If $z \in \partial M$ is a nearest point to $x$, $r_x$ achieves the minimum $h = r_x(z)$ at $z$. Thus $x$ lies on the normal geodesic from $z$ and $x = \exp_z(h\nu)$, $\exp$ being the standard exponential map on $TM$. The same holds for $y$ and hence $\mathcal{R} : M \to \mathcal{R}(M)$ is one-to-one. By definition it is onto. Since $M$ is compact, $\mathcal{R}$ is a homeomorphism. \(\square\)

3.5. Reconstruction of the Riemannian metric and the operator.

Let $f, g \in H_0^{s+1}(\partial M \times [0, T/2]), \ s \geq 0$. We define a bilinear form

$$\langle f, g \rangle = (JU^f(T), V^g(T)).$$

Let

$$\mathcal{R}(\varepsilon, p) = \mathcal{R}(M(\varepsilon, p)), \ \varepsilon > 0, \ p \in \mathbb{Z}_+^{I(\varepsilon)}. \quad (3.13)$$

Here $M(\varepsilon, p)$ is defined as in the proof of Theorem 3.8, i.e. $\mathcal{R}(\varepsilon, p)$ is the set of all boundary distance functions $r_x$ with $x \in M(\varepsilon, p) \subset M$.

Let $r_{x_0} \in \mathcal{R}(M \setminus \partial M)$. Then for any $\varepsilon$ there exists $p_\varepsilon \in \mathbb{Z}_+^{I(\varepsilon)}$ such that $x_0 \in M(\varepsilon, p_\varepsilon)$ and

$$\mathcal{R}(\varepsilon, p_\varepsilon) \longrightarrow \{r_{x_0}\} \text{ when } \varepsilon \to 0,$$

i.e. the Hausdorff distance between the above sets goes to 0 when $\varepsilon \to 0$. By Lemma 3.9, this yields that

$$M(\varepsilon, p_\varepsilon) \longrightarrow \{x_0\} \text{ when } \varepsilon \to 0. \quad (3.14)$$
Denote by \( g(\varepsilon), \varepsilon > 0 \) a family of functions in \( H^1_0(\partial M \times [0, T/2]) \) such that

i. \( \text{supp } V^g(\varepsilon)(T) \subset \text{cl } (M(\varepsilon, p_\varepsilon)) \).

ii. For any \( f \in H^{s+1}_0(\partial M \times [0, T/2]), \ s < m/2 < s + 1 \) there exists a limit

\[
W^{x_0}(f) = \lim_{\varepsilon \to 0} \langle f, g(\varepsilon) \rangle.
\]

Such families exist, indeed it is sufficient to take \( V^g(\varepsilon) \) to be \( C^\infty_0 \)-approximations to \( (0, \delta(-x_0)) \). On the other hand, assume that for every \( f \in H^{s+1}_0(\partial M \times [0, T/2]) \) the limit

\[
\lim_{\varepsilon \to 0} \langle f, g(\varepsilon) \rangle = \lim_{\varepsilon \to 0} (JU^f(T), V^g(\varepsilon)(T))
\]

exists. Then by Banach-Steinhaus theorem there is \( W^{x_0} \in (L^*)' \subset H^{s+1}_0(M)' \times H^s_0(M)' \) such that

\[
\lim_{\varepsilon \to 0} \langle f, g(\varepsilon) \rangle = (JU^f(T), W^{x_0}),
\]

where the right hand side is interpreted in the distribution sense. Assumption i. together with (3.14) imply that \( \text{supp } (W^{x_0}) \subset \{x_0\} \). Since any distribution supported in a point is a finite sum of derivatives of the delta-distribution, and since \( W^{x_0} \in H^s_0(M)' \times H^{s+1}_0(M)' \), \( s < m/2 < s + 1 \), it follows that there is a constant \( \kappa(x_0) \) that

\[
W^{x_0} = \left( \begin{array}{c} 0 \\ \kappa(x_0) \delta(\cdot - x_0) \end{array} \right).
\]

**Lemma 3.10.** Let \( DBD \) be given for \( T > 2t_* \). Assume that \((M, g)\) satisfies the Bardos-Lebeau-Rauch condition. Then it is possible to construct functions \( g(\varepsilon) \) such that

\[
W^{x_0}(f) = \kappa(x_0)u^f(x_0, t), \quad f \in H^{s+1}_0(\partial M \times [0, T/2]), \ t \geq 0, s < m/2 < s + 1
\]

and

\[
\kappa \in C^0(M), \quad \kappa|_{\partial M} = 1, \quad \kappa \neq 0 \text{ on } M.
\]

**Proof.** To prove the statement is sufficient to show that for any \( r^{x_0} \in \mathcal{R}(\text{int}(M)) \) it is possible to find a family \( g^{x_0}(\varepsilon), \varepsilon > 0 \) such that the corresponding \( W^{x_0} \) satisfy the following conditions

iii. \( W^{x_0} \neq 0 \) for any \( x_0 \in M \).

iv. The function \( r^{x_0} \mapsto W^{x_0}(f) \) has a continuous extension to \( \mathcal{R}(M) \) when \( f \in C^\infty_0(\partial M \times [0, T]) \).

v. For \( f \in C^\infty_0(\partial M \times [0, T]) \) and \( x_1 \in \partial M \)

\[
\lim_{x_0 \to x_1} W^{x_0}(f) = f(x_1, T).
\]

As we already know such sequence exists. Indeed, we can take functions \( g^{x_0}(\varepsilon) \) such that \( V^{g^{x_0}(\varepsilon)}(T) \) are smooth approximations to \( (0, \delta(-x_0))^t \). On the other hand, Corollary 3.4 makes possible to algorithmically verify conditions iii.-v. \( \square \)
Corollary 3.11. Let DBD and $r_{x_0} \in \mathcal{R}(M)$ be given. These data determine $\kappa(x)u^J(x_0, t)$ for any $t > 0$ and $f \in H^1_0(\partial M \times \mathbb{R}_+)$. 

Proof. The statement follows from Corollary 3.4 and Lemma 3.10. \qed

We want to emphasize that we do not know $\kappa(x)$ and, henceforth, can not reconstruct $u^J(x, t)$ using Lemma 3.10. However, we have the following

Theorem 3.12. The DBD determines a metric $E$ on $\mathcal{R}(M)$ such that $(\mathcal{R}(M), E)$ is isometric to $(M, g)$.

Proof. Let $r_x, r_y \in \mathcal{R}(\text{int}(M))$ and let $\mathcal{R}(\varepsilon, p_\varepsilon)$ (see (3.13)) be a sequence satisfying

$$\mathcal{R}(\varepsilon, p_\varepsilon) \rightarrow \{r_x\}$$

when $\varepsilon \to 0$. We denote $h_\varepsilon = \text{diam } M(\varepsilon, p_\varepsilon)$. By Corollary 3.7, we can construct the set

$$X(\varepsilon) = \{f \in H^2_0(\partial M \times [0, T/2]) : \text{supp } U^f(T) \subset M(\varepsilon, p_\varepsilon)\}.$$  \hspace{1cm} (3.17)

Let $\tau > 0$. Assume that $d(x, y) > \tau$. Then due to finite velocity of the wave propagation and the fact that $h_\varepsilon \to 0$ there is $\varepsilon_0$ such that for $\varepsilon < \varepsilon_0$ we have:

(A) There is a neighborhood $N$ of $y$ such that for any $f \in X(\varepsilon)$

$$U^f|_{\tau[\varepsilon\times T, T+\tau]} = 0.$$ 

Using Lemma 3.5 we can check if the property (A) is satisfied.

Let now $s(r_x, r_y)$ be the supremum of all $\tau > 0$ for which the property (A) is satisfied with some $\varepsilon > 0$. Then

$$s(r_x, r_y) \geq d(x, y).$$ \hspace{1cm} (3.16)

On the other hand, assume that $x$ and $y$ are so near to each other that $d(x, y) < d(x, \partial M)/2$ and there is an unique minimal geodesic $\gamma(t) = \exp_x(tv)$ from $x$ to $y$. Let $\tau > d(x, y)$. Then for every $\varepsilon > 0$ there is a solution $(u^J(x, T), 0)$, $f \in X(\varepsilon)$ such that $(x, T, v, 1) \in T^*(M \times \mathbb{R}_+)$ is in the wavefront set of $u^J$. By standard theory of propagation of singularities,

$$\text{singsupp } u^J \cap \{y\} \times [T, T + \tau] \neq \emptyset.$$ 

Thus the function $u^J$ can not vanish in any neighborhood of $y \times [T, T + \tau]$ and the property (A) is not satisfied with any $\varepsilon$. Thus $s(r_x, r_y) \leq d(x, y)$. Hence for $y$ sufficiently close to $x$ we have the equality in (3.16).

Define the metric

$$E(r_x, r_y) := \inf\{\sum_{j=0}^l s(r_{y_j}, r_{y_{j+1}}) : x_0 = x, y_l = y, y_j \in \text{int } (M), l \geq 1\}.$$ 

For any curve $\gamma \subset \text{int}(M)$, we see that the $E$-length of $\mathcal{R}(\gamma)$ is equal to the length of $\gamma$. Hence $E(r_x, r_y) = d(x, y)$ for any $x, y \in \text{int}(M)$. By continuing $E$ onto $\mathcal{R}(\partial M)$ we obtain $(\mathcal{R}(M), E)$ which is isometric to $(M, g)$. \qed

Thus $(\mathcal{R}(M), E)$ can be identified with $(M, g)$ as a metric space. In order to construct local coordinates on $\mathcal{R}(M)$, we start with constructing geodesics. By using triangular comparison theorems we can find the angles of intersecting geodesics. This defines normal coordinates near any $r_x \in \mathcal{R}(M)$ and, henceforth the differentiable structure on $\mathcal{R}(M)$.

Using this structure, we can go back to Lemma 3.10 and demand (see iv. in the proof) that $\kappa \in C^\infty(M)$.
Lemma 3.13. The functions $e^f(x,t) = \kappa(x)u^f(x,t)$, $x \in M$, $t \geq 0$ with $f \in H_0^{s+1}(\partial M \times [0,T/2])$ and $\kappa \in C^\infty(M)$ of form (3.15) determine $a_\kappa(x,D)$ and $b(x)$.

Proof. The functions $e^f(x,t) = \kappa(x)u^f(x,t)$ are the solutions of the initial boundary value problem (see (1.5))

$$
e^f_{tt} + be_t^f + a_\kappa(x,D)e^f = 0, \quad (3.17)$$

$$e^f|_{\partial M \times \mathbb{R}^+} = f; \quad e^f|_{t=0} = e^f_t|_{t=0} = 0.$$

However, Theorem 3.2 implies that for any $x_0 \in \text{int}(M)$ the vectors

$$(e^f(x_0,T), \partial_j(e^f(x_0,T)), \partial_k\partial_l(e^f(x_0,T)), e^f_t(x_0,t))_{j,k,l=1}^m$$

span the space $\mathbb{C}^{(m^2+3m+4)/2}$ when $f \in C_0^\infty(\partial M \times [0,T])$. Hence equation (3.17) may be used to determine $b$ and $a_\kappa(x,D)$.

Theorem 1.2 is proven. $\square$

4. Results for one measurement and further remarks..

In the first part of this section we analyse the possibility of the reconstruction of the response operator $R^{t_0}$ using only one measurement.

Theorem 4.1. For any $t_0 > 0$ there is $f \in H_0^1(\partial \Omega \times \mathbb{R}^+)$, $f|_{t=0} = 0$, such that $\partial_\nu u^f|_{\partial \Omega \times \mathbb{R}^+}$ determines $R^{t_0}$.

Proof. Our main tool is the consequence of energy inequality (see e.g. [10]),

$$||\partial_\nu u^f||_{L^2(\partial M \times [0,t])} \leq c_0e^{c_1 t}||f||_{H_0^1(\partial M \times [0,t])}, \quad f \in H_0^1(\partial M \times [0,T]), \quad (4.1)$$

where $c_0$ and $c_1$ are independent of $t$.

For $t_0 > 0$ let $(f_j : j = 1, \ldots)$ be an orthonormal basis of $H_0^1(\partial M \times [0,t_0])$. Let $g_n, n = 1, 2, \ldots$ be a sequence where each $f_j$ occurs infinitely many times. Consider

$$f(x,t) = \sum_{n=1}^{\infty} e^{cn^2}g_n(x,t-nt_0)$$

with $c > c_1 t_0$ where $c_1$ is the constant in (4.1). Assume that $\partial_\nu u^f|_{\partial M \times \mathbb{R}^+}$ is known. By inequality (4.1) we see that

$$||e^{-cn^2}\partial_\nu u^f(x,t+nt_0)|_{\partial M \times [0,t_0]} - (R^{t_0}g_n)(x,t)||_{L^2} \leq c'e^{-c_1 n t_0}.$$ 

As $ne^{-c_1 n t_0} \to 0$ when $n \to \infty$, this shows that we can determine all $R^{t_0}f_j$, $j = 1, 2, \ldots$. $\square$
Corollary 4.2. Let, in generic case, \((M, g)\) satisfy the Bardos-Lebeau-Rauch condition. There is \(f \in H^1_{\text{loc}}(\partial \Omega \times \mathbb{R}^+), f|_{t=0} = 0, \) such that \(\partial_\nu u^f|_{\partial \Omega \times \mathbb{R}^+}\) determines \(M,b\) and the equivalence class \([a(x,D)]\) uniquely.

In the self-adjoint case the Bardos-Lebeau-Rauch condition is unnecessary.

We conclude the paper with several remarks:

i. The Bardos-Lebeau-Rauch condition is always satisfied for \(M \subset \mathbb{R}^m\) with the metric \(g^{ij} = \delta^{ij}\) or its \(C^1\)-small perturbations (see e.g. [16]);

ii. In the case \(b = 0\) but \(a(x,D) \neq a^*(x,D)\) an analog of Theorem 1.1 states that given \(R^T\) for \(T > t_*\) determines \(R^t\) for all \(t\). Indeed, in this case we can use a sesquilinear form \(u^1(t)v^g(t) - u^f(t)v^d(t)\). Then an analog of lemma 3.3 states that given \(R^T\) it is possible to find the value of this form for \(t \leq T\). Further proof of Theorem 1.1 (with \(T > t_*\) instead of \(T > 2t_*\)) follows as in §3.

iii. The present work remains open the question what is the minimum time \(T\) needed to reconstruct the manifold and the operator. Indeed, in the case \(b = 0\), as we have just shown, \(T > t_*\) is sufficient. In the selfadjoint case \(T > 2r\) is sufficient where \(r\) is the geodesic radius of \((M,g)\), \(r \leq t_*/2\). Moreover, it is known that in the one-dimensional case when \(2r = t_*\) the case \(b \neq 0\) does need time \(T > 2t_*\).

v. Clearly the considerations of the paper remain valid for \((M, g)\) satisfying the Bardos-Lebeau-Rauch-Lebeau conditions for a part of boundary \(\Gamma \subset \partial M\).

iv. Corollary 1.3 remains open in the question if there is \(f \in H^0_0(\partial M \times \mathbb{R}^+),\) that is, a boundary source with finite energy which determines \(R^T\). By modifying the proof of Corollary 1.3 we see that this is true if \(c_1 < 0\) in inequality (4.1).

vi. Instead the boundary operator \(B = \partial_\nu - P_\nu\) we can use \(B = \partial_\nu - \beta\), where \(\beta\) is an arbitrary complex-valued \(C^\infty\)-function on \(\partial M\).

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