HYDRO_DYNAMIC TYPE SYSTEMS AND THEIR INTEGRABILITY
Introduction for Applied Mathematicians

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Abstract. Hydrodynamic type systems are systems of quasilinear equations of the first order. They naturally arise in continuum mechanics but also occur as a result of semi-classical approximations of nonlinear dispersive waves. The mathematical theory of one-dimensional hyperbolic quasilinear equations initiated by B. Riemann in XIX century has been developed into a rich and diverse area of applied mathematics including, e.g., theory of shock waves. Among the classical methods of integration of one-dimensional quasilinear hyperbolic equations are the method of characteristics and the hodograph method, the latter being applicable only to the two-component hydrodynamic systems. A relatively recent breakthrough in the theory of hydrodynamic type systems was made in 1980’s by S. Tsarev who proved Novikov’s hypothesis on the integrability of diagonalisable Hamiltonian systems of hydrodynamic type and formulated the generalised hodograph method. In these notes I will outline some of the basic ideas related to integrability of one-dimensional hydrodynamic type systems. The emphasis will be made on the applicable aspects of the theory.

1. Introduction

Hydrodynamic type systems are systems of the first-order quasilinear partial differential equations (see Def. 2.1. below). The systematic development of their mathematical theory was initiated by B. Riemann who, in particular, revealed the deep connection of the theory of hydrodynamic type systems with differential geometry. There have been numerous outstanding contributions to the theory of hydrodynamic type systems since Riemann’s works, some of the most important due to P.D. Lax who laid the foundations of the modern mathematical theory of shock waves (see e.g. [22]).

One can distinguish two principal general tools for the integration of systems of hydrodynamic type: the method of characteristics and the hodograph method (although some special classes of hydrodynamic type systems could be integrated by other methods). The method of characteristics gives explicit analytic results only for a single equation but is also very effective in the numerical integration of systems of hyperbolic equations (see the definition of hyperbolicity in the next section). The classical hodograph method is applicable only to two-component homogeneous hydrodynamic type systems, which are quite common in gas and fluid dynamics (e.g. the shallow-water equations). Although this method, due to its poor compatibility with Cauchy problems, is rarely applied to the solution of actual hydrodynamics problems, its principal significance is in the establishing of the integrability notion for the $2 \times 2$ hydrodynamic type systems: any local solution
of the two-component one-dimensional hydrodynamic type system is obtainable via the hodograph transform.

With the advent of soliton theory and the inverse scattering transform method in mid-1960s, the ideas of integrability penetrated the theory of nonlinear PDEs. The soliton equations (like the famous Korteweg-de Vries (KdV) and nonlinear Schrödinger (NLS) equations) are typically nonlinear PDEs which include dispersive terms (i.e. are not of hydrodynamic type). Parallel with the soliton theory, another method for the study of nonlinear dispersive waves was developed: the Whitham method of averaging (see [39]). Application of the Whitham averaging to a nonlinear dispersive PDE results in a system of equations of hydrodynamic type with the order of the system no less than the differential order of the original dispersive equation or system. For instance, the single-phase averaging of the KdV equation results in the system of three quasilinear equations to which the classical hodograph method is not applicable. On the other hand, it was clear that the systems obtained by the averaging of integrable dispersive equations should be integrable themselves in some sense.

The study of the integrability properties of the Whitham modulation equations by S. Novikov’s group in Moscow in 1980s has lead to the construction of the general theory of integrable system of hydrodynamic type. The seminal results were obtained by S. Tsarev in his PhD thesis (1985), in which the necessary and sufficient criteria of (local) integrability of hydrodynamic type systems were formulated and the method (the generalised hodograph transform) for their integration was proposed. Although originally formulated in geometric terms, the generalised hodograph method admits a simple hydrodynamic interpretation which may leave one wondering as to why this method had not been discovered some 150 years ago.

The generalised hodograph transform is a cornerstone of many modern theoretical works on integrable hydrodynamic type systems. At the same time, it has turned out to be a powerful tool for applications, especially in the theory of dispersive shock waves (DSWs) which are described by the Whitham modulation equations. In these notes I will make a brief introduction to the basic notions and concepts of the theory of one-dimensional hydrodynamic type systems from the perspective of an applied mathematician. A more detailed (and deep) description of various aspects of the theory can be found in the books and papers from the reference list (especially in [1]) although I am not aware of a single reasonably concise text covering the subject in the form accessible to a non-specialist.

2. HYDRODYNAMIC TYPE SYSTEMS: MAIN DEFINITIONS AND PROPERTIES

Systems of the first-order quasilinear partial differential equations (PDEs)

\begin{equation}
\dot{u}_i + A_{ij} u_j + B u_p + C u_q = f,
\end{equation}

where \( u(x,t) = \{ u^1, u^2, \ldots, u^N \} \in \mathbb{R}^N \), \( A \), \( B \) and \( C \) are \( N \times N \) matrices depending on \( u, x, t \) and \( f = f(u, x, t) \) is a given function, arise in the modelling of wave processes in continuum mechanics, plasma physics, magnetohydrodynamics etc.

In these notes we shall be interested in the homogeneous (\( f \equiv 0 \)) one-dimensional systems (2.1) with the coefficient matrix \( A \) not depending explicitly on \( x, t \).

**Definition 2.1** (Hydrodynamic type system [1]). A system of quasilinear equations

\begin{equation}
\dot{u}_i + A_{ij}(u) u_j = 0, \quad i, j = 1, \ldots, N
\end{equation}
is called a (one-dimensional) system of hydrodynamic type.

In (2.2) and below we assume the conventional summation rule w.r.t. repeated indices.

Example 2.1 (Shallow water equations)

\[ h_t + (hu)_x = 0; \]
\[ u_t + uu_x + h_x = 0. \]

System (2.3) is the hydrodynamic type system (2.2) with

\[ \begin{pmatrix} u^1 \\ u^2 \end{pmatrix} = \begin{pmatrix} h \\ u \end{pmatrix}, \quad A = \begin{pmatrix} u^2 & u^1 \\ 1 & u^2 \end{pmatrix} \]

Definition 2.2 (Hydrodynamic conservation law). If equation

\[ \frac{\partial P(u)}{\partial t} + \frac{\partial Q(u)}{\partial x} = 0 \]

is a consequence of the hydrodynamic type system (2.2) for any solution, then it is called a hydrodynamic conservation law of system (2.2).

Example 2.2. Equation \((hu)_t + (hu^2 + h^2/2)_x = 0\) is a hydrodynamic conservation law of the shallow water system (2.3).

Note: the number of independent conservation laws for a system (2.2) (if they exist) could be less, equal or greater than \(N\). E.g. the shallow water system (2.3) has an infinite number of conservation laws.

Definition 2.3 (Characteristic). The curve \(x(t)\) is called a characteristic of system (2.2) if

\[ \frac{dx}{dt} = V, \quad \det |A - VI| = 0, \]

where \(I\) is the \(N \times N\) identity matrix.

The eigenvalues \(V^\alpha\) are called the characteristic speeds. Importantly, since \(A = A(u)\) the characteristics depend on the solution \(u(x, t)\) of (2.2).

Let \(l^\alpha i\) be the left eigenvector of the matrix \(A\) defined for each eigenvalue \(V^\alpha\) by

\[ l^\alpha i (A - V^\alpha I) = 0, \quad i = 1, \ldots, n_\alpha, \]

where \(n_\alpha\) is the geometric multiplicity of the eigenvalue \(V^\alpha\). Let us consider a linear combination of the equations in the system (2.2) by multiplying it by the left eigenvector \(l^\alpha\).

Definition 2.4 (Characteristic relation). A consequence of the system (2.2)

\[ l^\alpha i \cdot (u_t + Au_x) = l^\alpha i \cdot (u_t + V^\alpha u_x) = 0, \]

is called a characteristic relation of the hydrodynamic type system (2.2).

Definition 2.5 (Hyperbolicity/Strict hyperbolicity). The system (2.2) is called hyperbolic if all the eigenvalues \(V^\alpha\) (the roots of the determinant equation in (2.6)) are real,

\[ V^1(u) \leq V^2(u) \leq \cdots \leq V^N(u), \]
and the eigenvectors $\mathbf{1}^\alpha_i$ form a basis in $\mathbb{R}^N$ (i.e. $\sum n_\alpha = N$). The system is strictly hyperbolic if all eigenvalues $V^\alpha$ are distinct (i.e. all $n_\alpha = 1$):

$$V^1(u) < V^2(u) < \cdots < V^N(u).$$

Corollary 2.1. For a hyperbolic system (2.2) there are $N$ independent characteristic relations (2.8) which form a system equivalent to (2.2) (the characteristic form of (2.2)). For a strictly hyperbolic system $\mathbf{1}^\alpha_i \equiv \mathbf{1}^\alpha$, $\alpha = 1, \ldots, N$.

The principal advantage of the characteristic form is that each equation in it contains differentiation only in a single (characteristic) direction of the $(x, t)$-plane, $\frac{d}{dx} = \frac{\partial}{\partial t} + V^\alpha \frac{\partial}{\partial x}$ so the PDE system (2.2) essentially transforms into a system of ODEs along the characteristic directions.

Definition 2.6 (Riemann invariants). Riemann invariants (if they exist) are the variables $r^j(u^1, u^2, \ldots, u^N)$ such that the system (2.2) assumes diagonal form

$$r^k_t + V^k(r) r^k_x = 0, \quad k = 1, \ldots, N.$$ (2.11)

Corollary 2.2. Any $r^j = \text{const}$ satisfies system (2.11) and, by $r^j = r^j(u^1, \ldots, u^N)$, specifies an algebraic relation between $N$ components of $u$ along $j$-th characteristic. This also implies principal availability of exact reductions for (2.11).

All statement regarding Riemann invariants below should be considered modulo their existence (which is not guaranteed!).

Remark 2.1. Each Riemann invariant $r^i$ is determined up to an arbitrary function of a single variable so that $F(r^i)$ is also a Riemann invariant for any differentiable function $F(x)$.

Remark 2.2. For non-strictly hyperbolic systems there could be several Riemann invariants associated with the same characteristic velocity $V^\alpha$. This can have non-trivial implications for the solutions of (2.2). From now on, unless explicitly specified, we shall be assuming that system (2.2) is strictly hyperbolic.

Existence of Riemann invariants. Comparing (2.8) and (2.11) one can see that the Riemann invariants exist if one can choose the left eigenvectors $\mathbf{1}^\alpha$ such that

$$l^\alpha_j = \frac{\partial r^\alpha}{\partial u^j}, \quad j = 1, \ldots, N$$ (2.12)

for every $\alpha = 1, \ldots, N$. Thus, essentially, the problem of the existence of Riemann invariants is the problem of the existence of the integrating factor $\mu(u)$ for the characteristic form of the system (2.2). Therefore, Riemann invariants always exist for $N = 2$ (Pfaff’s theorem on integrability of differential forms) but generally do not exist for $N \geq 3$.

Example 2.3 (Riemann form of the shallow water system). The shallow water system (2.3) can be represented in the Riemann form (2.11) with

$$r^{1,2} = u \pm 2\sqrt{h}, \quad V^{1,2} = u \pm \sqrt{h}.$$ (2.13)

Remark 2.3. Generally the number of Riemann invariants of a hyperbolic system of $N$ equations could be less than $N$. In that case, only part of system (2.2) assumes diagonal form.
The existence of Riemann invariants for strictly hyperbolic systems (2.2) for $N \geq 3$ can be established by evaluating the so-called Haantjes tensor \cite{16}, which is constructed from the elements of the coefficient matrix $A$ (see \cite{8}). Vanishing of the Haantjes tensor is a necessary and sufficient condition for diagonalisability of a hydrodynamic type system. While there is no general method for the actual construction of the Riemann invariants for a given non-diagonal system (2.2), there could be efficient methods for their computation in the systems having certain symmetry properties. For the important class of hydrodynamic type systems obtained by the Whitham averaging \cite{39} of nonlinear dispersive PDEs the existence of Riemann invariants was shown to be related to the integrability of the original equations via the inverse scattering transform (see \cite{9} for the corresponding KdV theory).

**Definition 2.7** (Genuine nonlinearity/Linear degeneracy). The characteristics family of the diagonal system (2.11) corresponding to the characteristic velocity $V^i$ is called genuinely nonlinear if

$$\partial_i V^i \neq 0,$$

where $\partial_i \equiv \frac{\partial}{\partial r_i}$.

If (2.14) holds for all $i$ then the system (2.11) is called genuinely nonlinear. The characteristic family $\frac{dr}{dt} = V^i$ is called linearly degenerate if $\partial_i V^i = 0$. The system (2.11) is called linearly degenerate if $\partial_i V^i = 0$ for all $i = 1, \ldots, N$.

**Example 2.4.** The shallow water system (2.3) is genuinely nonlinear. Indeed, from (2.13) we have $V_1 = \frac{3}{4} r_1 + \frac{1}{4} r_2$, $V_2 = \frac{3}{4} r_2 + \frac{1}{4} r_1$ which immediately implies $\partial_i V^i = \frac{3}{4} \neq 0$.

**Example 2.5.** The hydrodynamic type system $u_t + vu_x = 0$, $v_t + uv_x = 0$ is linearly degenerate. Actually any two-component linearly degenerate hydrodynamic type system can be reduced to this form.

**Example 2.6.** Equation $u_t + u^2 u_x = 0$ is genuinely nonlinear everywhere except in any neighbourhood of $u = 0$.

**Corollary 2.3.** The notions of genuine nonlinearity/linear degeneracy can be generalised to non-diagonal systems (2.2). The $k$-characteristic family of the system (2.2) is genuinely nonlinear if \cite{22}

$$\frac{\partial V^k}{\partial u} \cdot p^k \neq 0,$$

where $p^k$ is the right eigenvector of the matrix $A$ corresponding to the eigenvalue $V^k$.

**Exercise 2.1.** Show that the definitions (2.14) and (2.15) are equivalent for diagonalisable systems (2.2).

Obviously, the negation of (2.15) represents the condition of linear degeneracy of the $k$-characteristic family etc.

**Remark 2.4.** Similar to non-strict hyperbolicity, linear degeneracy of several or all characteristic families can have important implications for solutions of (2.2).
3. Integration of hyperbolic systems

We assume that system (2.2) is strictly hyperbolic, genuinely nonlinear and diagonalisable. Thus there are \( N \) Riemann invariants \( r_j(u^1, \ldots, u^N), j = 1, \ldots, N \) so that the system assumes the form (2.11), and we also have \( \partial_i V^i \neq 0 \) for all \( i = 1, \ldots, N \).

3.1. Simple waves.

Let \( r^j = r^j_0 = \text{const}, \forall j \neq k \) in (2.11). Then for \( r^k(x, t) \) we obtain the equation
\[
\frac{\partial}{\partial t} r^k + V^k(r^k) \frac{\partial}{\partial x} r^k = 0, \quad V'(r) \neq 0,
\]
where \( r \equiv r^k \) and \( V(r) \equiv V^k(r^k) \). Obviously, there are \( N \) simple-wave reductions of the diagonal system (2.11). Equation (3.1) can be obtained directly from the original non-diagonal form (2.2) of the hydrodynamic type system by setting \( r \equiv u^i \), and looking for a solution of the system in the form \( u^i = u^i(r) \) for all \( i \) (the simple-wave ansatz).

Exercise 3.1. Without assuming the existence of the Riemann invariant form show that the simple-wave reduction (3.1) of the system (2.2) exists only if the corresponding characteristic family is genuinely nonlinear (see (2.15)).

The general solution of (3.1) has the form
\[
x - V(r)t = W(r),
\]
where \( W(r) \) is an arbitrary function. If one solves the Cauchy problem for (3.1) with the initial condition \( r(x, 0) = r_0(x) \), the function \( W(r) \) has the meaning of the inverse function to the initial profile \( r_0(x) \) (provided such an inverse exists). Solution (3.2) can be readily obtained by the integration of (3.1) along the characteristic \( dx/dt = V(r) \) (see e.g. [39]) so it is often called the characteristic solution. Generally, solution (3.2) implies occurrence of the gradient catastrophe, \( |r_x| \to \infty \) for some \( t = t_b \) and thus, does not exist globally. Here, however, our main concern will be with local integrability of the hydrodynamic type systems, which for the simple-wave equation (3.1) is realised by the explicit construction of the general solution (3.2).

3.2. Interaction of simple waves: hodograph solution.

Now we relax the simple-wave constraint and consider a more general reduction of the diagonal system (2.11) in which all but two of the Riemann invariants are constant. W.l.o.g. we assume that the two varying invariants are \( r^1 \) and \( r^2 \) and consider the system of two equations
\[
\begin{align*}
\frac{\partial}{\partial t} r^1 + V^1(r^1, r^2) \frac{\partial}{\partial x} r^1 &= 0, \\
\frac{\partial}{\partial t} r^2 + V^2(r^1, r^2) \frac{\partial}{\partial x} r^2 &= 0,
\end{align*}
\]
where \( V^1 \neq V^2 \) (strict hyperbolicity). System (3.3) can be naturally interpreted in terms of the interaction of two simple waves. To integrate this system we take advantage of the classical hodograph transform (see e.g. [39], [32]) which is achieved by interchanging the roles of the dependent and independent variables. To this end we consider \( x = x(r^1, r^2), t = t(r^1, r^2) \) and compute the derivatives
\[
\begin{align*}
\partial_1 x &= \frac{1}{\partial t^2} r^1, & \partial_2 x &= -\frac{1}{\partial t^2} r^1, & \partial_1 t &= -\frac{1}{\partial t^2} r^2, & \partial_2 t &= \frac{1}{\partial t^2} r^2,
\end{align*}
\]
where $J$ is the Jacobian: $J = r^1_t r^2_x - r^1_x r^2_t$. We assume that $J \neq 0$ (important!). Then, owing to the absence of undifferentiated terms, the hydrodynamic type system (3.3) transforms into the system of two linear (hodograph) PDEs:

$$\partial_t x - V^2(r^1, r^2) \partial_t t = 0, \quad \partial_t x - V^1(r^1, r^2) \partial_t t = 0. \tag{3.5}$$

Thus we have reduced the problem of integration of the system of quasilinear PDEs to integrating the system of linear PDEs (e.g. by separation of variables or other methods). An essential part of constructing the solution to the original system (3.3) is the inversion of the hodograph solution $x(r^1, r^2)$, $t(r^1, r^2)$ which is not always possible (e.g. in the vicinity of the wave-breaking points). Indeed, by solving (3.5) and inverting the hodograph solution $x(r^1, r^2), t(r^1, r^2)$ one generally obtains only a local solution of (3.3). However, the fact that any smooth non-constant local solution of (3.3) can be obtained in this way constitutes integrability of system (3.3) via the hodograph transform.

**Remark 3.1.** The simple-wave solution cannot be obtained by the classical hodograph transform due to degeneracy of the mapping $(x, t) \mapsto (r^1, r^2)$ (and thus, vanishing of the Jacobian $J$).

The hodograph method is known to be poorly compatible with the Cauchy problem for (3.3) (see e.g. [39]) so it has not been much used in classical fluid dynamics. It has turned out, however, that it is ideally compatible with the nonlinear free boundary problems arising in the theory of dispersive shock waves.

Below we present a somewhat modernised version of the hodograph method [1] which will later be generalised to the systems with the number of components greater than two. To this end we introduce in (3.5) new ‘characteristic’ dependent variables $W^{1,2}(r^1, r^2)$ instead of $x$ and $t$:

$$W^{1,2}(r) = x - V^{1,2}(r)t, \tag{3.6}$$

so that (3.5) assumes the form

$$\frac{\partial W^2}{W^2 - W^1} = \frac{\partial V^2}{V^2 - V^1}, \quad \frac{\partial W^1}{W^2 - W^1} = \frac{\partial V^1}{V^2 - V^1}. \tag{3.7}$$

**Proposition 3.1.** Any smooth non-constant local solution of system (3.3) can be obtained via (3.7), (3.6).

Note that the hodograph solution in the form (3.6) represents a natural generalisation of the simple-wave characteristic solution (3.2), the crucial difference being that in (3.6), unlike in (3.2), $W^{1,2}$ are not arbitrary functions but must satisfy linear PDEs (3.7). If, for instance, $\partial_t V^2 = \partial_t V^1$ the system (3.7) can be reduced to a single second-order linear PDE. For the equations of isentropic gas dynamics with the polytropic pressure law $P \sim \rho^\gamma$ the resulting second-order hodograph PDE is the Euler-Poisson-Darboux equation (see e.g. [32]).

**Proposition 3.2.** A hydrodynamic type system

$$r^1_t + W^1(r)r^1_x = 0, \quad r^2_t + W^2(r)r^2_x = 0, \tag{3.8}$$

where $\tau$ is the new ‘time’ (i.e., now $r^i = r^i(x, t, \tau)$) and $W^{1,2}(r)$ are solutions of (3.7), determines a symmetry (commuting flow) of system (3.3) so that $r^i_{\tau \tau} = r^i_{\tau \tau}$,

$i = 1, 2$. All the hydrodynamic symmetries are obtainable in this way.

Proposition (3.2) is readily proved by a direct computation of the mixed derivatives of $r^i$ using (2.11) and (3.8).
4. Integrability and the Generalised Hodograph Transform

So what happens when \( N \geq 3 \)? Clearly, the classical hodograph construction outlined in the previous section is not applicable since the mapping \((x,t) \mapsto (r^1, r^2, \ldots, r^N)\) is no longer one-to-one.

A remarkable discovery made by Tsarev in 1985 was that the hodograph construction in the symmetrised form (3.5), (3.7) is still applicable to diagonal hydrodynamic type systems (2.11) with \( N \geq 3 \) modulo one important constraint: for the system to be integrable its characteristic speeds \( V_j(r) \) must satisfy the identity

\[
\partial_j \frac{\partial_i V^i}{V^k - V^j} = \partial_k \frac{\partial_i V^i}{V^j - V^i}, \quad i \neq j \neq k
\]

for each three distinct characteristic speeds. The hydrodynamic type systems satisfying (4.1) are called semi-Hamiltonian.

A semi-Hamiltonian hydrodynamic type system possesses infinitely many conservation laws parameterised by \( N \) arbitrary functions of a single variable. Its general local solution for \( \partial_x r^i \neq 0, \ i = 1, \ldots, N \) is given by the generalised hodograph formula [34]

\[
x - V^i(r)t = W^i(r), \quad i = 1, 2, \ldots, N,
\]

where functions \( W^i(r) \) are found from the linear system of PDEs:

\[
\frac{\partial_i W^j}{W^i - W^j} = \frac{\partial_i V^j}{V^i - V^j}, \quad i, j = 1, \ldots, N, \quad i \neq j.
\]

One can see that the system of linear PDEs (4.3) is overdetermined for \( N \geq 3 \). The semi-Hamiltonian property (4.1) is nothing but the condition of compatibility of system (4.1) and thus, provides the criterion of integrability of diagonal hydrodynamic type system (2.11) in the above generalised hodograph sense. Remarkably, the proof of Tsarev’s theorem stating that any smooth solution of the diagonal system (2.11) is locally obtainable via (4.2), (4.3), is rather straightforward (see [1]) and does not involve any sophisticated mathematical techniques. Note that for \( N = 2 \) the conditions (4.1) do not exist so that any 2 × 2 diagonal system is integrable. Also, similar to the case \( N = 2 \), it can be proved that any solution \( W^i(r) \) of system (4.3) determines the symmetry

\[
r^i_t + W^i(r)r^i_x = 0
\]

of the initial system (2.11), i.e. (2.11) and (4.4) commute, i.e. \( r^i_t = r^i_{tt} \). One of the applications of this property will be outlined in the next section.

**Remark 4.1.** The original hypothesis by Novikov was that, if a hyperbolic hydrodynamic type system is: a) Hamiltonian, and b) diagonalisable, then it is integrable. This hypothesis was proved by Tsarev but it has turned out that the requirement for the system to be Hamiltonian is too restrictive and a milder, semi-Hamiltonian, condition (4.1) is actually required.

**Remark 4.2.** [Theorem (attributed to Tsarev)] The semi-Hamiltonian condition (4.1) can be replaced by the existence of \( N \) independent hydrodynamic conservation laws (2.5), i.e. if a hydrodynamic type system (2.2) has \( N \) Riemann invariants and \( N \) hydrodynamic conservation laws, then it is semi-Hamiltonian i.e. integrable.

**Remark 4.3.** The theory of semi-Hamiltonian linearly degenerate (see Def. 2.7) systems of hydrodynamic type was developed in [28], [6]. In particular, the following
useful theorem holds [28]: If a semi-Hamiltonian system (2.11) possesses $N$ conservation laws $u_i^t = (u^iv^j)_x$ such that $v_i(u(r)) = V^i(r)$, $i = 1, \ldots, N$, then this system is linearly degenerate.

**Remark 4.4.** While the theory of linearly degenerate integrable systems is well developed, almost nothing is known in general about integrability of the systems which are not strictly hyperbolic (i.e. have multiple characteristic speeds, see Def. 2.5).

5. **Application: Dispersive resolution of the ‘cubic’ nonlinear wave-breaking**

As was mentioned in the Introduction, the development of the integrability theory for hydrodynamic type system was initially motivated by the interest in the Whitham modulation equations and, in particular, in the analytical description of dispersive shock waves (DSWs). The study of DSWs using the Whitham equations was initiated by Gurevich and Pitaevskii [15] who considered two generic problems in the framework of the KdV equation: dispersive resolution of an initial discontinuity (the dispersive Riemann problem) and dispersive resolution of the universal ‘cubic’ gradient catastrophe. Gurevich and Pitaevskii constructed analytic solution of the former problem while the latter one was solved by Potemin [31] using Krichever’s algebro-geometrical approach [20]. In this approach, instead of integrating the Tsarev equations (4.3) for $W_j$’s, one finds the particular family of the averaged commuting flows (4.4) associated with the ‘higher’ KdV equations in the isospectral hierarchy. In this section, we obtain Potemin’s solution by a direct application of the generalised hodograph transform (4.2) and integrating Tsarev’s equations (4.3) (see e.g. [19]). Importantly, the solution of this problem makes an essential use of the integrability of the hydrodynamic-type KdV-Whitham modulation system.

5.1. **Problem formulation and pre-breaking dynamics.**

Consider the initial-value problem for the KdV equation:

$$ u_t + 6uu_x + \epsilon^2 u_{xxx} = 0, \quad u(x, 0) = u_0(x), \quad x \in \mathbb{R}, \quad t > 0, $$

where $u_0(x)$ is a sufficiently smooth decreasing function, $u'_0(x) < 0$, $|u'_0(x)| = O(1)$, with a single inflection point; and $\epsilon \ll 1$.

Initially, for some interval $0 < t < t_b$ one can neglect the dispersive term in (5.1) and approximate the KdV solution by the solution of the Riemann-Hopf (dispersionless KdV) equation $u_t + 6uu_x = 0$ with the same initial condition $u_0(x)$ (this is, actually, a rigorous result [23]):

$$ x - 6ut = W(u), $$

where $W(u)$ is the inverse function of $u_0(x)$ (see (3.2)).

The nonlinear evolution $u(x, t)$ specified by (5.2) leads to the steepening of the profile of $u(x)$ and eventually, to a gradient catastrophe: $|u_x| \to \infty$ at $(x_b, t_b, u_b)$, where $W''(u_b) = 0$, $t_b = -\frac{1}{6}W'(u_b)$ and $x_b = 6u_b t_b + W(u_b)$ (see e.g. [39]):

By making the shift $t \to t - t_b$, $u \to u - u_b$ and passing to the reference frame $x \to x - u_b t$ one can always set the wave-breaking point at $x_b = t_b = u_b = 0$. Then expanding the solution (5.2) near this point (note that after the rescaling $W(0) = W''(0) = 0$ and assuming $W'''(u) \neq 0$ we obtain, on retaining the first nonvanishing
term and applying convenient (invariant) rescaling, the universal "cubic" behaviour of the simple wave near the point of the gradient catastrophe, i.e. for $|t| \ll 1$ (see e.g. [30]):

$$x - 6ut = -u^3,$$

One can readily show that the curve $u(x,t)$ defined by (5.3) is single-valued if $u_x(0, t) < 0$, which happens for $t < 0$ (pre-breaking). In contrast, $u(x,t)$ is triple-valued for $t > 0$. This function is the solution to the Cauchy problem for the Riemann-Hopf equation:

$$u_t + 6uu_x = 0, \quad u(x,0) = -x^{1/3}.$$  

The IVP (5.4) is invariant with respect to the scaling

$$u \rightarrow Cu, \quad t \rightarrow C^2 t \quad x \rightarrow C^3 x \quad \text{where} \quad C = \text{const},$$

which implies that $u(x,t)$ specified by (5.3) is a ‘generalised similarity’ solution:

$$u = t^{1/2}U(\zeta), \quad \zeta = xt^{-3/2}$$

valid for $t > 0$. Substituting (5.6) into (5.3) we obtain a cubic equation for $U(\zeta)$:

$$U^3 - 6U + \zeta = 0,$$

which has three roots for $\zeta > 0$. The three-valued solution $u(x,t)$ is, however, non-physical, so we have to include the dispersive term in (5.1) to regularise the gradient catastrophe at $t = t_b$ and prevent the wave-breaking. Now, having established the universal cubic behaviour near the wave-breaking point, we shall be interested in the solution of the full KdV IVP (5.1) with $u_0(x) = -x^{1/3}$ (but now without the restriction $|t| \ll 1$).

5.2. Post-breaking dynamics and the KdV-Whitham equations.

For $t > t_b$ the dispersive regularisation of the wave-breaking singularity occurs via the generation of an expanding nonlinear wavetrain – a dispersive shock wave (DSW) [18]. The DSW is ‘built into’ the evolving profile (5.3) so that outside the DSW the solution (5.3) is single-valued. This does not imply, however, that the DSW edges coincide with the edges of the three-valued formal solution of (5.3) at $t > t_b$. The DSW is asymptotically described by the modulated KdV travelling

![Figure 1](image_url)

**Figure 1.** The DSW forming due to the dispersive regularisation of the ‘cubic’ gradient catastrophe in the KdV equation. Dashed line: the modulation solution $r_3(x,t) \geq r_2(x,t) \geq r_1(x,t)$, which does not coincide with the formal three-valued solution $u(x,t)$ (5.3) of the Riemann-Hopf equation after the wave-breaking.
wave (cnoidal) solution which locally (i.e. on the scale $\Delta x, \Delta t = O(\epsilon)$), is described by the formula
\begin{equation}
(5.8) \quad u(x, t) = r_1 + r_2 - r_3 + 2(r_2 - r_1) \operatorname{cn}^2[2\epsilon^{-1}(r_3 - r_1)^{1/2}(x - ct - x_0)]m, \nonumber
\end{equation}
where $\operatorname{cn}(\xi|m)$ is the Jacobi elliptic cosine, $r_1 \leq r_2 \leq r_3$ are parameters, so that
the phase velocity $c$ and the modulus $m$ are
\begin{equation}
(5.9) \quad c = 2(r_1 + r_2 + r_3), \quad m = \frac{r_2^2 - r_1^2}{r_3^2 - r_1^2}, \nonumber
\end{equation}
and $x_0$ is an arbitrary initial phase. Parameters $r_i$ vary slowly ($\Delta x$, $\Delta t = O(1)$) throughout the rapidly oscillating DSW region. At the leading edge the modulus $m = 1$ so that the cnoidal solution transforms into a soliton, while at the trailing edge $m = 0$ and the wave degenerates into the vanishing amplitude linear wave (see Fig. 1). The slow evolution of $r_i$’s is governed by the Whitham modulation equations, which, for the KdV equation, have diagonal (Riemann) form \[38\]
\begin{equation}
(5.10) \quad r_k t + V_k(r) r_k x = 0, \quad k = 1, 2, 3. \nonumber
\end{equation}
The characteristic speeds are expressed in terms of the complete elliptic integrals of the first and second kinds and may be compactly represented as [13]
\begin{equation}
(5.11) \quad V^i = (1 - \frac{L}{\partial L})c, \nonumber
\end{equation}
where
\begin{equation}
(5.12) \quad L = \int_{r_1}^{r_2} \frac{d\lambda}{\sqrt{(\lambda - r_1)(r_2 - \lambda)(r_2 - r_1)}} = \frac{2K(m)}{(r_2^3 - r_1^3)^{1/2}}, \nonumber
\end{equation}
$K(m)$ being the complete elliptic integral of the first kind. Using the representation (5.11), (5.12) for the characteristic speeds and definitions 2.5 and 2.7 it is not difficult to show that

**Proposition 5.1.** The KdV-Whitham system (5.10), (5.11) is: (i) strictly hyperbolic and (ii) genuinely nonlinear.

**Remark 5.1.** System (5.10) is obtained by a single-phase averaging of the KdV equation (5.1) over the family of periodic solutions (5.8). Solution (5.8) is a simplest representative of a remarkable class of multiphase (multi-periodic) KdV solutions which are finite-gap potentials of the associated linear Schrödinger operator in the KdV Lax pair (see e.g. [27]). The Whitham equations obtained by the $g$-phase ($g$-gap) averaging of the KdV equation can be represented in the diagonal form with the Riemann invariants $r_1, r_2, \ldots, r_{2g+1}$ being the endpoints of spectral gaps – a highly nontrivial result obtained by Flaschka, Forest and McLaughlin in [9]. The “multi-gap” KdV-Whitham equations are also strictly hyperbolic and genuinely nonlinear [26].

**Remark 5.2.** It was shown by Lax and Levermore [23] and Venakides [36] that the Whitham equations also describe the evolution of certain weak limits of the KdV solution in the semi-classical, zero dispersion ($\epsilon \to 0$) limit for $t > t_b$. As a matter of fact, for $t > t_b$, the limit $\lim_{\epsilon \to 0} u$ does not coincide with the solution $u(x, t)$ of the “classical”, dispersionless ($\epsilon \equiv 0$) limit of the KdV equation, see (5.2). The link between the Lax-Levermore-Venakides and Flaschka-Forest-McLaughlin results was established in [37].
5.3. The generalised hodograph solution.

To obtain the solution \( r^i(x, t), \ i = 1, 2, 3 \) describing the modulation of the periodic wave (5.8) in a DSW one should generally solve certain nonlinear free-boundary problem for the Whitham equations (5.10) so that the mean (averaged over the period) value \( \overline{u}(x, t) \) of the travelling wave solution (5.8)) would continuously match with the ‘external’ smooth solution \( u(x, t) \) (5.2) at the trailing \((m = 0)\) and leading \((m = 1)\) DSW edges (see [15], [2]). However, in the particular case of the cubic wave-breaking described by the generalised similarity solution (5.6) one can take advantage of the invariance of the KdV-Whitham equations (5.10), (5.11) with respect to the same transformation (5.5). Indeed, the characteristic speeds \( V^i(\mathbf{r}) \) (5.11) are homogeneous functions of \( r^1, r^2, r^3 \), i.e. \( V^i(\mathbf{C}) = C^{\alpha\nu}V^i(\mathbf{r}) \), with the index of homogeneity \( \alpha = 1 \). Then turning to the generalised hodograph solution (4.2), (4.3) and the necessity to match the functions \( W^i(\mathbf{r}) \) with \( W(u) = -u^3 \) in (5.3) outside the DSW region one can conclude that we are interested in the homogeneous solutions \( W^i(\mathbf{r}) \) of the Tsarev equations (4.3) with the index of homogeneity \( \alpha = 3 \).

The homogeneous solutions of the Tsarev equations (4.3) are most easily found using an additional transformation motivated by the representation (5.11) for the characteristic speeds [21], [13], [33]:

\[
W_i = (1 - \frac{L}{\partial_x L} \partial_t) g, \quad i = 1, 2, 3,
\]

reducing the system (4.3) to the system of the classical Euler-Poisson-Darboux (EPD) equations for the unknown (scalar) function \( g(\mathbf{r}) \)

\[
2(r_i - r_j) \frac{\partial^2 g}{\partial r_i \partial r_j} = \frac{\partial g}{\partial r_i} - \frac{\partial g}{\partial r_j}, \quad i, j = 1, \ldots, 3, \quad i \neq j.
\]

It is not difficult to show that the overdetermined system (5.14) is compatible.

One can see by direct verification that the function

\[
G(\lambda, r_1, r_2, r_3) = \frac{\phi(\lambda)}{\sqrt{(\lambda - r_1)(\lambda - r_2)(\lambda - r_3)}},
\]

where \( \phi(\lambda) \) is an arbitrary function, satisfies the EPD system (5.14) identically (i.e. for any \( \lambda \)). Thus \( G(\lambda, r_1, r_2, r_3) \) is the generating function for the solutions of the EPD system and, therefore, via (4.2), (5.13), for the (local) solutions of the KdV-Whitham system. Here we are interested in the particular homogeneous solutions \( W(\mathbf{r}) \) with the index of homogeneity \( \alpha = 3 \).

Choosing \( \phi(\lambda) = \lambda^{3/2} \) and expanding the generating function \( G(\lambda, r_1, r_2, r_3) \) for \( \lambda \gg 1 \) we obtain

\[
G = 1 + \frac{g_1}{\lambda} + \frac{g_2}{\lambda^2} + \ldots,
\]

where

\[
g_1 = 2s_1, \quad g_2 = 6s_1^2 - 8s_2, \quad g_3 = 5s_1^3 - 12s_1s_2 + 8s_3, \quad \ldots,
\]

and

\[
s_1 = r^1 + r^2 + r^3, \quad s_2 = r^1r^2 + r^1r^3 + r^2r^3, \quad s_3 = r^1r^2r^3, \quad \ldots
\]

are the symmetric polynomials. Each \( g_\alpha(\mathbf{r}) \), \( \alpha = 1, 2, 3, \ldots \) is a symmetric homogeneous function of \( r_1, r_2, r_3 \) with the homogeneity index \( \alpha \). The hodograph
modulation solutions generated by $g_3$ have the form

\[(5.19) \quad x - V^i(r)t = (1 - {L \over \partial L} \partial_i)(Cg_3), \quad i = 1, 2, 3,\]

where $C$ is an arbitrary constant (due to linearity of the EPD equation). The value of this constant is found from the condition of the continuous matching with the solution (5.3) at the edges of the DSW, which requires that solutions (5.19) and (5.3) must agree when $r^1 = r^2 = r^3 \equiv u$ [33]. Then one readily finds that $C = -1/35$. Since the functions $W^i(r)$ in the right-hand side of (5.19) are homogeneous with the index $\alpha = 3$ the modulation solution $r^i(x, t)$, $i = 1, 2, 3$ specified by (5.19) is self-similar, $r^i = t^{-1/2}R^i(x/t^{3/2})$ and so is defined for all $t > 0$ (cf. (5.4) – (5.6)). It describes a multivalued curve $\{r^1(x, t), r_2(x, t), r^3(x, t)\}$ (see Fig. 1) defined within the expanding interval $[x^-, x^+)$, where

\[(5.20) \quad x^- = -12\sqrt{3}t^{3/2} \quad \text{(trailing edge),} \quad x^+ = \frac{4}{9}\sqrt{15}t^{3/2} \quad \text{(leading edge).}\]

We stress one more time that the triple-valued curve $\{r^1(x, t), r_2(x, t), r^3(x, t)\}$ does not coincide with the formal triple-valued solution (5.3) of the Riemann hoph equation for $t > 0$.

Substituting the modulation $r^j(x, t)$ defined by (5.19) into the ‘carrier’ travelling wave solution (5.8) one obtains an asymptotic solution describing the DSW. We note that in the DSW solution the initial phase $x_0$ in (5.8) also becomes a slowly varying function, $x_0(x, t)$ (the modulation phase shift, see, e.g., [3]). It can be shown that $x_0(x, t) = -{4\sqrt{3} \over 9}g_3$.

At the trailing edge $x^- (t)$ of the DSW one has $r^1 = r^2$ (i.e. $m = 1$, see (5.9)) so that the solution (5.8) degenerates into the vanishing amplitude linear wave, while at the leading edge $x^+ (t)$ one has $r_2 = r_3$ ($m = 1$) and it transforms into a solitary wave (see Fig. 1).

**Remark 5.3.** The generalised hodograph transform enables one, in principle, to obtain the modulation solutions for the KdV equation with reasonably arbitrary initial data (see [14], [33], [5], [12]).

### 6. Concluding remarks

The theory of hydrodynamic type systems is a rich, diverse and influential area of modern applied mathematics and mathematical physics. In these brief notes, the main attention was paid to one-dimensional integrable systems of hydrodynamic type and their application to the description of dispersive shock waves. Needless to say that there are many other important directions and applications of this theory. E.g. nothing or almost nothing has been said about the theory of classical and non-classical shocks (see [22], [24]), Hamiltonian theory of hydrodynamic type systems [1] and dispersionless Lax pairs [11].

Below we list several directions of the recent development in the theory of integrable hydrodynamic type systems. As a matter of fact, this list (as well as the list of references) is far from being complete and is heavily influenced by the personal research interests of the author.

- Generalisation of the notion of integrability of $(1 + 1)$ hydrodynamic type systems to multidimensions [40], [7], infinite-component systems (hydrodynamic chains) [29] and kinetic equations [10], [4]:
• Description of the wave-breaking for integrable 2+1 hydrodynamic type systems [25];
• Modulation theory of DSWs for the Kadomtsev-Petviashvili equation. This is an outstanding problem;
• Integrability of non-strictly hyperbolic systems (another outstanding problem).

REFERENCES

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