On strongly interacting internal waves in a rotating ocean and coupled Ostrovsky equations

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(Dated: 14 May 2013)

In the weakly nonlinear limit, oceanic internal solitary waves for a single linear long wave mode are described by the KdV equation, extended to the Ostrovsky equation in the presence of background rotation. In this paper we consider the scenario when two different linear long wave modes have nearly coincident phase speeds, and show that then the appropriate model is a system of two coupled Ostrovsky equations. These are systematically derived for a density-stratified ocean. Some preliminary numerical simulations are reported which show that, in the generic case, initial solitary-like waves are destroyed and replaced by two coupled nonlinear wave packets, being the counterpart of the same phenomenon in the single Ostrovsky equation.
I. INTRODUCTION

Internal solitary waves commonly occur in the coastal ocean, marginal seas and fjords. It is now widely accepted that the canonical model for their description is the Korteweg-de Vries (KdV) equation, see the reviews by Grimshaw \(^1\) and Helfrich and Melville \(^2\) for instance. When expressed in a reference frame moving with the linear long wave speed \(c\), it is

\[
A_t + \nu A A_x + \lambda A_{xxx} = 0. \tag{1}
\]

Here \(A(x, t)\) is the amplitude of the linear long wave mode \(\phi(z)\) corresponding to the linear long wave phase speed \(c\), which is determined from the modal equation

\[
(\rho_0 W^2 \phi_z)_z + \rho_0 N^2 \phi = 0, \tag{2}
\]

\[
\phi = 0 \quad \text{at} \quad z = -h, \quad \text{and} \quad W^2 \phi_z = g \phi \quad \text{at} \quad z = 0. \tag{3}
\]

Here \(\rho_0(z)\) is the stable background density stratification, \(\rho_0 N^2 = -g \rho_0 z\), \(W = c - u_0\) where \(u_0(z)\) is the background shear flow, and it is assumed there are no critical levels, that is \(W \neq 0\) for any \(z\) in the flow domain. The coefficients \(\nu\) and \(\lambda\) are given by

\[
I\nu = 3 \int_{-h}^{0} \rho_0 W^2 \phi_z^3 \, dz, \quad I\lambda = \int_{-h}^{0} \rho_0 W^2 \phi^2 \, dz, \quad I = 2 \int_{-h}^{0} \rho_0 W \phi_z^2 \, dz. \tag{4}
\]

However, oceanic internal waves are often observed to exist for several inertial periods, and hence the effect of the Earth’s background rotation needs to be taken into account. Thus, although the effect of background rotation is small for an individual wave, the effect can be significant for the wave evolution. Indeed, recently Helfrich \(^3\), Grimshaw and Helfrich \(^4\) and Grimshaw and Helfrich \(^5\), have shown that the long-time effect of rotation is the destruction of the initial internal solitary wave by the radiation of small-amplitude inertia-gravity waves, and the eventual emergence of a coherent steadily propagating nonlinear wave packet. The simplest model equation which takes account of background rotation is the Ostrovsky equation, which is an adaptation of the KdV equation (1) given by, see Ostrovsky \(^6\), Grimshaw \(^7\), Grimshaw et al \(^8\) and Grimshaw et al \(^9\),

\[
\{ A_t + \nu A A_x + \lambda A_{xxx} \}_x = \gamma A. \tag{5}
\]

The background rotation is represented by the coefficient \(\gamma\), which in the absence of a shear flow is given by

\[
\gamma = \frac{f^2}{2c}. \tag{6}
\]
where $f$ is the Coriolis parameter. For oceanic internal waves $\lambda \gamma > 0$, see (4) and (51), and then it is known that equation (5) does not support steady solitary wave solutions (see Grimshaw and Helfrich\textsuperscript{5} and the references therein). Although the additional term on the right-hand side of (5) is a linear long-wave perturbation to the KdV equation, it has the effect of removing the spectral gap on which solitary waves exist for the KdV equation. Indeed, the linear dispersion relation of the Ostrovsky equation (5) for the phase velocity $c_p$ and the group velocity $c_g$ as a function of wavenumber $k$ are given by

$$c_p = \frac{\gamma}{k^2} - \lambda k^2, \quad c_g = \frac{d\omega}{dk} = -\frac{\gamma}{k^2} - 3\lambda k^2.$$  \hspace{1cm} (7)

For the KdV equation (1) ($\gamma = 0$) there is a gap in the spectrum for all $c_p > 0$ where solitary waves can exist. But there is no such gap for the Ostrovsky equation, and hence no solitary waves are expected to occur since $\gamma \lambda > 0$ for internal waves. In the opposite case when $\gamma \lambda < 0$, which can arise in other physical applications, solitary waves do exist, see Obregon and Stepanyants\textsuperscript{10}.

The derivation of the KdV equation (1) and the Ostrovsky equation (5) assume that only a single long wave mode is operative. Eckart\textsuperscript{11} first called attention to the phenomena that for internal waves it is possible for the phase speeds of different modes to be nearly coincident, and mentioned that there will then be a resonant transfer of energy between the waves. In this scenario, the KdV equation is replaced by two coupled KdV equations, see Gear and Grimshaw\textsuperscript{12} and Grimshaw\textsuperscript{13}, describing a strong interaction between internal solitary waves of different modes. Various families of solitary waves can be expected from coupled KdV equations such as pure solitary waves, generalised solitary waves and envelope solitary waves depending on the structure of the linear dispersion relation, see the review by Grimshaw\textsuperscript{13}. In this paper we extend the derivation of the coupled KdV equations to take account of background rotation, and also a background shear flow. We find that then the single Ostrovsky equation (5) is replaced by two coupled Ostrovsky equations, each equation having both linear and nonlinear coupling terms.

The structure of this paper is as follows. In section II A, II B we demonstrate how a pair of coupled Ostrovsky equations can be systematically derived from the complete set of equations of motion for an inviscid, incompressible, density stratified fluid with boundary conditions appropriate to an oceanic situation, using the asymptotic multiple-scales expansion method. In sections III, we analyse the linear dispersion relation for this coupled
system, and based on that analysis, we present some preliminary numerical simulations in section IV using a pseudo-spectral method, described in Appendix A 2. Using parameters based on a three-layer model of the oceanic stratification, described in section II C, we show that typically initial solitary-like waves in the coupled system are destroyed, and replaced by nonlinear envelope wave packets, a two-component counterpart of the outcome for the single Ostrovsky equation (5). We add some discussions and concluding remarks in the last section.

II. FORMULATION

A. Asymptotic derivation

We consider two-dimensional flow of an inviscid, incompressible fluid on a rotating $f$-plane. In the basic state the fluid has a density stratification $\rho_0(z)$, a corresponding pressure $p_0(z)$ such that $p_0 = -g\rho_0$ and a horizontal shear flow $u_0(z)$ in the $x$-direction, see Figure 1. When $u_0 \neq 0$, this basic state is maintained by a body force. Then the equations of motion relative to this basic state are,

$$\rho_0(u_t + u_0 u_x + wu_{0z}) + p_z = -\rho_0(\rho_0 + \rho)(uw_x + wu_z - fv) - \rho(u_t + u_0 u_x + wu_{0z}),$$

$$\rho_0(v_t + u_0 v_x + fu) + \rho fu_0 = -(\rho_0 + \rho)(uw_x + wv_z) - \rho(v_t + u_0 v_x) - \rho fu,$$

$$p_z + g\rho = -(\rho_0 + \rho)(w_t + (u_0 + u)w_x + wv_z),$$

$$g(\rho_t + u_0 \rho_x) - \rho_0 N^2 w = -g(u \rho_x + w \rho_z),$$
\[ u_x + w_z = 0. \] 

(12)

Here, the terms \((u_0 + u, w)\) are the velocity components in the \((x, z)\) directions, \(\rho_0 + \rho\) is the density, \(p_0 + p\) is the pressure, \(t\) is time, \(N(z)\) is the buoyancy frequency, defined by \(\rho_0 N^2 = -g\rho_0 z\) and \(f\) is the Coriolis frequency. The boundary conditions to the above problem are given by

\[ w = 0, \quad \text{at} \quad z = -h, \] 

(13)

\[ p_0 + p = 0, \quad \text{at} \quad z = \eta, \] 

(14)

\[ \eta_t + (u_0 + u)\eta_x = w, \quad \text{at} \quad z = \eta. \] 

(15)

The constant \(h\) gives the undisturbed depth of the fluid, and \(\eta\) gives the displacement of the free surface from its undisturbed position \(z = 0\). In our subsequent derivation, we use a new variable \(\zeta\) as the vertical particle displacement which is related to the vertical speed, \(w\). It is defined by the equation

\[ \zeta_t + (u_0 + u)\zeta_x + w\zeta_z = w, \] 

(16)

and satisfies the boundary condition

\[ \zeta = \eta \quad \text{at} \quad z = \eta. \] 

(17)

The derivation of coupled Ostrovsky equations follows a similar strategy to the derivation of coupled KdV equations (see Grimshaw\(^\text{13}\) and the references therein). Grimshaw\(^\text{13}\) used a Lagrangian formulation, but here we use an equivalent Eulerian formulation.

First we note that at the leading linear long wave order, and in the absence of any rotation, the solution for \(\zeta\) is given by an expression of the form \(A(x - ct)\phi(z)\) where the modal function is given by (2), (3). In general there is an infinite set of solutions for \([\phi(z), c]\). When all these speeds are distinct, then the asymptotic expansion can proceed for each mode separately, yielding a single Ostrovsky equation for each mode, see Ostrovsky\(^\text{6}\), Grimshaw\(^\text{7}\) and Grimshaw et al\(^\text{8}\). Here, instead, we are concerned with the case when there are two modes with nearly coincident speeds \(c_1 = c\) and \(c_2 = c + \epsilon^2 \Delta\), \(\epsilon \ll 1\), where \(\Delta\) is the detuning parameter. Importantly, the modal functions \(\phi_1(z), \phi_2(z)\) are distinct, and each satisfy the system (2), (3), that is

\[ (\rho_0 W_i^2 \phi_{iz})_{z} + \rho_0 N^2 \phi_i = 0, \quad i = 1, 2 \] 

(18)

\[ \phi_i = 0 \quad \text{at} \quad z = -h, \quad \text{and} \quad W_i^2 \phi_{iz} = g\phi_i \quad \text{at} \quad z = 0. \] 

(19)
Here $W_i = c_i - u_0(z)$ where $c_i$ is the long wave speed corresponding to the mode $\phi_i(z), i = 1, 2$. It is readily shown from the two modal systems (18), (19) that

$$\int_{-h}^{0} \rho_0 (W_1^2 - W_2^2) \dot{\phi}_1 \dot{\phi}_2 \, dz = 0.$$  

Since in general $W_1 - W_2 = c_1 - c_2 \neq 0$, it follows that the two modes satisfy an orthogonality condition,

$$\int_{-h}^{0} \rho_0 [c_1 + c_2 - 2u_0] \phi_1 \phi_2 \, dz = 0, \quad \text{so that} \quad \int_{-h}^{0} \rho_0 W \phi_1 \phi_2 \, dz \approx 0. \quad (20)$$

Note that here, and in the sequel, $W_i = W = c - u_0(z)$ with an error of order $\epsilon^2$.

Thus we introduce the scaled variables

$$\tau = \epsilon \alpha t, \quad s = \epsilon(x - ct), \quad f = \alpha \tilde{f}$$

where $\alpha = \epsilon^2$ and seek a solution of the form

$$\begin{align*}
[\zeta, u, \rho, p] &= \alpha [\zeta_1, u_1, \rho_1, p_1] + \alpha^2 [\zeta_2, u_2, \rho_2, p_2] + \cdots, \\
[w, v] &= \alpha \epsilon [w_1, v_1] + \alpha^2 \epsilon [w_2, v_2] + \cdots.
\end{align*} \quad (22) \quad (23)$$

Substituting these expansions into the system (8) - (12), and assuming that two waves $A_1$ and $A_2$ are present at the leading order, we obtain

$$\begin{align*}
\zeta_1 &= A_1(s, \tau) \phi_1(z) + A_2(s, \tau) \phi_2(z), \\
u_1 &= A_1 \{W \phi_1\}_z + A_2 \{W \phi_2\}_z, \\
w_1 &= -A_{1s} W \phi_1 - A_{2s} W \phi_2, \\
p_1 &= \rho_0 A_1 W^2 \phi_1 + \rho_0 A_2 W^2 \phi_2, \\
gp_1 &= \rho_0 \rho^2 \zeta_1, \\
v_1 &= \tilde{f}(B_1 \Phi_1 + B_2 \Phi_2), \quad \rho_0 W \Phi_{1,2} = \rho_0 W \phi_{1z,2z} - (\rho_0 u_0) z \phi_{1,2}, \quad B_{1s,2s} = A_{1,2}. \quad (24) \quad (25) \quad (26) \quad (27) \quad (28) \quad (29)
\end{align*}$$

Note that the exact solution of the linearised equations should contain the exact expressions $W_1$ and $W_2$ in the terms related to the first and second waves, respectively, rather than just $W$. In fact, $W_1 = W$ through our choice of $c_1 = c$, but there is an $O(\epsilon^2)$ difference between $W_2$ and $W$ since $c_2 = c + \epsilon^2 \Delta$. This difference between the exact and leading order solutions necessitates the introduction of correction terms at the next order, in order to recover the distinct modal equations for the functions $\phi_1$ and $\phi_2$. 

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Thus, collecting terms of the second order for each equation, and calculating the correction terms originating from the leading order, the following set of equations are obtained,

\[
\begin{align*}
\rho_0(-Wu_{2s} + u_{0z}w_2) + p_{2s} &= -\rho_0(u_{1\tau} + u_{1}u_{1s} + w_{1}u_{1z}) + \rho_1(Wu_{1s} - u_{0z}w_1) + \rho_0\tilde{f}v_1, \quad (30) \\
\rho_0(\tilde{f}u_{2s} - Wv_{2s}) + \rho_2\tilde{f}u_0 &= -\rho_0(v_{1\tau} + u_{1}v_{1s} + w_{1}v_{1z}) + \rho_1Wv_{1s} - \rho_1\tilde{f}u_1, \quad (31) \\
p_{2s} + g\rho_2 &= \rho_0Ww_{1s} + 2\Delta A_2\{\rho_0W\phi_{2s}\}_z, \quad (32) \\
-gW\rho_2 - \rho_0N^2w_2 &= -g(\rho_{1\tau} + u_{1}\rho_{1s} + w_{1}\rho_{1z}), \quad (33) \\
u_{2s} + w_{2s} &= 0, \quad (34) \\
W\zeta_{2s} + w_2 &= \zeta_{1\tau} + u_{1}\zeta_{1s} + w_{1}\zeta_{1z}. \quad (35)
\end{align*}
\]

Note that the extra term proportional to \( A_2 \) in (32) comes from the afore-mentioned difference between \( W_2 \) and \( W \) in the leading order pressure term (27) creating in effect a contribution to \( p_1 \). There is no analogous term in (30) as there is a cancellation between the corrections to \( u_1 \) and \( p_1 \). The boundary conditions (13) - (15), (17) are treated in analogous manner to yield

\[
\begin{align*}
w_2 &= 0 \quad \text{at} \quad z = -h, \quad (36) \\
p_2 - \rho_0g\eta_2 + p_{1z}\eta_1 - \frac{1}{2}\rho_{0z}g\eta_1^2 - 2\Delta \rho_0W\phi_{2z}A_2 &= 0 \quad \text{at} \quad z = 0, \quad (37) \\
w_2 + w_{1z}\eta_1 - \eta_{1\tau} + W\eta_{2s} - u_{0z}\eta_1\eta_{1s} - u_{1}\eta_{1s} &= 0 \quad \text{at} \quad z = 0, \quad (38) \\
\zeta_2 + \zeta_{1z}\eta_1 - \eta_2 &= 0 \quad \text{at} \quad z = 0. \quad (39)
\end{align*}
\]

Eliminating all variables in favour of \( \zeta_2 \) yields

\[
\begin{align*}
\{\rho_0W^2\zeta_{2sz}\}_z + \rho_0N^2\zeta_{2s} &= M_2 \quad \text{at} \quad -h < z < 0, \quad (40) \\
\zeta_2 &= 0 \quad \text{at} \quad z = -h, \quad \rho_0W^2\zeta_{2sz} - \rho_0g\zeta_{2s} &= N_2 \quad \text{at} \quad z = 0, \quad (41)
\end{align*}
\]

where \( M_2, N_2 \) are known expressions containing terms in \( A_i \) and their derivatives. The full expressions are listed in the Appendix A.1, see (A1), (A4).

Two compatibility conditions need to be imposed on the system (40), (41), given by

\[
\int_{-h}^{0} M_2\phi_{1,2} \, dz - [N_2\phi_{1,2}]_{z=0} = 0. \quad (42)
\]

where \( \phi_{1,2} \) are evaluated at the leading order. The outcome is the coupled Ostrovsky equations

\[
I_1(A_{1\tau} + \mu_1A_1A_{1s} + \lambda_1A_{1sss} - \gamma_1B_1)
\]
\[ + \nu_1 [A_1 A_2]_s + \nu_2 A_2 A_{2s} + \lambda_{12} A_{2ss} - \gamma_{12} B_2 = 0, \]  
\[ I_2 (A_{2r} + \mu_2 A_2 A_{2s} + \lambda_2 A_{2ss} + \Delta A_{2s} - \gamma_2 B_2) + \nu_2 [A_1 A_2]_s + \nu_1 A_1 A_{1s} + \lambda_{21} A_{1ss} - \gamma_{21} B_1 = 0, \]  
where \( B_{1s} = A_1, B_{2s} = A_2 \), and the coefficients are given by

\[ I_{i\mu_i} = 3 \int_{-h}^{0} \rho_0 (c - u_0)^2 \phi_i^3 dz, \]  
\[ I_{i\lambda_i} = \int_{-h}^{0} \rho_0 (c - u_0)^2 \phi_i^2 dz, \]  
\[ I_i = 2 \int_{-h}^{0} \rho_0 (c - u_0) \phi_i^2 dz, \]  
\[ \lambda_{12} = \lambda_{21} = \int_{-h}^{0} \rho_0 (c - u_0)^2 \phi_1 \phi_2 dz, \]  
\[ \nu_1 = 3 \int_{-h}^{0} \rho_0 (c - u_0)^2 \phi_1^2 \phi_2 dz, \]  
\[ \nu_2 = 3 \int_{-h}^{0} \rho_0 (c - u_0)^2 \phi_2^2 \phi_1 dz, \]  
\[ I_{i\gamma_i} = \tilde{f}^2 \int_{-h}^{0} \rho_0 \Phi_i \phi_i dz, \]  
\[ \gamma_{ij} = \tilde{f}^2 \int_{-h}^{0} \rho_0 \Phi_i \phi_j dz. \]

Here \( i, j = 1, 2 \). If there is no shear flow, that is \( u_0 = 0 \), then \( \gamma_1 = \gamma_2 = \tilde{f}^2/2c \) and \( \gamma_{12} = \gamma_{21} = 0 \). The expression (51) also generalises the expression (6) to the case when there is a background shear flow. Also note the Cauchy-Schwartz inequality \( \lambda_{12}^2 < I_1 I_2 \lambda_1 \lambda_2 \),

**B. Coupled Ostrovsky equations**

The coupled Ostrovsky equations possess three conservation laws

\[ \int_{-\infty}^{\infty} A_1 \, ds = 0, \quad \int_{-\infty}^{\infty} A_2 \, ds = 0, \quad \text{when} \quad I_1 I_2 \gamma_1 \gamma_2 - \gamma_{12} \gamma_{21} \neq 0 \]  
\[ \int_{-\infty}^{\infty} [I_1 A_1^2 + I_2 A_2^2] \, ds = \text{constant}, \quad \text{when} \quad \gamma_{12} = \gamma_{21}. \]

All three hold in the main case of interest here when there is no shear flow, since then \( \gamma_{12} = \gamma_{21} = 0 \). Note that then \( I_1 I_2 > 0 \), and so (54) guarantees stability. But if \( I_1 I_2 < 0 \), implying there is a critical level, then (54) indicates possible instability.

We scale the dependent and independent variables as in Gear and Grimshaw \(^{12}\), that is

\[ A_1 = \frac{u}{\mu_1}, \quad A_2 = \frac{v}{\mu_2}, \quad s = \lambda_1^{1/2} X, \quad \tau = \lambda_1^{1/2} T, \]  

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assuming that \( \lambda_2 > 0, \lambda_1 \neq 0, \mu_{1,2} \neq 0 \) without loss of generality. Then equations (43), (44) become

\[
(u_T + uu_X + u_{XXX} + n(uv)_X + m(vv)_X + \alpha(vv)_X)_X = \beta u + \gamma v, \quad (56)
\]
\[
(v_T + vv_X + \delta v_{XXX} + \Delta v_X + p(uu)_X + q(uu)_X + \lambda uu_{XXX})_X = \mu v + \nu u, \quad (57)
\]

where

\[
\begin{align*}
    n &= \frac{\nu_1}{I_1 \mu_2}, & m &= \frac{\mu_1 \nu_2}{I_1 \mu_2^2}, & \alpha &= \frac{\lambda_{12} \mu_1}{\lambda_1 I_1 \mu_2}, & \beta &= \gamma_1 \lambda_1, & \gamma &= \frac{\gamma_{12} \mu_1 \lambda_1}{I_1 \mu_2}, \\
    \delta &= \frac{\lambda_2}{\lambda_1}, & p &= \frac{\nu_2}{I_2 \mu_1}, & q &= \frac{\mu_2 \nu_1}{I_2 \mu_1^2}, & \lambda &= \frac{\lambda_{21} \mu_2}{\lambda_1 I_2 \mu_1}, & \mu &= \gamma_2 \lambda_1, & \nu &= \frac{\gamma_{21} \mu_2 \lambda_1}{I_2 \mu_1}. \quad (58)
\end{align*}
\]

Especially note that

\[
\frac{q}{n} = \frac{p}{m} = \frac{\lambda}{\alpha} = \frac{\gamma_{12} \nu}{\gamma_{21} \gamma} = \frac{I_1 \mu_2^2}{I_2 \mu_1^2}, \quad \frac{\alpha \lambda}{\delta} = \frac{\lambda_{12}^2}{\lambda_1 \lambda_2 I_1 I_2} < 1. \quad (59)
\]

The counterparts of the conservation laws are

\[
\int_{-\infty}^{\infty} u \, dx = 0, \quad \int_{-\infty}^{\infty} v \, dx = 0, \quad \text{when} \quad \beta \mu - \gamma \nu \neq 0 \quad (60)
\]
\[
\int_{-\infty}^{\infty} [\lambda u^2 + \alpha v^2] \, dx = \text{constant, when} \quad \gamma_{12} = \gamma_{21}, \quad \text{that is} \quad \alpha \nu = \lambda \gamma. \quad (61)
\]

C. Three-layer example

In order to illustrate the general theory described above, we consider a simple example, extending that considered by Gear and Grimshaw $^{12}$. This is a three-layer stratification, see figure 2,

\[
N = N_1 + (N_2 - N_1)H(z + h_2 + h_3) + (N_3 - N_2)H(z + h_3). \quad (62)
\]
Here $N$ is a constant, $N_{1,2,3}$, in each of three layers of depths $h_{1,2,3}$ where $h = h_1 + h_2 + h_3$, counted from the bottom to the top, and $H(z)$ is the Heaviside function. We also assume there is no shear flow, that is $u_0(z) = 0$. This is not meant to be a realistic model of a typical ocean stratification, but nonetheless representative of a double thermocline. With some effort, it could be developed into a more realistic model, but that would require numerical evaluation of the modal equations. The present example is designed to be amenable to an analytic solution, so that the expressions for all coefficients can be found explicitly. Using the Boussinesq and rigid lid approximations, (18), (19) become

$$\phi_{zz} + \frac{N^2}{c^2}\phi = 0, \quad \text{and} \quad \phi = 0 \quad \text{at} \quad z = -h, 0.$$  \hspace{1cm} (63)

The modal equation (63) holds in each layer, and $\phi, \phi_z$ are continuous at the layer boundaries. This is the extension of the case studied by Gear and Grimshaw \textsuperscript{12} who also put $h_1 = h_2 = h_3$ and $N_2 = 0$; their Figure 2 shows a near resonance at $N_3 = 0.46N_1$. Here we also put $N_2 = 0$, but leave $h_{1,2,3}$ undetermined at this stage. The solution is

$$\phi = A_1 \frac{\sin (N_1(z + h)/c)}{\sin (N_1h_1/c)}, \quad -h < z < -h + h_1,$$

$$\phi = -A_1 \frac{z + h_3 - h_2}{h_2} + A_3 \frac{z + h_2 + h_3 - h_1}{h_2}, \quad -h_2 - h_3 < z < -h_3,$$

$$\phi = -A_3 \frac{\sin (N_3z/c)}{\sin (N_3h_3/c)}, \quad -h_3 < z < 0,$$

where $A_1, A_3$ are the amplitudes at $z = -h + h_1, z = -h_3$ respectively, and the continuity of $\phi$ is already satisfied. Then, continuity of $\phi_z$ yields

$$A_1 N_1 \cot (N_1h_1/c) = -A_1 \frac{c}{h_2} + A_3 \frac{c}{h_2}, \quad -A_3 N_3 \cot (N_3h_3/c) = A_3 \frac{c}{h_2} - A_1 \frac{c}{h_2}. \hspace{1cm} (67)$$

The determinant of this $2 \times 2$-system (67) yields $c$. Thus, we get that

$$D = D_1 D_3 - D_2^2 = 0,$$

where

$$D_1 = N_1 \cot \left( \frac{N_1h_1}{c} \right) + \frac{c}{h_2}, \quad D_3 = N_3 \cot \left( \frac{N_3h_3}{c} \right) + \frac{c}{h_2}, \quad D_2 = \frac{c}{h_2}. \hspace{1cm} (69)$$

In order to obtain a resonance, that is a double solution $c$ with two distinct modes we require that $D_1 = D_3 = D_2 = 0$ simultaneously in order to ensure that there are two independent solutions for $A_1, A_3$. Formally, this requires that we take the limit $h_2 \to \infty$, and then $c = c_{res}$, where

$$\frac{N_1h_1}{c_{res}} = \frac{N_3h_3}{c_{res}} = (n + \frac{1}{2})\pi, \quad n = 0, 1, 2, \cdots.$$  \hspace{1cm} (70)
Clearly this requires that \( N_1 h_1 = N_3 h_3 \), which is satisfied in the symmetric case \( h_1 = h_3, N_1 = N_3 \). In this limit there is an exact resonance, with a double solution for \( c_{\text{res}} \) and we can choose \( A_1, A_3 \) arbitrarily. A sensible choice is \( A_1 = 1, A_3 = 0 \) and \( A_1 = 0, A_3 = 1 \) so that the two modes with the same speed correspond to a lower interface mode, and an upper interface mode respectively. In practice, we assume that \( h_2 \) is finite, but \( h_{1,3} \ll h_2 \). Then the detuning parameter \( \Delta \) is small, but non-zero. Letting \( c = c_{\text{res}} + \Delta \) we find that

\[
\Delta = -\frac{c_{\text{res}}^2 (N_1 + N_3)}{h_2 N_1 N_3 (n + 1/2) \pi}.
\]

At leading order the modes defined by (64), (65), (66) are now given by

\[
\begin{align*}
\phi_1 &= (-1)^n \sin \left\{ \frac{(n + 1/2) \pi (z + h)}{h_1} \right\}, \quad -h < z < -h + h_1, \\
\phi_2 &= (-1)^{n+1} \sin \left\{ \frac{(n + 1/2) \pi z}{h_3} \right\}, \quad -h_3 < z < 0, \\
\end{align*}
\]

Note that formally, as \( h_2 \to \infty \), \( \phi_{1,2} = 1 \) in the near field where \( z + h_2 + h_3 > 0, z + h_3 < 0 \) respectively when \( z \) is fixed as \( h_2 \to \infty \). All coefficients in the coupled Ostrovsky system can now be evaluated, here for \( n = 0 \). Thus, we get that

\[
c_{\text{res}} = C, \quad C = \frac{2 N_1 h_1}{\pi} = \frac{2 N_3 h_3}{\pi},
\]

\[
I_1 = \frac{N_1^2 h_1}{C} = \frac{\pi^2 C}{4 h_1}, \quad I_2 = \frac{N_3^2 h_3}{C} = \frac{\pi^2 C}{4 h_3},
\]

\[
I_1 \nu_1 = 2 N_1^2, \quad I_2 \nu_2 = -2 N_3^2,
\]

\[
\lambda_1 = C^2 \left( \frac{h_1}{2} + \frac{h_2}{3} \right), \quad \lambda_2 = C^2 \left( \frac{h_3}{2} + \frac{h_2}{3} \right), \quad \lambda_{12} = 2 \lambda_{12} = \frac{C^2 h_2}{6}.
\]

Note that here the nonlinear coupling coefficients \( \nu_{1,2} \) are zero, but \( \lambda_{12} \neq 0 \) so the coupling is purely through the linear dispersion terms. There is an apparent serious deficiency here in that \( \lambda_{1,2,12} \) all scale with \( h_2 \), and \( h_2 \to \infty \). In this limit \( I_1 \lambda_1 \sim I_2 \lambda_2 \sim 2 \lambda_{12} \). However a rescaling of time and space in the coupled Ostrovsky equations can remove this, as seen in the rescaled equations (56), (57).
The simplest case here is the symmetric case when $N_1 = N_3, h_1 = h_3$, when the symmetry indicates that $\mu_1 = -\mu_2, \lambda_1 = \lambda_2$. Then, except for the detuning parameter $\Delta$, the coupled Ostrovsky system is symmetric. This reduces the parameter space considerably, as then $m = n = p = q = 0, \delta = 1, \alpha = \lambda = -1/2, \beta = \mu, \gamma = \nu = 0$, leaving only two parameters, $\Delta, \beta$, which can be varied independently. The non-symmetric case when $N_1 \neq N_3$ has $p = n = q = m = 0, \gamma = \nu = 0$, and $\delta = h_3/h_1, \beta = \mu$, while $\alpha = -h_3/2h_1 = -\delta/2, \lambda = -1/2$, so that $\alpha \lambda = h_3/4h_1$. Hence there are now three parameters $\Delta, \beta, \delta = h_3/h_1$, which can be varied independently.

III. LINEAR DISPERSION RELATION

The linear dispersion relation is obtained by seeking solutions of the linearised equations (56), (57) in the form

$$u = u_0 e^{ik(X-c_pT)} + c.c., \quad v = v_0 e^{ik(X-c_pT)} + c.c.,$$

where $k$ is the wavenumber and $c_p$ is the phase speed and $c.c.$ denotes the complex conjugate. This leads to

$$\begin{align*}
(c_p - C_1(k))u_0 + (\alpha k^2 - \frac{\gamma}{k^2})v_0 &= 0, \\
(\lambda k^2 - \frac{\nu}{k^2})u_0 + (c_p - C_2(k))v_0 &= 0,
\end{align*}$$

where $C_1(k) = -k^2 + \frac{\beta}{k^2}, \quad C_2(k) = \Delta - \delta k^2 + \frac{\mu}{k^2}.$ (76)

The determinant of this $2 \times 2$ system yields the dispersion relation

$$(c_p - C_1(k))(c_p - C_2(k)) = D(k) = (\alpha k^2 - \frac{\gamma}{k^2})(\lambda k^2 - \frac{\nu}{k^2}).$$ (77)

Solving this dispersion relation we obtain the two branches of the dispersion relation,

$$c_p = c_{p1, p2} = \frac{C_1 + C_2}{2} \pm \frac{1}{2} \sqrt{4D + (C_1 - C_2)^2}.$$ (78)

Here $C_{1,2}(k)$ are the linear phase speeds of the uncoupled Ostrovsky equations, obtained formally by setting the coupling term $D(k) = 0$. If $D(k) > 0$ for all $k$, then both branches are real-valued for all wavenumbers $k$, and the linearised system is spectrally stable. This will be assumed henceforth, and we note that this is assured in the absence of a background shear flow, as then $\gamma = \nu = 0$ and $\alpha \lambda > 0$ so that $D(k) > 0$ for all $k$. 

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The case when there is no background shear is the main concern in this paper, and in that case $\gamma = \nu = 0$, and we can assume also that $c > 0, I_1 > 0, I_2 > 0$ without loss of generality. It follows that then $\lambda_{1,2} > 0$, so that $\delta > 0$, $\beta = \mu > 0$ and $0 < \alpha \lambda < \delta$. In particular the coupling coefficient $D(k) = \alpha \lambda k^4 > 0$ for all $k > 0$. Also we recall that $\Delta < 0$ without loss of generality. Then (78) reduces to

$$c_{p1,g2} = \frac{\beta}{k^2} + \frac{\Delta}{2} - \frac{(1 + \delta)k^2}{2} \pm \frac{1}{2} \sqrt{[\Delta + (1 - \delta)k^2]^2 + 4\alpha \lambda k^4}. \quad (79)$$

The group velocities are given by $c_g = d(kc_p)/dk$,

$$c_{g1,g2} = -\frac{\beta}{k^2} + \frac{\Delta}{2} - \frac{3(1 + \delta)k^2}{2} \pm \frac{[\Delta + (1 - \delta)k^2][\frac{1}{2} \Delta + \frac{3}{2}(1 - \delta)k^2] + 6\alpha \lambda k^4}{\sqrt{[\Delta + (1 - \delta)k^2]^2 + 4\alpha \lambda k^4}}. \quad (80)$$

It is useful now to examine the limits $k \to 0, \infty$. Thus

$$c_{p1,g2} \to \frac{\beta}{k^2} + (0, \Delta) + O(k^2), \quad \text{as} \quad k \to 0, \quad (81)$$

$$c_{p1,g2} \to E_{1,2}k^2, \quad 2E_{1,2} = -(1 + \delta) \pm \{(1 - \delta)^2 + 4\alpha \lambda\}^{1/2}, \quad \text{as} \quad k \to \infty. \quad (82)$$

Note that since $0 < \alpha \lambda < \delta$, $E_2 < E_1 < 0$. It follows that both branches have no spectral gaps, and we infer that there are no solitary waves. Note that the two branches coalesce as $k \to 0$, implying that in this limit there is a strong coupling between the two modes. This can be compared with the case for no rotation, when $\beta = 0$, and there is a spectral gap for both modes when $c > 0$ where KdV solitary waves for mode 1 exist, and a spectral gap for mode 2 when $c > \Delta$ and then generalised solitary waves exist, in resonance with mode 1. But when there is rotation $\beta > 0$, both gaps are removed, which is the same scenario that arises for the single Ostrovsky equation.

A typical dispersion curve is shown in Figure 3, which is based on the parameter values obtained from the three-layer example discussed above in section II C. The significant features are that there is no spectral gap, and that both group velocity curves have a turning point with maximum speed. Hence, based on the results for the single Ostrovsky equation obtained by Grimshaw and Helfrich, we expect that an initial solitary-like wave initial conditions will collapse through the radiation of inertia-gravity waves, followed by the emergence of two nonlinear wave packets associated with each of these maximum group velocities.
FIG. 3. Typical dispersion curves (79) with $\delta = 1, \alpha = \lambda = -0.5, \beta = \mu = 0.5$. The solid curves show the phase speed, and the dashed curves show the group velocity.

IV. NUMERICAL RESULTS

The coupled Ostrovsky equations (56), (57) are solved numerically with the scheme described in Appendix A.2 and with the initial conditions specified in Appendix A.3, which model a solitary wave solution of the coupled KdV system obtained by removing the rotation terms. The derivation of this initial data is described in A 3 a, A 3 b when $\Delta$ is $O(1)$ and the initial amplitudes $a, b$ of $u,v$ are small, and by A 3 c when $\Delta$ is small but $a, b$ are not small. Note that then in order for nonlinear effects to be significant initially, we require that $(u^2/2)_{XX} > \beta u$ and $(v^2/2)_{XX} > \mu v$. For the initial conditions A 3 a and A 3 b these imply that $a^2 > 6\beta$ and $b^2 > 6\delta\mu$, and $a^2 > 6\beta|1 + \alpha|$ and $b^2 > 6\mu|\delta + \lambda|$ respectively. Although the initial conditions A 3 a and A 3 b are firmly based on a valid bifurcation analysis, in practice the initial conditions were too small to generate significant nonlinear effects, and consequently the emergence of the expected nonlinear wave packets was suppressed. Consequently, all the results shown here used the initial condition A 3 c. A summary of some typical simulations is set out in Table I.

We will discuss three cases based on the three-layer example of section II C. These are the symmetric case $N_1 = N_3$ when $m = n = p = q = 0, \alpha = \lambda = -1/2$ and $\delta = 1$, and two non-symmetric cases $N_1 \neq N_3$ with $N_3/N_1 = 4$, where $\delta = h_3/h_1 = 1.5, 0.2$, $\alpha = -h_3/2h_1 = -0.75, -0.1$ respectively. Note that $\lambda = -1/2$ in all cases. As diagnostics,
we compare the speeds of the numerically found wave packets with the maximum group speeds $c_{g1}, c_{g2}$ from the linear dispersion relation, see (80), and also the modal structure, that is the ratio $r_{1,2} = |u_0|/|v_0|$ computed from either (74) or (75) for the corresponding wave numbers. A summary of the results from our simulations are shown in Table II. Overall, there is good agreement with the predicted speeds $c_{g1, g2}$ and ratios $r_{1,2}$ and those found in the simulations, $C_{g1, g2}$ and $R_{1,2}$. In the following subsections, we show some typical plots.

![Table I](image)

**TABLE I. Simulation parameters**

![Table II](image)

**TABLE II. Numerically determined group velocities, $C_{g1, g2}$ and modal ratio $R_{1,2}$ versus theoretical predictions $c_{g1, g2}$ and ratio $r_{1,2}$ for each case.**
A. Symmetric case

Here, when $\Delta = 0$, the initial condition used has $u = v$ at $t = 0$. But in this special case, $u = v$ is an exact solution for all time, each satisfying a single Ostrovsky equation. Hence in this case only one mode is activated and the solution is the same as that found for a single Ostrovsky equation by Grimshaw and Helfrich $^4$. As $|\Delta|$ is increased, the signature of a second wave packet emerges. A typical result is shown in Figures 4 and 5. Two wave packets are clearly visible in the plots for the $v$-component, but the second packet is barely discernible in the plots for the $u$-component, where its predicted amplitude is too small for it to be distinguished from the background field of radiating waves.

B. Non-symmetric cases

When $\delta \neq 1$, the symmetry is broken even when $\Delta = 0$, and now two wave packets are clearly seen. Typical plots are shown when $\delta = 1.5, \beta = 0.1$ and $\Delta = -0.1$ in Figures 6 and 7, and for $\delta = 0.2, \beta = 0.5$ and $\Delta = -0.1$ in Figures 8 and 9. Numerical results for $\Delta = -0.01$ and the same values of $\delta$ and $\beta$ are qualitatively similar. The comparison between the predicted and numerical values shown in Table II is now very good. In some simulations, the results were qualitatively similar to Figure 4, when two wave packets can be seen in the $v$-component, but one of them is too small to be seen in the $u$-component.

V. DISCUSSION

In this paper we have systematically derived coupled Ostrovsky equations for the amplitudes of two different linear long wave modes with nearly coincident phase speeds from the full set of Euler equations for stratified incompressible fluid with background shear flow on an $f$-plane, with a free surface and rigid bottom boundary conditions. This extends the derivation of coupled KdV equations by Gear and Grimshaw $^{12}$ and that of a single Ostrovsky equation by Grimshaw $^7$. As an example, we have evaluated the coefficients in these equations for the case of a three-layer stratification, this being an extension of the special case considered by Gear and Grimshaw $^{12}$. Within the scope of this and similar models of the density stratification, both branches of the dispersion relation of the linearised equations resemble the dispersion curve of a single Ostrovsky equation, with no spectral gaps, and with
FIG. 4. Numerical simulations for coupled Ostrovsky equations using initial conditions from (A 3 c) with $a = b = 5, \Delta = -0.5, \delta = 1, \beta = 1$.

FIG. 5. Same as Figure 4, but a cross-section at $t = 100$. 
FIG. 6. Numerical simulations for coupled Ostrovsky equations using initial conditions from (A 3 c) with $a = 0.75, b = 3, \Delta = -0.1, \delta = 1.5, \beta = 0.1$.

FIG. 7. Same as Figure 6, but a cross-section at $t = 200$. 
FIG. 8. Numerical simulations for coupled Ostrovsky equations using initial conditions from (A 3 c) with \( a = 3, b = -1, \Delta = -0.1, \delta = 0.2, \beta = 0.5 \).

FIG. 9. Same as Figure 8, but a cross-section at \( t = 100 \).
an extremum in both group velocity curves. Importantly, in all the cases we show here these extrema are distinct. Hence, based on the results for the single Ostrovsky equation obtained by Grimshaw and Helfrich \(^4\), we expect to observe the emergence of two separated nonlinear wave packets associated with the extrema in these group velocity curves. This generic outcome has been confirmed in our numerical simulations, using a pseudo-spectral code, and initiated using solitary-type initial conditions. As for the single Ostrovsky equation, we would expect each of these wave packets to be described by an extended nonlinear Schrödinger equation, although the derivation of that asymptotic reduction is beyond the scope of this article. Of course, at early times, for example around \(t = 10\) to \(20\) in Figure 4, there is an interaction between these wave packets, and indeed, there is continuing small-amplitude radiation from the leading wave packet which does then interact with the second wave packet. But this does not seem to greatly affect its coherence as a stable and persistent nonlinear wave packet.

The Ostrovsky equation, and a system of coupled Ostrovsky equations, belong to the class of universal mathematical models of nonlinear wave theory, and a study of the behaviour of their solutions is valuable for a variety of applications. In addition to the oceanographic applications already cited above, we can also mention the two-directional generalisation of the Ostrovsky equation derived by Gerkema \(^{14}\). The latter equation is related to the dynamics of a modified Toda lattice on an elastic substrate, considered by Yagi and Kawahara \(^{15}\), where the emergence of nonlinear wave packets was also found, independently of the analogous results obtained by Grimshaw and Helfrich \(^4\) for the Ostrovsky equation. Recently, a system of coupled Boussinesq equations has been derived as a model for long nonlinear waves in a layered solid waveguide with a soft bonding layer in Khusnutdinova et al \(^{16}\). A nonsecular weakly-nonlinear solution of the initial-value problem for this system has been constructed, under certain conditions, in terms of solutions of coupled and uncoupled Ostrovsky equations for unidirectional waves, see Khusnutdinova and Moore \(^{17}\). While wave packets described by the single Ostrovsky equations were clearly seen in the numerical simulations for the coupled Boussinesq equations in the latter case, generalised solitary waves were generated in the case described in terms of solutions of the coupled Ostrovsky equations. Therefore, in general, solutions of the coupled Ostrovsky equations can differ from the scenario described in the present paper, depending on the structure of the linear dispersion relation, thus inviting further studies of the general system of coupled Ostrovsky equations.
Nevertheless, the three-layer model studied here suggests that the emergence of two wave packets is the typical outcome for the oceanic situation.

VI. ACKNOWLEDGEMENTS

A. Alias is supported by Universiti Malaysia Terengganu and the Ministry of Higher Education of Malaysia. We are grateful to the referees for some helpful and insightful comments.

REFERENCES


Appendix A: Appendix

1. Expressions

The expression for $M_2$ in (40) is

$$M_2 = M_2^1 + M_2^2 + M_{12}^2 [A_1 A_2]_s + 2 \{ \rho_0 (c - u_0) \phi_{2z} \} z \Delta A_{2s} - \tilde{f}_2 \{ B_1 \rho_0 \Phi_1 + B_2 \rho_0 \Phi_2 \}_z,$$  \hspace{1cm} (A1)

where

$$M_2^1 = 2 \{ \rho_0 (c - u_0) \phi_{jz} \} z A_{j\tau} - \rho_0 (c - u_0)^2 \phi_{jsss} + (3 \{ \rho_0 (c - u_0)^2 \phi_{jz} \} z$$

$$+ 2 \{ \rho_0 (c - u_0)^2 \phi_{jz} \} z \phi_{jz} - 2 \{ \rho_0 (c - u_0)^2 (\phi_{jz} \phi_{jz}) \} z \} A A_{js}, \hspace{0.5cm} j = 1, 2,$$  \hspace{1cm} (A2)

and

$$M_{12}^1 = 3 \{ \rho_0 (c - u_0)^2 \phi_{1z} \phi_{2z} \} z - \rho_0 (c - u_0)^2 (\phi_{1z} \phi_{2zz} + \phi_{2z} \phi_{1zz})$$

$$- \{ \rho_0 (c - u_0)^2 (\phi_{1z} \phi_{2zz} + \phi_{2z} \phi_{1zz}) \} z.$$  \hspace{1cm} (A3)

Similarly, the expression for $N_2$ in (41) is

$$N_2 = N_2^1 + N_2^2 + N_{12}^2 [A_1 A_2]_s + 2 \rho_0 (c - u_0) \phi_{2z} \Delta A_{2s} - \tilde{f}_2 \{ B_1 \rho_0 \Phi_1 + B_2 \rho_0 \Phi_2 \}_z,$$  \hspace{1cm} (A4)

where

$$N_2^1 = 2 \rho_0 (c - u_0) \phi_{jz} A_{j\tau} + (3 \rho_0 (c - u_0)^2 \phi_{jz}^2 - 2 \rho_0 (c - u_0)^2 \phi_{j} \phi_{jzz}) A_{j} A_{js}, \hspace{0.5cm} j = 1, 2,$$  \hspace{1cm} (A5)

and

$$N_{12}^1 = 3 \rho_0 (c - u_0)^2 \phi_{1z} \phi_{2z} - \rho_0 (c - u_0)^2 (\phi_{1z} \phi_{2zz} + \phi_{2z} \phi_{1zz}).$$  \hspace{1cm} (A6)


The coupled Ostrovsky equations (56) and (57) are solved numerically using a pseudo-spectral (PS) method in the periodic domain. PS method is an alternative to the finite
difference or finite element methods, and it has been used to solve many nonlinear evolution
equations and systems of coupled equations, see for example Chan and Kerhoven 18, Canuto
et al 19, Nouri and Sloan 20, Boyd 21, Huang and Zhang 22, Rashid 23, Hou and Li 24, Klein
25, Gulkac and Ozis 26, Rashid and Ismail 27, Yaguchi et al 28 and references therein. This
method was used by Grimshaw and Helfrich 4 for the single Ostrovsky equation, although we
note that explicit finite difference schemes can be used as well, see Obregon and Stepanyants
29 for instance.

In the past several decades, Nouri and Sloan 20 have compared six Fourier pseudo-spectral
methods for the KdV equation which differ in terms of the time discretisation. One of the
most efficient methods tested was the semi-implicit scheme of Chan and Kerhoven 18. They
integrated the KdV equation in time in Fourier space using two Fast Fourier Transforms
(FFT) per time step. Here, we extend this scheme to solve the coupled Ostrovsky equations.

Thus, we formulate the problem over a periodic domain \(-L < x < L\), where \(L\) is
sufficiently large. Before we proceed with the numerical discretization, we add a linear
damping region ("sponge layer") at each end of the domain to prevent the possibility of
radiated waves re-entering the region of interest and interfering with the main wave structure.
Equations (56) and (57), in the absence of a background shear flow, and including the sponge
layer, are written as follows,

\[
\begin{align*}
[u_T + uu_X + u_{XXX} + n(uv)_X + m vv_X + \alpha v_{XXX} + r(x)u]_X &= \beta u, \quad (A7) \\
[v_T + vv_X + \delta v_{XXX} + \Delta v_X + p(uv)_X + q uu_X + \lambda u_{XXX} + r(x)v]_X &= \mu v, \quad (A8)
\end{align*}
\]

where the sponge layer, \(r(x)\) is defined by

\[
r(x) = \frac{\nu}{2}\left\{ (1 + \tanh \kappa (x - 3L/4) ) + (1 - \tanh \kappa (x + 3L/4) ) \right\}
\]

for some constants \(\nu, \kappa\). For instance, \(\kappa L = 12\) and the value of \(\nu\) is chosen so that the
damping occurs quickly.

First we transform the solution interval \([-L, L]\) to \([0, 2\pi]\) using \(\xi = sX + \pi, s = \frac{\pi}{L}\). Then
equations (A7) and (A8) will be transformed to

\[
\begin{align*}
[u_T + suu_\xi + s^3u_{\xi\xi\xi} + ns(uv)_\xi + msvv_\xi + \alpha s^3v_{\xi\xi\xi} + r(x)u]_\xi &= \frac{\beta}{s} u, \quad (A9) \\
[v_T + ssvv_\xi + s^3\varepsilon_{\xi\xi\xi} + \Delta sv_\xi + ps(uv)_\xi + qssuu_\xi + \lambda s^3u_{\xi\xi\xi} + r(x)v]_\xi &= \frac{\mu}{s} v. \quad (A10)
\end{align*}
\]
The sponge layer terms are treated in the same manner as the nonlinear terms. It is now convenient to use the notation,

\[ suu_\xi = w_a_\xi, \quad w_a = \frac{su^2}{2}, \quad svv_\xi = w_b_\xi, \quad w_b = \frac{sv^2}{2}, \quad s(uv)_\xi = w_c_\xi, \quad w_c = suv, \]

\[ r(x)u = R_a, \quad r(x)v = R_b. \]

The interval \([0, 2\pi]\) is discretized by \(N\) equidistant points with the spacing \(\Delta \xi = 2\pi/N\), generating the values \(u(\xi_j, T)\) and \(v(\xi_j, T)\) at \(\xi = \xi_j = j\Delta \xi, j = 0, 1, \cdots , N - 1\). Then \(N\) is chosen to be even and a power of two, so that \(u(\xi_j, T)\) and \(v(\xi_j, T)\) are transformed by a Discrete Fourier Transform (DFT), so that

\[ \hat{u}(\kappa, T) = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} u(\xi_j, T)e^{-i\kappa \xi_j}, \quad -\frac{N}{2} \leq \kappa \leq \frac{N}{2} - 1, \quad \kappa \neq 0 \]

\[ \hat{v}(\kappa, T) = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} v(\xi_j, T)e^{-i\kappa \xi_j}, \quad -\frac{N}{2} \leq \kappa \leq \frac{N}{2} - 1, \quad \kappa \neq 0. \] (A11)

Note that \(\kappa\) is an integer, which can be interpreted as a discretized and scaled version of a wavenumber. The inverse formulas for the discrete transform are

\[ u(\xi_j, T) = \frac{1}{\sqrt{N}} \sum_{\kappa=-N/2}^{N/2-1} \hat{u}(\kappa, T)e^{i\kappa \xi_j}, \quad j = 0, 1, \cdots , N - 1, \]

\[ v(\xi_j, T) = \frac{1}{\sqrt{N}} \sum_{\kappa=-N/2}^{N/2-1} \hat{v}(\kappa, T)e^{i\kappa \xi_j}, \quad j = 0, 1, \cdots , N - 1. \] (A12)

These transformations (A11) and (A12) can be performed efficiently using the Fast Fourier Transform algorithm (FFT). The DFT of equations (A9), (A10) with respect to \(\xi\) yields

\[ \hat{u}_T + i\kappa \hat{w}_a - i\kappa^3 s^3 \hat{u} + i\kappa \hat{w}_c + i\kappa \hat{w}_b - i\alpha \kappa^3 s^3 \hat{v} + R_a = -\frac{i\beta}{\kappa s} \hat{u}, \]

\[ \hat{v}_T + i\kappa \hat{w}_b - i\delta \kappa^3 s^3 \hat{v} + i\Delta \kappa s \hat{v} + i\kappa \hat{w}_c + i\kappa \hat{w}_a - i\lambda \kappa^3 s^3 \hat{u} + R_b = -\frac{i\mu}{\kappa s}. \] (A13) (A14)

We use the time discretizations

\[ \hat{u}_T(\kappa, T) \approx \frac{\hat{u}(\kappa, T + \Delta T) - \hat{u}(\kappa, T - \Delta T)}{2\Delta T}, \]

\[ \hat{u}(\kappa, T) \approx \frac{\hat{u}(\kappa, T + \Delta T) + \hat{u}(\kappa, T - \Delta T)}{2}, \]

\[ \hat{v}_T(\kappa, T) \approx \frac{\hat{v}(\kappa, T + \Delta T) - \hat{v}(\kappa, T - \Delta T)}{2\Delta T}, \]

\[ \hat{v}(\kappa, T) \approx \frac{\hat{v}(\kappa, T + \Delta T) + \hat{v}(\kappa, T - \Delta T)}{2}. \] (A15)
Finally, we obtain the forward scheme for the coupled Ostrovsky equations in the form

\[\hat{u}(\kappa, T + \Delta T) = \frac{1}{[1 - i\kappa^3 s^3 \Delta T + i\beta \Delta T/\kappa s]} \{[1 + i\kappa^3 s^3 \Delta T - i\beta \Delta T/\kappa s] \hat{u}(\kappa, T - \Delta T) - 2i\kappa \Delta T \hat{w}_a - 2i\kappa \Delta Tn \hat{w}_c - 2i\kappa \Delta T \hat{w}_b + 2i\kappa^3 s^3 \Delta T \alpha \hat{v} - 2\Delta TR_a \},\]  

(A16)

\[\hat{v}(\kappa, T + \Delta T) = \frac{1}{[1 - i\kappa^3 s^3 \Delta T \delta + i\kappa s (\Delta T) \Delta + i\mu \Delta T/\kappa s]} \{[1 + i\kappa^3 s^3 \Delta T \delta - i\kappa s (\Delta T) \Delta - i\mu \Delta T/\kappa s] \hat{v}(\kappa, T - \Delta T) - 2i\kappa \Delta T \hat{w}_b - 2i\kappa \Delta T p \hat{w}_c - 2i\kappa \Delta T q \hat{w}_a + 2i\kappa^3 s^3 \Delta T \lambda \hat{u} - 2\Delta TR_b \}.\]  

(A17)

This scheme is then implemented with de-aliasing using the truncation 2/3-rule by Orszag in Boyd to remove the aliasing error. This error is due to the pollution of the numerically calculated Fourier transform by higher frequencies because of the truncation of the series, see Canuto et al for details. Hou and Li also compared the behaviour of PS methods using 2/3 dealiasing rule between a high order Fourier smoothing in order to remove the aliasing errors. Here, it becomes important in dealing with the nonlinear and the sponge layer terms in the equations. The values of \(N\) and \(\Delta t\) were obtained through numerical experimentation. The solution in physical space is obtained by the inverse discrete Fourier transform (A12).

3. Initial conditions

The initial conditions that we use are obtained as approximate solitary wave solutions of the coupled KdV equations, that is (43), (44) with the rotation terms omitted, \(\beta = \mu = \gamma = \nu = 0\). That is,

\[u_T + uu_X + u_{XXX} + n(uv)_X + mvv_X + \alpha v_{XXX} = 0,\]  

(A18)

\[v_T + vv_X + \delta v_{XXX} + \Delta v_X + p(uv)_X + qwu_X + \lambda u_{XXX} = 0.\]  

(A19)

Solitary wave solutions of these equations are found by seeking solutions which depend only on \(X - c_s T\), where \(c_s\) is the solitary wave speed to be found as part of the solution. Thus (A18), (A19) reduce to

\[-c_s u + \frac{u^2}{2} + u_{XX} + nuv + \frac{mv^2}{2} + \alpha v_{XX} = 0,\]  

(A20)

\[-c_s v + \frac{v^2}{2} + \delta v_{XX} + \Delta v + puv + \frac{qu^2}{2} + \lambda u_{XX} = 0.\]  

(A21)
We use two approaches. First, we use the dynamical systems approach for small-amplitude waves, where the solutions bifurcate from the linear long wave speeds, which are 0 and $\Delta$ respectively. Recall that $\Delta < 0$, so the bifurcation from 0 will yield a KdV solitary wave, and the bifurcation from $\Delta$ a generalised solitary wave.

a. Bifurcation from 0

In the linear long wave limit, $v \to 0$, and $u$ is arbitrary. We then expand as follows,

$$ u = \epsilon^2 A(\xi) + \epsilon^4 A_2(\xi) + \cdots, \quad v = \epsilon^4 B_2(\xi) + \cdots, \quad c_s = \epsilon^2 c_2 + \cdots, \quad \xi = \epsilon X. \quad (A22) $$

We obtain the following equations by collecting the terms of $O(\epsilon^4)$

$$ -c_2 A + \frac{A^2}{2} + A_{\xi \xi} = 0, \quad \Delta B_2 + \frac{q A^2}{2} + \lambda A_{\xi \xi} = 0; \quad (A23) $$

$$ A = a \text{sech}^2(\gamma \xi), \quad c_2 = \frac{a}{3} = 4\gamma^2, \quad B_2 = -\frac{\lambda a^2}{3\Delta} \text{sech}^2(\gamma \xi) + \frac{(\lambda - q)a^2}{2\Delta} \text{sech}^4(\gamma \xi). \quad (A24) $$

Hence in the original coordinates the initial condition is

$$ u = a \text{sech}^2(\gamma X), \quad a = 12\gamma^2, \quad v = -\frac{\lambda a^2}{3\Delta} \text{sech}^2(\gamma X) + \frac{(\lambda - q)a^2}{2\Delta} \text{sech}^4(\gamma X), \quad (A25) $$

where $a$ is a disposable parameter, ideally small. This asymptotic solution requires that $\Delta \neq 0$ is order unity. Note that the nonlinear term $(u^2/2)_{XX}$ has a maximum absolute value of $2a^2\gamma^2 = a^3/6$. When instead $\Delta \sim O(a)$, then equations (A18), (A19) are strongly coupled, and the expressions (A25) cannot be used.

b. Bifurcation from $\Delta$

In the linear long wave limit, $c_s \to \Delta$, $u \to 0$, and $v$ is arbitrary. The expansion is now given by,

$$ u = \epsilon^4 A_2(\xi) + \cdots, \quad v = \epsilon^4 B(\xi) + \epsilon^4 B_2(\xi) + \cdots, \quad c_s = \Delta + \epsilon^2 c_2 + \cdots, \quad \xi = \epsilon X. \quad (A26) $$

By collecting the terms of $O(\epsilon^4)$, we obtain

$$ -c_2 B + \frac{B^2}{2} + \delta B_{\xi \xi} = 0, \quad -\Delta A_2 + \frac{m B^2}{2} + \alpha B_{\xi \xi} = 0; \quad (A27) $$

$$ B = b \text{sech}^2(\gamma \xi), \quad c_2 = \frac{b}{3} = 4\delta \gamma^2, \quad A_2 = \frac{\alpha b^2}{3\delta \Delta} \text{sech}^2(\gamma \xi) + \frac{(\delta m - \alpha)b^2}{2\delta \Delta} \text{sech}^4(\gamma \xi). \quad (A28) $$

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Hence in the original coordinates the initial condition is
\[
v = b \sech^2(\gamma X), \quad b = 12\delta \gamma^2, \quad u = \frac{\alpha b^2}{3\delta \Delta} \sech^2(\gamma X) + \frac{(\delta m - \alpha)b^2}{2\delta \Delta} \sech^4(\gamma X), \quad (A29)
\]
where \( b \) is a disposable parameter, ideally small. Formally, these are generalised solitary waves, with an exponentially small radiating tail, but that is ignored here, as we only require an initial condition. Note that the nonlinear term \((v^2/2)_{XX}\) has a maximum absolute value of \(2b^2\gamma^2_2 = b^3/6\delta\). This asymptotic solution again requires that \(\Delta \neq 0\) is order unity.

c. Weak coupling

The asymptotic solutions described above are not suitable when either \(\Delta\) is small, or when the amplitudes \(a, b\) are not small. However, if the coupling is weak, that is \(m, n, p, q, \alpha, \lambda\) are all small, then the leading order approximation is just the free solitary wave solutions of each uncoupled equation. Here we propose a small modification of these solutions, that is,
\[
u = b \sech^2(\gamma_2 X), \quad \frac{b}{3} = 4(\delta + \lambda)\gamma_2^2.
\]

This should be useful especially when \(m = n = p = q = 0\) and \(\Delta, \alpha, \lambda\) are small. This was implemented with the constraint that \(\gamma_1 = \gamma_2\). In the symmetric case when \(\delta = 1, \alpha = \lambda\), this is an exact solution when \(\Delta = 0\), and it is this feature which has motivated the incorporation of the terms \(\alpha, \lambda\) in these expressions. Note that here the nonlinear terms \((u^2/2)_{XX}, (v^2/2)_{XX}\) have maximum absolute values of \(2a^2\gamma_1^2 = a^3/6(1 + \alpha)\) and \(2b^2\gamma_2^2 = b^3/6(\delta + \lambda)\) respectively.

d. Pedestal

Solutions of the coupled Ostrovsky equations must satisfy the zero mass constraints,
\[
\int_{-L}^{L} u(X,T) \, dX = 0, \quad \int_{-L}^{L} v(X,T) \, dX = 0.
\]
If \(u(X,0) = u_0(X)\) and \(v(X,0) = v_0(X)\) then also,
\[
\int_{-L}^{L} u_0(X) \, dX = 0, \quad \int_{-L}^{L} v_0(X) \, dX = 0.
\]
Thus if the initial condition described above is, say $\tilde{u}_0(x)$, then this must be corrected to have zero mass by adding a negative pedestal $\tilde{d}$ as follows,

$$u_0(X) = \tilde{u}_0(X) - \tilde{d}(X), \quad \int_{-L}^{L} \tilde{u}_0(X) \, dX = \int_{-L}^{L} \tilde{d}(X) \, dX.$$  

Note that $\tilde{d}(X)$ cannot be a constant here due to the presence of the sponge layer in equations (A7) and (A8).

For instance, consider the case of bifurcation from $\Delta$, where the $v$-mode is given by

$$\tilde{v}_0 = b \text{sech}^2(\gamma X),$$

$$v_0(X) = \tilde{v}_0(X) - \tilde{d}_v(X), \quad \int_{-L}^{L} \tilde{d}_v(X) \, dX = \int_{-L}^{L} \tilde{v}_0(X) \, dX \approx \frac{2b}{\gamma}, \quad \gamma L \gg 1.$$  

Then we choose $\tilde{d}_v(X)$ as follows,

$$\tilde{d}_v(X) = \frac{d_{0v}}{2} \{\tanh (\kappa_0(X + L/2)) - \tanh (\kappa_0(X - L/2))\}, \quad \kappa_0 L/4 \gg 1.$$

For example, let $\kappa_0 L = 12$. Then

$$\int_{-L}^{L} \tilde{d}_v(X) \, dX \approx d_{0v} L, \quad \text{so that} \quad d_{0v} = \frac{2b}{\gamma L}.$$  

Next, for $u$-mode given by

$$\tilde{u}_0 = \frac{\alpha b^2}{3\delta \Delta} \text{sech}^2(\gamma X) + \frac{(\delta m - \alpha)b^2}{2\delta \Delta} \text{sech}^4(\gamma X),$$

we use the same method, that is we let

$$\tilde{d}_u(X) = \frac{d_{0u}}{2} \{\tanh (\kappa_0(X + L/2)) - \tanh (\kappa_0(X - L/2))\}, \quad \kappa_0 L/4 \gg 1,$$

$$d_{0u} L = \int_{-L}^{L} \tilde{u}_0(X) \, dX = \int_{-L}^{L} \left\{\frac{\alpha b^2}{3\delta \Delta} \text{sech}^2(\gamma X) + \frac{(\delta m - \alpha)b^2}{2\delta \Delta} \text{sech}^4(\gamma X)\right\} \, dX,$$

so that

$$d_{0u} = \frac{2\alpha b^2}{3\delta \Delta \gamma L} + \frac{2(\delta m - \alpha)b^2}{3\delta \Delta \gamma L} = \frac{2m b^2}{3\delta \Delta \gamma L}.$$  

Now, the initial conditions with a negative pedestal are,

$$u_0(x) = \tilde{u}_0(x) - \tilde{d}_u(X),$$

$$v_0(x) = \tilde{v}_0(x) - \tilde{d}_v(X).$$  

An analogous procedure was followed for the other initial conditions.