Rogue Wave Modes for the Long Wave-Short Wave Resonance Model

Kwok Wing CHOW*(1), Hiu Ning CHAN (1), David Jacob KEDZIORA (2),
Roger Hamilton James GRIMSHAW (3)

(1) = Department of Mechanical Engineering, University of Hong Kong,
Pokfulam, Hong Kong
(2) = Research School of Physics and Engineering, Australian National University,
Canberra ACT 0200, Australia
(3) = Department of Mathematical Sciences, Loughborough University,
Loughborough LE11 3TU, United Kingdom

* = Corresponding author

Email: kwchow@hku.hk Fax: (852) 2858 5415

Submission date: March 2013

JPSJ Classification: 02; 05; 47. PACS: 02.30.Jr; 05.45.Yv; 47.35.Fg

ABSTRACT

The long wave-short wave resonance model arises physically when the phase velocity of a long wave matches the group velocity of a short wave. It is a system of nonlinear evolution equations solvable by the Hirota bilinear method

1
and also possesses a Lax pair formulation. ‘Rogue wave’ modes, algebraically localized entities in both space and time, are constructed from the breathers by a singular limit involving a ‘coalescence’ of wavenumbers in the long wave regime. In contrast with the extensively studied nonlinear Schrödinger case, the frequency of the breather cannot be real and must satisfy a cubic equation with complex coefficients. The same limiting procedure applied to the finite wavenumber regime will yield mixed exponential-algebraic solitary waves, similar to the classical ‘double pole’ solutions of other evolution systems.

**KEYWORDS:** Breathers, algebraic solitons, rogue waves.
1. Introduction

Resonance often occurs in a nonlinear wave system when special criteria among wavenumbers and frequencies are met. Long wave-short wave resonant interaction is a classic example, where the phase velocity of a long wave matches the group velocity of a short wave. Significant interactions and energy transfer can then occur. A physically important example is the waveguide configuration for a two-layer fluid,\(^{1-5}\) where the short wave envelope on the surface and the long wave on the interface make up an intriguing dynamical system.

Rogue or freak waves are unexpectedly large displacements of a sea surface from an otherwise calm sea state, and have received intensive study.\(^ {6-12}\) The focusing nonlinear Schrödinger equation, which governs the evolution of a weakly nonlinear wave packet in deep water, has frequently been employed as a model. A particular localized mode, sometimes known as the ‘Peregrine breather’ in the literature, relaxes to a plane wave in the far field but possesses a single sharp maximum in amplitude for one localized value in space at one specific instance in time.\(^ {13,14}\) This algebraic soliton thus serves as a plausible model for rogue waves. Elegant theoretical extensions to incorporate higher order nonlinearities have been performed, and the analytic structures of these
solutions have been elucidated.\textsuperscript{15, 16} Concurrently, similar entities in the optical context have been investigated and even demonstrated experimentally to some extent.\textsuperscript{17, 18}

The objective here is to study ‘explode-decay’ type wave modes for the long wave-short wave resonant interaction model. This is valuable from the general perspective of nonlinear science, as new families of exact solutions will be obtained in this paper for this nonlinear evolution system. Furthermore, the time scale for such long-short wave resonance is $\varepsilon^{4/3}t$, where $t$ is the time scale of rapid oscillations in the packet and $\varepsilon$ is a small, non-dimensional parameter measuring the strength of the wave amplitude. The corresponding time scale for the nonlinear Schrödinger (NLS) equation is $\varepsilon^2t$ and thus effects of long-short wave resonance can be observed sooner in an asymptotic sense.

Theoretically, a relatively novel analytical technique will be employed. Many studies of rogue waves in the literature utilize the Darboux transformation.\textsuperscript{13, 15} Here the rogue wave modes will be derived from multi-soliton or multi-breather obtainable from the Hirota bilinear transformation.\textsuperscript{19} Mathematically, this is accomplished by taking a singular limit for solitary modes with nearly identical wavenumbers, augmented by special phase factors, and thus an appropriate name might be a ‘coalescence of wavenumbers’.\textsuperscript{20, 21} Alternatively these results can also be derived in terms of a ‘double pole’ (or
‘multiple pole’) solution in the language of the inverse scattering transform,\textsuperscript{22, 23} where a double pole in the reflection coefficient also leads to these exponential-algebraic modes. If these procedures are performed in the long wave regime (wavenumber tending to zero), one recovers these rogue waves / purely algebraic modes.

The structure of the paper can now be explained. To illustrate our method, a special ‘coalescence of wavenumbers’ limit will be taken for the pulsating, or ‘breather’, solution of the nonlinear Schrödinger equation to recover the Peregrine soliton in Section 2. In Section 3, we start by presenting the bilinear transform and Lax pair of the long-short interaction system, with a detuning parameter incorporated. A breather is then derived through the bilinear method. Although such a breather can also be obtained by a Bäcklund transformation and the dressing method,\textsuperscript{24} the bilinear method is conceptually simpler. Furthermore, the present effort corrects and generalizes previous work by demonstrating that the frequency parameter must be complex and cannot be real.\textsuperscript{25} The first contribution here is to calculate the rogue wave (spatiotemporally localized) mode explicitly, by taking a ‘coalescence of wavenumbers’ in the long wave limit. In Section 4, this mechanism is generalized to the finite wavenumber regime and exact exponential-algebraic modes are then obtained accordingly. Such modes will correspond to a ‘double
pole’ solution arising from an inverse scattering mechanism. Conclusions are drawn in Section 5.

2. The Rogue Wave Solution of the Nonlinear Schrödinger Equation

A breather of the NLS equation,

\[ iA_t + A_{xx} + \sigma A^2 A^* = 0, \]

where \( \sigma \) tunes the effect of nonlinear focusing in the system, can be obtained from the bilinear calculations,\(^ {26}\)

\[ A = \alpha \exp(i\sigma \alpha^2 t)[1 + g_1/f], \]

\[ (iD_t + D_x^2)g_1 . f + D_x^2 f . f = 0, \quad D_x^2 f . f = \sigma \alpha^2 [g_1 g_1^* + f(g_1 + g_1^*)] \]

\[ f = \exp(px) + \exp(-px) + s \exp(i\omega t + i\zeta) + s \exp(-i\omega t - i\zeta), \]

\[ g_1 = \lambda \exp(i\omega t + i\zeta) + \mu \exp(-i\omega t - i\zeta), \]

\[ s = \left[ \frac{1}{1 + \frac{p^2}{2\sigma \alpha^2}} \right]^{1/2}, \quad \omega = p\sqrt{p^2 + 2\sigma \alpha^2}, \]

\[ \lambda = \frac{sp}{\sigma \alpha^2} \left( p + \sqrt{p^2 + 2\sigma \alpha^2} \right), \quad \mu = \frac{sp}{\sigma \alpha^2} \left( p - \sqrt{p^2 + 2\sigma \alpha^2} \right). \]

The parameter \( \zeta \), originally arising from the flexibility in choosing the starting point in time, will now be exploited to take on arbitrary values. In
particular, on choosing \( \exp(i\zeta) = -1 \) and taking the limit of \( p \) approaching zero, one obtains

\[
A = \alpha \exp\left(i\alpha^2 t\right) \left\{ 1 - \frac{2\left(1+2i\alpha^2 t\right)}{\alpha^2 (x^2 + 2\alpha^2 t^2 + \frac{1}{2\alpha^2})} \right\},
\]

the familiar Peregrine breather ‘rogue wave’ solution of the NLS equation.

3. The Long Wave–Short Wave Resonance Model

The nonlinear evolution systems of the short wave envelope \( S \) and the induced long wave \( L \) are derived by multiple scale asymptotic expansion of the underlying fluid dynamics equations. In scaled coordinates, the equations are given by

\[
iS_t - S_{xx} = LS, \quad L_t + \Delta L_x = -\sigma(SS^*)_x,
\]

(1)

where \( \Delta \) is a detuning parameter measuring the deviation from exact resonance. The theoretical formulation will be treated by examining two aspects, namely, the Hirota bilinear transform and the Lax pair.

3.1 The Hirota bilinear transform

Using the dependent variable transformation \( f \) real,

\[
S = g/f, \quad L = 2(\log f)_{xx},
\]

(2)
the Hirota bilinear form of eq. (1) is \((C = \text{constant})^{2-5}\)

\[(iD_t - D_x^2) g \cdot f = 0, \quad (D_xD_t + \Delta D_x^2 - C) f \cdot f = -\sigma gg^*. \quad (3)\]

3.2 The Lax pair

Integrable nonlinear systems can often be investigated by converting them into auxiliary linear systems, with the Lax pairs being classic examples. These pairs form the basis for a variety of methods, e.g. the Darboux scheme. They usually take the form:

\[R_x = U \cdot R,\]
\[R_t = V \cdot R.\]

Both the compatibility condition and zero-curvature equation, related by

\[R_{xt} = R_{tx}\]
\[\Rightarrow U_t \cdot R + U \cdot R_t = V_x \cdot R + V \cdot R_x\]
\[\Rightarrow U_t \cdot R + U \cdot V \cdot R = V_x \cdot R + V \cdot U \cdot R\]
\[\Rightarrow U_t - V_x + [U, V] = 0,\]

provide a representation of the nonlinear system through the matrices \(U\) and \(V\).

The NLS equation and its conjugate form,

\[iA_t + A_{xx} + \sigma|A|^2A = 0, \quad -iA^*_t + A^*_{xx} + \sigma|A^*|^2A^* = 0,\]

are encapsulated by
\[
U = \begin{bmatrix}
  i\lambda & i\sqrt{\frac{\sigma}{2}}A \\
  i\sqrt{\frac{\sigma}{2}}A & -i\lambda
\end{bmatrix}, \quad V = \begin{bmatrix}
  2i\lambda^2 - \frac{i}{2}\sigma|A|^2 & i\sqrt{2}\sigma\lambda A^* + \sqrt{\frac{\sigma}{2}}A_x \\
  i\sqrt{2}\sigma\lambda A - \sqrt{\frac{\sigma}{2}}A_x & -2i\lambda^2 + \frac{i}{2}\sigma|A|^2
\end{bmatrix}.
\]

Likewise, the long wave-short wave resonance model can be written as,

\[
iS_t - S_{xx} - LS = 0, \quad -iS_t^* - S_{xx}^* - LS^* = 0, \quad L_t = -\sigma(SS^*)_x,
\]

where for simplicity we have concentrated on the case \(\Delta = 0\). The above system can be generated from

\[
U = \begin{bmatrix}
  i\lambda & \sqrt{\frac{\sigma}{2}}S & -iL \\
  0 & 0 & -\sqrt{\frac{\sigma}{2}}S^* \\
  -i & 0 & -i\lambda
\end{bmatrix}, \quad V = \begin{bmatrix}
  \frac{i}{3}\lambda^2 & \sqrt{\frac{\sigma}{2}}\lambda S - i\sqrt{\frac{\sigma}{2}}S_x & \frac{i}{2}\sigma|S|^2 \\
  \sqrt{\frac{\sigma}{2}}S^* & -\frac{2i}{3}\lambda^2 & \sqrt{\frac{\sigma}{2}}\lambda S^* - i\sqrt{\frac{\sigma}{2}}S^*_x \\
  0 & -\sqrt{\frac{\sigma}{2}}S & \frac{i}{3}\lambda^2
\end{bmatrix}.
\]

Importantly, the spectral problem for the long wave-short wave resonance model is of the third order, whereas the corresponding problem for the NLS equation is of the second order.

3.3 The breather solution for the long wave-short wave resonance model

The breather solution can in principle be generated from either of these formulations. We shall adopt the Hirota approach, as the algebra is slightly simpler. The appropriate expansion is \((p \text{ real, } \Omega \text{ complex})\)
\[ f = 1 + \exp(ipx - \Omega t + \zeta^{(1)}) + \exp(-ipx - \Omega^* t + \zeta^{(2)}) \]
\[ + M \exp(-\Omega t - \Omega^* t + \zeta^{(1)} + \zeta^{(2)}), \]

\( (\rho_0 = \text{constant} = \text{amplitude in the far field}) \)

\[ g = \rho_0[1 + a_1 \exp(ipx - \Omega t + \zeta^{(1)}) + a_2 \exp(-ipx - \Omega^* t + \zeta^{(2)}) \]
\[ + Ma_1 a_2 \exp(-\Omega t - \Omega^* t + \zeta^{(1)} + \zeta^{(2)}) \]. \quad (4) \]

The major distinction from the corresponding case of the NLS equation is the presence of a complex frequency \( \Omega \). \( \zeta^{(1)}, \zeta^{(2)} \) are arbitrary phase factors. Using the bilinear equations (3), the parameters, \( a_1, a_2 \) are given by

\[ a_1 = -\frac{(p^2 + i\Omega)}{(p^2 - i\Omega)}, \quad a_2 = -\frac{(p^2 + i\Omega^*)}{(p^2 - i\Omega^*)}, \quad (5) \]

and thus the relations \( a_1^* = 1/a_1, a_2^* = 1/a_2 \) follow. The dispersion relation is

\[ (\Omega - \Delta p i)(\Omega^2 + p^4) = 2i\sigma \rho_0^2 p^3, \quad (6) \]

and hence for small \( p \), the asymptotic expansion for \( \Omega \) is

\[ \Omega = p[\Omega_0 + \Omega_2 p^2 + O(p^4)], \]

where the leading order frequency \( \Omega_0 \) satisfies

\[ (\Omega_0 - \Delta i)\Omega_0^2 = 2i\sigma \rho_0^2. \quad (7) \]

The coefficient \( M \) is given by

\[ M = 1 + 4p^4/[(\Omega + \Omega^*)^2]. \]
To obtain a rogue wave mode, we perform a long wave limit expansion similar to that done in Section 2. On choosing \( \exp(\zeta^{(1)}) = \exp(\zeta^{(2)}) = -1 \), we obtain rogue wave modes where the short wave component is

\[
S = \rho_0 \left\{ 1 + \frac{-4 + 2x(\Omega_0 - \Omega_0^*) + 2it(\Omega_0^2 + (\Omega_0^*)^2)}{|\Omega_0|^2 \left[ \left( x + it \frac{\Omega_0 - \Omega_0^*}{2} \right)^2 + \left( \frac{\Omega_0 + \Omega_0^*}{4} \right)^2 t^2 + \frac{4}{(\Omega_0 + \Omega_0^*)^2} \right] } \right\},
\]

while the long wave component is given by

\[
L = \frac{4}{\left[ \left( x + it \frac{\Omega_0 - \Omega_0^*}{2} \right)^2 + \left( \frac{\Omega_0 + \Omega_0^*}{4} \right)^2 t^2 + \frac{4}{(\Omega_0 + \Omega_0^*)^2} \right] } - \frac{2 \left[ 2x + it(\Omega_0 - \Omega_0^*) \right]^3}{\left[ \left( x + it \frac{\Omega_0 - \Omega_0^*}{2} \right)^2 + \left( \frac{\Omega_0 + \Omega_0^*}{4} \right)^2 t^2 + \frac{4}{(\Omega_0 + \Omega_0^*)^2} \right]^2}.
\]

\( \Omega_0 \) is given by eq. (7). The denominators in eqs. (8, 9) are clearly nonsingular. This set of solutions constitute purely algebraic mode exhibiting ‘explode-decay’ behavior (Figs. 1, 2).

4. The Double Pole Solution

This ‘coalescence of wavenumbers’ approach can also be applied to the case where the pre-existing wave is of a finite wavelength. A more physically
realistic picture can be obtained by incorporating another horizontal spatial
dimension ($y$). To simplify the algebraic complexity, we shall now ignore the
effect of detuning and restrict our attention to a 2-soliton of the long wave-short
wave model, where the governing equations and an exact expression are known
from earlier works,\(^3\) \((F\) real\)

\[iS_t + iS_y - S_{xx} - LS = 0, \quad L_i = -\sigma(SS^*)_x, \quad (10)\]

\[S = G/F, \quad L = 2(\log F)_{xx},\]

\[G = \exp(\eta_1) + \exp(\eta_2) + a(1, 2, 1^*) \exp(\eta_1 + \eta_2 + \eta_1^*) + a(1, 2, 2^*) \exp(\eta_1 + \eta_2 + \eta_2^*),\]

\[F = 1 + a(1, 1^*) \exp(\eta_1 + \eta_1^*) + a(1, 2^*) \exp(\eta_1 + \eta_2^*) + a(2, 1^*) \exp(\eta_2 + \eta_1^*) + a(2, 2^*) \exp(\eta_2 + \eta_2^*) + a(1, 2, 1^*, 2^*) \exp(\eta_1 + \eta_2 + \eta_1^* + \eta_2^*),\]

\[\eta_n = p_n x + q_n y - \Omega_n t + \eta_n^{(0)}, \quad \Omega_n = ip_n^2 + q_n, n = 1, 2,\]

\[a(i, j^*) = [(p_i + p_j^*)(\Omega_i + \Omega_j^*)]^{-1}, \quad a(i, j) = (p_i - p_j)(\Omega_i - \Omega_j),\]

\[a(i^*, j^*) = [a(i, j)]^*, \quad a(i, j, k^*) = a(i, j) a(i, k^*) a(j, k^*),\]

\[a(i, j, k^*, l^*) = a(i, j) a(i, k^*) a(i, l^*) a(j, k^*) a(j, l^*) a(k^*, l^*).\]

On using

\[p_1 = p + i\epsilon, \quad p_2 = p - i\epsilon, \quad q_1 = q + i\epsilon, \quad q_2 = q - i\epsilon, \quad \eta_1^{(0)} = 1/\epsilon, \quad \eta_2^{(0)} = -1/\epsilon,\]

we obtain the double pole solution of eq. (10) by letting $\epsilon \to 0$ as
\[ G = \exp \left( -ip^2 t \right) \]

\[
\begin{align*}
2 \left[ (x - my + mt) + 2pti \right] & \exp \left( px + qy - qt \right) \\
\quad + \frac{-2p + im}{2p^2q^2} \left[ \left( \frac{2p}{q} + 2pt \right) + i \left( \frac{1}{p} + \frac{m}{q} - x - my + mt \right) \right] & \exp \left[ 3 \left( px + qy - qt \right) \right]
\end{align*}
\]

\[ F = 1 + \frac{1}{2pq} \exp \left[ 2 \left( px + qy - qt \right) \right] \left[ 2 \left( x + my - mt - \frac{1}{p} - \frac{m}{q} \right) + 4p^2 \left( t + \frac{1}{2q} \right) \right] \]

\[ \frac{4p^2 + m^2}{4} \exp \left[ 4 \left( px + qy - qt \right) \right] \]

which is illustrated in Fig. 3 for some typical parameter values. The taller soliton catches up with the shorter one, and they exchange identity without actual merging. The analytic structure of \(|S|\) resembles an ordinary 2-soliton and a heuristic explanation can be offered. The long-short system, just like the NLS equation, admits a 2-soliton breather, where two solitary pulses with nearly equal frequencies pulsate periodically. This ‘coalescence of wavenumbers’ destroys this beating behavior and the oscillatory character of the amplitudes.
disappears, as confirmed in Fig. 3. The analytical structure of eq. (11) is similar to those double pole solutions of the NLS and the modified Korteweg-de Vries equations, but details of the inverse scattering mechanism applied to eq. (10) and a comparison with the present result will be left for future investigations.

5. Conclusions

The long wave-short wave resonance model arises as a simplified model of certain circumstances in the dynamics of the upper ocean, as well as other physical contexts. Theoretically it admits both the Hirota bilinear transform and Lax pair formulations. In this paper, both purely algebraic and exponential-algebraic modes are derived. The critical difference between the nonlinear Schrödinger equation and the present long-short wave system is the nature of the dispersion relation, namely, the breather solution of the latter must have a complex frequency solvable from a cubic polynomial. Purely algebraic modes are obtained from a singular limit in the long wave regime. The exponential-algebraic modes are derived from a similar procedure applied in the realm of finite wavenumber.

There are many directions for future works from both the theoretical and practical perspectives. A higher order breather solution can be constructed, and from there analytic structures of multiple rogue wave modes can be studied.
The effects of the detuning parameter on the evolution and stability of wavetrains can be investigated numerically, and will provide useful insight into the dynamics of the upper ocean. Furthermore, a breather periodic in time and spatially localized in space can be calculated, similar to the situation for the NLS equation, but technical details remain to be worked out. Finally, this whole mechanism can in principle be applied to other integrable nonlinear evolution equations, generating exact solution and information for further research in nonlinear evolution equations.

**Acknowledgements**

Partial financial support has been provided by the University of Hong Kong Incentive Award Scheme.
References


**Figures Captions**

(1) Fig. 1: Intensity of the short wave envelope $|S|^2$ (eq. (8)) of the rogue wave mode versus $x$ and $t$: $\Delta = 0.1, \sigma = 1, \rho_0 = 1$, (a) three dimensional view; (b) top view.

(2) Fig. 2: The long wave $L$ (eq. (9)) of the rogue wave mode versus $x$ and $t$: $\Delta = 0.1, \sigma = 1, \rho_0 = 1$, (a) three dimensional view; (b) top view.

(3) Fig. 3: The interaction of solitary pulses in a ‘double pole’ solution (eq. (11)), $|S|^2 = |G/F|^2$, $p = 1, q = 0.1, m = 1$: (a) $t = -20$ (just before the interaction), (b) $t = 0$ (just after the interaction).
Fig. 2(a)
Fig. 2(b)
Fig. 3(a)