Interaction of a Large Amplitude Interfacial Solitary Wave of Depression with a Bottom Step

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Abstract

The dynamics and energy balance of the transformation of a large amplitude interfacial solitary wave of depression transformed at the bottom step are studied. Three simulations are described using the non-hydrostatic extension of the Princeton Ocean Model (POM), based on the fully nonlinear Navier-Stokes equations in the Boussinesq approximation. The first simulation is when the ratio of the step height to the lower layer thickness is about 0.4 and the incident wave amplitude is less than the limiting value estimated for a Gardner soliton. It shows the applicability of the weakly nonlinear model (the extended Korteweg-de Vries or Gardner equation) to describe the transformation of a strongly nonlinear wave in this case. In the second simulation the incident wave amplitude is increased and is then described by the Miyata-Choi-Camassa solitary wave solution. In this case, the process of wave transformation is accompanied by shear instability and the formation of Kelvin-Helmholtz billows that results in a thickening of the interface layer. In the third simulation, the ratio of the step height to the thickness of the lower layer is 0.8, and then the same Miyata-Choi-Camassa solitary wave passes over the step, but undergoes stronger reflection and mixing between the layers although Kelvin-Helmholtz instability is absent. The energy budget of the wave transformation is calculated. It is shown that the energy loss in the vicinity of the step
grows with an increase of the ratio of the incident wave amplitude to the thickness of the lower layer over the step.

**Introduction**

A two-layer representation of the ocean density stratification is a useful approximation to apply analytical and numerical methods in the study of internal solitary waves. Rigorous mathematical results of the existence and properties of an interfacial solitary wave were found by Amick and Turner\(^1\) and by Tung, Chan and Kubota\(^2\). For small wave amplitudes the dynamics of interfacial solitary waves can be described using the extended Korteweg-de Vries, or Gardner, equation which is an integrable equation\(^3-6\). In particular, as the wave amplitude is increased to a limiting value, its width is also increased and the limiting solitary wave has a “table-top” shape. The same properties are obtained using the Miyata, Choi-Camassa and Ostrovsky-Grue models for strongly nonlinear but weakly dispersive solitary waves\(^7-10\), but these strongly nonlinear models predict a different value for the limiting wave amplitude. Numerical calculations in the framework of the full Euler equations confirm the conclusions of these analytical theories\(^11-13\).

The effects of slowly varying bottom depth can be incorporated into these analytical theories and the transformation of a solitary wave over variable depth has been investigated in detail in both, the weakly nonlinear approximation\(^14-19\) and in a fully nonlinear model\(^20\). The case when the bottom topography varies rapidly is more difficult for theoretical analysis. Nevertheless the transformation of a weakly nonlinear solitary wave at a bottom step seems to be quite well described in the framework of a weakly nonlinear asymptotic theory, see Grimshaw et al\(^21\). In this paper it is shown that in the vicinity of the step the wave transformation can be described using the linear long-wave theory for interfacial waves, from which the coefficients for wave reflection and transmission are calculated. The reflected and
transmitted waves in the vicinity of a step have solitary-wave shapes, but their parameters do not satisfy the steady-state solution. Hence they fission into secondary waves (internal solitons), and the using inverse scattering technique allows for an estimate of the number and amplitudes of these secondary solitons. For moderate and large incident wave amplitudes, this process has been studied numerically in the recent paper by Maderich et al\textsuperscript{22}, which considered the case when the interface was close to the bottom and hence the incident wave was a wave of elevation. In these numerical experiments, the ratio of the initial wave amplitude to the layer thickness is varied up to one half, and nonlinear effects are then essential. In general, the characteristics of the generated solitary waves obtained in the fully nonlinear simulations are in reasonable agreement with the predictions of the theoretical model of Grimshaw et al\textsuperscript{21}, which is based on matching linear shallow-water theory in the vicinity of the step with solutions of the Gardner equation for waves far from the step.

In this present paper, the problem considered is that when the interface is close to the surface and hence the incident interfacial solitary wave is a wave of depression. We pay particular attention to the effects mixing, Kelvin-Helmholtz instability and wave energy dissipation. It is important to mention that Kelvin-Helmholtz instability of large amplitude solitary waves has been clearly observed in laboratory experiments in tanks of constant depth\textsuperscript{11,23,24} and has also been studied theoretically\textsuperscript{25,26}. The formation of instability at the interface has been modeled in the framework of the fully nonlinear Euler equations\textsuperscript{24} and good comparison with experiments had been shown. Similar phenomena are observed in the ocean when the stratification is real\textsuperscript{27,28}. The manifestation of such effects when a large amplitude solitary wave interacts with a bottom step has some specific features which we shall describe in this paper. We consider here the case when the incident wave is a depression interfacial solitary wave because its interaction with a bottom step and the consequent energy dissipation due to mixing is better pronounced than for an elevation interfacial solitary wave.
Theoretical formulas for interfacial solitary waves of moderate and large amplitudes are presented in Section 1. Then the non-hydrostatic numerical model for a stratified fluid is briefly described in Section 2. The numerical results for the transformation of an interfacial depression solitary wave interacting with a bottom step are presented in Section 3. The energy dissipation due to these processes is discussed in Section 4. Our results are summarized in the Discussion.

1 Analytical models for interfacial solitary waves

The configuration for a two-layer stratification is shown in Fig. 1, where the upper and lower layers have thicknesses $h_1$ and $h_2$ with total depth $H = h_1 + h_2$, and the densities $\rho_1$ and $\rho_2$, respectively. The difference between densities, $\Delta \rho = \rho_2 - \rho_1$ is assumed to be small compared to either undisturbed density $\rho_{1,2}$, that is we use the Boussinesq approximation. The interface lies near the surface ($h_1 < h_2$) and as is well-known, the solitary wave has negative polarity, and so is a wave of depression. An solitary wave with interface displacement $\eta(x,t)$ approaches the bottom step from the right. For convenience we say that the incident wave approaches from deep to shallow water but of course both depths are smaller the wavelength. The thickness of lower layer before the step is $h_2$, and after the step is $h_2^+$. 

Fig.1. Sketch of the problem. Dashed lines shows cross-sections where energy fluxes are calculated (see Section 5).
The analytical description of interfacial solitary waves of weak and moderate amplitudes can be carried out using the extended Korteweg-de Vries (Gardner) equation\textsuperscript{3-5}.

\[
\frac{\partial \eta}{\partial t} + (c_o + \alpha \eta + \alpha_1 \eta^2) \frac{\partial \eta}{\partial x} + \beta \frac{\partial^3 \eta}{\partial x^3} = 0
\]  

(1)

where \(t\) is time, and \(x\) is the horizontal coordinate. The coefficients of equation (1) are (in the Boussinesq approximation)

\[
c_o = \sqrt{g \Delta \rho h_1 h_2 / \rho_2 (h_1 + h_2)}
\]  

(2)

\[
\begin{align*}
\alpha &= \frac{3c_0}{2} \frac{h_1 - h_2}{h_1 h_2} \\
\beta &= \frac{c_0 h_2 h_1^4}{6} \\
\alpha_1 &= -\frac{3c_0}{8h_1 h_2^2 (h_1^2 + h_2^2 + 6h_1 h_2)}.
\end{align*}
\]  

(3)

Here \(c_0\) is the linear long-wave phase speed of an interfacial wave, \(g\) is the gravity acceleration and \(\alpha, \alpha_1\) and \(\beta\), describe the coefficients of quadratic and cubic nonlinearity, and dispersion respectively. The steady-state solution of the Gardner equation describing the interface solitary wave is

\[
\eta(x,t) = \frac{D}{1 + B \cosh(\gamma (x - V t))},
\]  

(4)

where

\[
D = \frac{6 \beta \gamma^2}{\alpha}, \quad B^2 = 1 + \frac{6 \alpha_1 \beta \gamma^2}{\alpha^2}, \quad V = \beta \gamma^2
\]  

(5)

and \(\gamma\) is a free parameter, inverse to the solitary wave length. The solitary wave amplitude is
\[ a = \frac{D}{1 + B}, \]  

(6)

and its sign is negative if \( h_1/h_2 < 1 \) (wave of depression). The wave amplitude varies from small values (where the Gardner equation coincides with the Korteweg-de Vries equation) to the limiting amplitude

\[ a_{\text{lim}} = -4h_1h_2 \frac{h_2 - h_1}{h_1^2 + h_2^2 + 6h_1h_2}, \]  

(7)

when the solitary wave has a “table-top” shape.

The Miyata-Choi-Camassa equations describe the shallow-water solitary waves (MCC solitary waves) in the approximation of weak dispersion but with no limitation of nonlinearity, see Miyata\textsuperscript{7-8} and Choi and Camassa\textsuperscript{9}. The shape of the solitary wave of amplitude \( a \) is determined from the ordinary nonlinear equation for the interfacial displacement \( \eta \) (in the Boussinesq approximation)

\[ \left( \frac{d\eta}{dX} \right)^2 = \frac{3c_0^2}{c^2h_1h_2(h_1 - h_2)} \times \frac{\eta^2(\eta - b_1)(\eta - b_2)}{(\eta - b_3)} \]  

(8)

where \( X = x - ct \), and

\[ b_*= \frac{h_1h_3}{h_2 - h_1}, \quad c = c_0 \sqrt{\frac{(h_1 - a)(h_1 + a)}{h_1h_2}}, \]  

(9)
and the parameters $b_1$ and $b_2$ are the roots of the quadratic algebraic equation

$$b^2 + q_1 b + q_2 = 0,$$

(10)

where

$$q_1 = h_2 - h_1, \quad q_2 = h_1 h_2 \left( \frac{c^2}{c_0^2} - 1 \right).$$

(11)

The implicit solution $X = X(\eta)$ may be obtained by integrating equation (8) and it is a combination of elliptic integrals\(^9\). The amplitude of the solitary wave varies from zero to the limiting value $A_{\text{lim}}$

$$A_{\text{lim}} = \frac{h_1 - h_2}{2}.$$

(12)

If the difference between the thicknesses of both layers is weak, the formula (12) coincides with “Gardner” formula (7). For large amplitudes formula (12) predicts a larger value for the limiting solitary wave amplitude than (7).

To describe our numerical results two parameters are introduced. The parameter of nonlinearity $\epsilon_{nl} = \alpha a / c_0 + |\alpha_1| a^2 / c_0$ in the weakly nonlinear asymptotic theory (the Gardner equation) and characterizes the waves above a flat bottom. Strictly speaking, it should be much less than unity for applicability of the Gardner model. Nevertheless, in practice this model may be used even for $\epsilon_{nl} \geq 1$, see Ostrovsky and Grue\(^{10}\) and Maderich et al\(^{22}\).

Another parameter $\mu = \frac{|a|}{h_{zs}}$, see Brovchenko et al\(^{29}\), characterizes the interaction of the wave with the step. Here $a_-$ is the amplitude of the incident wave before the step and $h_{zs}$ is the
height of the lower layer over the step. The limit $\mu \to 0$ means that wave amplitude is small and $\mu = 1$ when the incident wave amplitude is equal to the height of the lower layer and the disturbed interface touches the step. The numerically computed waves are compared with the analytical solutions of Gardner and MCC models.

2 Numerical model based on the Navier-Stokes equations

The numerical model using the Navier–Stokes equations is fully described in Maderich et al.\textsuperscript{22} It is applied here in a two-dimensional mode with horizontal coordinate $x$ and vertical coordinate $z$, see Fig. 1. The computational tank parameters are as follows. The total length is 30 m, while the length of the deep part is 16 m and the step position is at $x = 14$ m. The background stratification in the flume is modelled by two layers with upper and bottom layer salinities $S_{up} = 2$ and $S_{bot} = 15$ at constant temperature of $20^\circ$ C, respectively. The density jump $\Delta \rho/\rho_2$ is equal 0.01. The vertical profile $S(z)$ in the transition zone is approximated by

$$S(z) = \frac{S_{up} + S_{bot}}{2} - \frac{S_{bot} - S_{up}}{2} \tanh \left( \frac{(z - h_1)}{dh} \right)$$

where the thickness of the upper layer is $h_1 = 4$ cm. The interface initial thickness for $dh = 0.2$ cm is much less than the thickness of both layers. In the simulations we visualized the interface as an isohaline with salinity equal 8.5. Three simulation runs were done. All the runs were carried out with the thickness of the lower layer in the deeper part of tank is $h_2 = 28$ cm. The height of the step is chosen to be 8 cm for runs 1 and 2, and 16 cm for run 3. The numerical experiments were carried out with molecular values of kinematic viscosity $\nu = 1.14 \cdot 10^{-6}$ m$^2$s$^{-1}$ and diffusivity of salt $\chi = 10^{-9}$ m$^2$s$^{-1}$. Non-slip boundary conditions at the
bottom and end walls were used, whereas at the free surface the viscous stresses were set to zero. The flux of salinity through the flume boundaries was also set to zero. The computational grid is 2400×120. In the numerical experiments the solitary wave was generated by a collapse of mixed volume at the right-end side of the tank. Further details of the method are described by Maderich et al\textsuperscript{22}.

3 Numerical simulations of wave transformation at a step

We present our results in a dimensionless form where the horizontal $x$ and vertical $z$ coordinates and interface displacement $\eta$ are normalized on the height of the upper layer $h_1$ and time $t$ is normalized to $\tau$ by

$$\tau = t / \sqrt{\rho_s h_1 / (\Delta \rho g)} .$$

This allows us to apply the results to an oceanic situation using appropriate scaling. The input run parameters are presented in Table 1. Here and in the sequel “-” and “+” denote values of variables before and after step, respectively. The characteristics of incident solitary waves in Table 1 are estimated in the vicinity of the step at the cross-section $x_r = 14.2$ m (Fig. 1). To demonstrate the nonlinearity of the interfacial solitary wave we calculate the limiting values of the solitary wave amplitude according to the Gardner equation $a_{\text{lim}}$ (7) and Camassa-Choi theory $A_{\text{lim}}$ (12), and they also given in Table 1.
Table 1 The parameters of runs

<table>
<thead>
<tr>
<th>Run</th>
<th>$h_1$ (cm)</th>
<th>$h_2$ (cm)</th>
<th>$h_{2+}$ (cm)</th>
<th>$a.$ (cm)</th>
<th>$a_{lim.}$ (cm)</th>
<th>$A_{lim.}$ (cm)</th>
<th>$h_{2+}/h_2$</th>
<th>$a. / a_{lim.}$</th>
<th>$a.$ / $A_{lim.}$</th>
<th>$\varepsilon_{nl}$</th>
<th>$\mu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4</td>
<td>28</td>
<td>20</td>
<td>-7.3</td>
<td>-7</td>
<td>-12</td>
<td>0.71</td>
<td>1.64</td>
<td>0.90</td>
<td>0.55</td>
<td>4.0</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>28</td>
<td>20</td>
<td>-8.8</td>
<td>-7</td>
<td>-12</td>
<td>0.71</td>
<td>2.21</td>
<td>1.21</td>
<td>0.74</td>
<td>6.5</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>28</td>
<td>12</td>
<td>-8.8</td>
<td>-7</td>
<td>-12</td>
<td>0.43</td>
<td>2.21</td>
<td>1.21</td>
<td>0.74</td>
<td>6.5</td>
</tr>
</tbody>
</table>

Run 1

The incident wave formed initially by the collapse mechanism has an amplitude -7 cm that is about half of the amplitude of a limiting solitary wave in the full nonlinear model (12), $A_{lim} = -12$ cm, and it is equal the amplitude of the limiting Gardner solitary wave for this configuration, $a_{lim} = -7$ cm. The shape of this wave is close to the shape of an MCC solitary wave but is a little bit narrower as shown on Fig. 2.

Fig. 2. The shape of the initially formed solitary wave at $\tau = 166$ in Run 1.

As expected, it is far from the table-top solitary wave using in the same scale for comparison.
The solitary wave amplitude decreases with distance from the source due to several effects; dissipation, diffusion leading to thickening of interface layer and interaction with the step near the wave front. Near the step (at $x_r$), the wave amplitude is equal -6.6 cm and its shape is well described by both the Gardner and MCC models (Fig. 3 and 4a). Nevertheless the MCC solitary wave is still a little bit wider.

![Diagram](image)

**Fig.3.** The comparison of the incident solitary wave shape before the step at $\tau = 276$ with the shape of Gardner and MCC solitary waves in Run 1.

The “interaction” parameter is small for this run ($\mu = 0.33$) and we assume that the Gardner equation theory may be applied to describe the reflection and transmission of this incident wave\(^{21}\). The reflection coefficient $R$ from the step computed with use of the formula from linear theory\(^{21,22}\) $R = (1 - c_+/c_+)/(1 + c_+/c_+)$ is very small and it is equal to 0.01. This is why reflection is not visible in Fig. 4b. The transmission coefficient according to the linear theory is $T = 2/(1 + c_+/c_+)$ is equal 1.01. The transmitted wave amplitude immediately after the step equals -6.7 cm (Fig. 4c), coinciding with “linear” predicted amplitude $Ta$. There is no wave disturbance at the step and there is no transformation of energy due to any instability.
Then, the transmitted pulse transforms into a solitary wave and a weak oscillatory tail (Fig. 4d). The amplitude of the transmitted solitary wave is -5.1 cm (24% less than on a step) and its shape is well described by both models (Fig. 5). It should be noted here that the transmitted wave is almost the same as the incident wave because the reflection coefficient is only 1%; nevertheless it transforms strongly in the left-hand side of tank. There is a noticeable change in nonlinearity (more than 10%), and a large changing in dispersion (30%), whereas the speed of wave propagation is changed by only 2%. In this run, weakly nonlinear theory is used to model the solitary wave transformation after the step. The Gardner model used has been described by Holloway et al\textsuperscript{17}, Grimshaw et al\textsuperscript{21} and Maderich et al\textsuperscript{22}. First we note that only one solitary wave is predicted by this theory\textsuperscript{21} for these model parameters, and the numerical modeling of this. The comparison of the solitary wave shapes obtained by both models is shown in Fig.5; the solitary wave modeled by the Gardner model is the solid line and the solitary wave modeled from the full nonlinear model is the dashed line. There is essentially no difference in both solitary wave positions during the transformation.
Fig. 5. The comparison of the solitary wave transformation after step in the Gardner model (solid line) and full nonlinear model (dashed line).

The transmitted solitary wave in the framework of the Gardner equation theory has an amplitude of -5.5 cm that is only 7% more than solitary wave amplitude modelled in the full system of equations (-5.1 cm) but it is a little wider. This disagreement between the asymptotic theory and the Navier-Stokes model is of the same order as that obtained by Maderich et al.\textsuperscript{22} for an incident solitary wave of elevation of moderate amplitude and related to a loss of energy in the Navier-Stokes model due small viscosity. Thus, the process of a depression solitary wave passing over a step agrees quite well with the theoretical scenario of the linear wave transformation at a step and subsequent soliton formation in the transmitted wave field using the Gardner model\textsuperscript{21,22}. This is a surprising result because the nonlinearity of the incident wave characterised by parameter $\varepsilon_{nl} = 4$ is quite strong.

\textbf{Run2}

In the next run the amplitude of incident solitary wave at $x_r$ is -8.8 cm which is less than the limiting value of the solitary wave amplitude in the MCC model (-12 cm). For such amplitudes the Gardner model is not applicable because the limiting amplitude in this model is only -7.3 cm. The computed shape of the solitary wave is well described by the MCC model (Fig. 6).
The interaction parameter here $\mu = 0.44$ and it is bigger than the same parameter in Run 1. Although the amplitude of the incident wave in Run 2 only 1.35 times bigger than in Run 1, the process of wave transformation at the step in Run 2 differs qualitatively from the process in Run 1. As is shown on Fig. 7(b, c), the wave transformation in the vicinity of the step is accompanied by the formation of high-frequency tail.

Fig. 7. The solitary wave transformation at the step in the Run 2 in successive times.
Let us consider this process more detail. Fig. 8 shows the transformation of the salinity field during the wave propagation over the step. The wave becomes unstable at its trough and the billows grow at its rear end. The formation of such billows due to Kelvin-Helmholtz instability is well known and has been studied for large-amplitude interfacial waves\textsuperscript{11,24}. In the solitary wave trough the Richardson number is

\[ Ri = -\frac{g}{\rho_0} \frac{\partial \rho}{\partial z} \left( \frac{\partial U}{\partial z} \right)^2, \tag{15} \]

where \( U \) is the horizontal velocity. It can be estimated roughly as,

\[ Ri = \frac{g\Delta \rho}{\rho_0} \frac{\Delta h}{(\Delta U)^2}, \tag{16} \]

where \( \Delta h \) is the thickness of the interface layer with a linear distribution of density and \( \Delta U \) is the shift of the horizontal velocity on the interface. The vertical structure of the salinity and horizontal velocity in the wave trough are presented in Fig. 9 and they are used to compute \( \Delta h \) and \( \Delta U \). In the trough of the solitary wave the interface subsided on the wave amplitude (Fig. 9a).
The observed \( \Delta h \sim 1.5 \) cm exceeds the initial value of the interface thickness (~0.5 cm) due to diffusion and the mixing processes initiated by the wave passage. The horizontal velocity shear \( \Delta U \) estimated from Fig. 9b is about 9 cm/s. As a result, the Richardson number is \( Ri \sim 0.18 \), less than the critical value 0.25 of the Miles-Howard theory\textsuperscript{30}. The characteristic scale of the Kelvin-Helmholtz billows is about \( \lambda_{KH} \approx 11.5 \) cm. The ratio of unstable billow scale to the interface thickness \( \lambda_{KH} / \Delta h = 7.7 \) which agrees well with the theoretical estimate\textsuperscript{30} \( \lambda_{KH} / \Delta h = 7.5 \), and the results of laboratory experiments for an interfacial solitary wave of large amplitude\textsuperscript{23}, where \( \lambda_{KH} / \Delta h = 7.9 \). Therefore, the Kelvin-Helmholtz instability is one of the possible mechanisms for the formation of a secondary solitary wave from a large amplitude wave passing over the step.
As a result, the transmitted wave decreases its amplitude due to the Kelvin-Helmholtz instability (Fig. 8). This mechanism was parameterized by Bogucki and Garrett\textsuperscript{32} in the framework of the Korteweg-deVries theory. When the wave amplitude and shear decreases the wave is stabilized (Fig. 7c and 8d) and then transforms to a solitary wave with a dispersive tail (Fig. 7d). The amplitude of the formed solitary wave is about 6.5 cm and it is well described by the MCC solution (Fig. 10). Note that the Gardner model predicts a limiting value for the wave amplitude after the step of 5.7 cm, and therefore cannot describe the transmitted solitary wave.
Fig. 10. The comparison of the shape of the transmitted solitary wave at $\tau = 517$ with the shape of MCC solitary wave in Run 2.

The reflection coefficient $R$ in Run 2 is the same as that in Run 1 and is equal to 0.01. Thus the reflected solitary wave here also has a very small with amplitude of about 0.8 cm, in good agreement with linear theory.

**Run 3**

In this run the step height is 16 cm, twice bigger than used in the previous runs. The initial solitary wave is the same as in run 2 (amplitude -8.8 cm), this shape was reproduced early in Fig. 6. The depth of the lower layer after the step is 12 cm and the interaction of the incident wave with step is strong in this case and the interaction parameter is 0.74 and this shape was reproduced early in Fig. 6. It means that the wave trough goes down very close to the step and we may expect a strong interaction between the wave and the step.

The transformation of this solitary wave at the step is shown in Figs. 11 and 12. It is clearly evident that wave touches the step through its trough (Fig. 11b). This leads to the creation of large eddy and consequent strong mixing as is shown on Fig. 12. Due to this the reflection is large and the transmitted wave has a smaller amplitude than in Run 2 (cf. Fig 5 and Fig 11a). In linear theory the reflection coefficient $R$ here is also small, 0.07. In fact, the reflection is large and the amplitude of the reflected wave is about -4 cm (Fig. 11c) what is much larger than $R A_{\text{incident}}$. 

Fig. 11. Solitary wave transformation on the step in the Run 3.

Hence the linear theory for wave reflection cannot be used for this case. Accordingly, the amplitude of transmitted wave is less than $T_A^{\text{incident}}$ ($T = 1.04$) and is about $-3.5$ cm (Fig. 11c). Interestingly, the shape of the transmitted wave at the time $\tau = 370$ (Fig. 12) is close to the shape of a table-top solitary wave in both theoretical models but it is not a steady wave and a dispersive tail is generated. At the end of the numerical tank (Fig. 11d) the amplitude of the secondary solitary wave is about $-2.5$ cm and its shape is close to both the Gardner and MCC solitary waves (Fig. 13), and is far from a table-top shape in Fig. 12.
Fig. 12. The comparison of the transmitted wave shape at $x I$ with the shape of the Gardner and MCC solitary waves in Run 3 at $\tau = 370$.

![Graph showing wave comparison](image)

Fig. 13. The comparison of the transmitted wave shape at large distance from the step with the shape of the Gardner and MCC solitary waves in Run 3 at $\tau = 630$.

The wave in Figs 11c and 12 is an example of unsteady thick solitary “formation” which is discussed by Grimshaw et al.\textsuperscript{33} in connection with the damping of a table-top solitary wave. Kelvin-Helmholtz instability was absent in this Run 3 due to the strong interaction of the incident wave with the step and the reflection of a large amplitude solitary wave from the step (Fig. 11c) that essentially decreases the amplitude of transmitted wave and the shear velocity.

Process of the wave interaction with the bottom step is detailed on Fig. 14. The character of the flow regimes can be characterized by the composite Froude number

$$Fr^2 = \frac{U_1^2}{g' h_{1*}} + \frac{U_2^2}{g' h_{2*}},$$

where $g' = g \Delta \rho / \rho_0$ and it is shown in Fig. 15.
Fig. 14. The velocity vectors $\vec{U}$ and vorticity $\omega$ near the step when incident wave passes through it.

The interaction process may be divided into several stages. In the first stage (Fig. 14a) the front of the incident wave is deformed by flow forming in the lower layer. The Froude number at the step grows to a value 0.4 and the flow is sub-critical. In the second and third stages (Fig. 14b, c) this flow becomes critical and supercritical (Fig. 15) at time $\tau = 286-291$. This flow entrains the wave trough into the bottom layer, with the formation of a strong eddy below the step and a weaker eddy of opposite sign above it. This pair of eddies pulls fluid from the upper layer into the lower layer. Then at the fourth stage the pair of eddies reflects from the bottom step leading to intensive mixing of stratified water in the neighbourhood of the step. This numerical modelling agrees qualitatively with the laboratory experiment by Brovchenko et al$^{28}$ on the interaction of a solitary wave of large amplitude with long rectangular obstacle.
Fig. 15. The composite Froude number of flow at successive times.

4 Wave energy budget

In this section we describe the energy budget of the transformation of an interfacial solitary wave at a step. The energy density consists of kinetic energy density

\[ E_k(x, z, t) = \frac{1}{2} \rho_0 (U^2 + W^2) , \]  

(18)

where \( U \) is the horizontal and \( W \) is the vertical velocities, correspondingly, and the potential energy density

\[ E_p(x, z, t) = \rho(x, z, t)gz . \]  

(19)

For estimations of energy budget transformations we calculate that part of the potential energy available for conversion into kinetic energy (APE), which may be estimated by the following formula\(^{27,34-36}\),
\[ E_\phi(x, z, t) = g \int_z^{z'} (\bar{\rho}(z') - \rho) dz', \]  

(20)

when the reference density \( \bar{\rho}(z, t) \) profile is invertible with inverse \( z^*(\rho, t) \). In open system, such as that considered here for a propagating solitary wave, the undisturbed far field density distribution can be used as reference profile\(^3^7\). The sum of \( E_K \) and \( E_A \) is called the pseudoenergy density \( E_{PSE} \). When dissipation can be ignored and the reference density is time independent, the evolution of depth integrated pseudoenergy is described by the equation

\[ \frac{\partial}{\partial t} \int_{-H}^{0} E_{PSE} dz + \frac{\partial}{\partial x} F(x, t) = 0, \]  

(21)

where \( F(x, t) \) is the depth integrated pseudoenergy flux

\[ F(x, t) = \int_{-H}^{0} (E_{PSE} + p) dz \]  

(22)

where \( p \) is pressure disturbance due to passing wave.

Our goal is to estimate the balance of the total energy after the wave has crossed the step. As in the papers by Grimshaw et al\(^2^1\) and Maderich et al\(^2^2\), we estimate the total energy of the incident, reflected and transmitted waves before the step and after the step by integrating (21) over the horizontal coordinate \( x \) on each side of the step from the tank wall to the positions of the chosen sections shown in Fig. 1. Two sections near the step (before and after it) have been chosen to exclude the zone of mixing at the positions \( x_r \) and \( x_f \). We
assume that the energy losses take place mainly near the step. The energy fluxes of the incident and reflected waves are estimated at the section \(x_i\) and the flux of the transmitted wave is estimated at the section \(x_f\). The energy fluxes \(F(x,t)\) are shown on Fig. 16 versus time for the three numerical experiments. Then volume integration of these flows outside the mixing zone allows us to estimate the energy of the incident (\(PSE_{in}\)), reflected (\(PSE_{ref}\)) and transmitted (\(PSE_{tr}\)) waves, e.g.

\[
\begin{align*}
PSE_{in} &= \int_{x_r}^{L} \int_{0}^{H} E_{PSE} \, dx \, dt = \int_{t_i}^{t_2} F(x_r,t) \, dt, \\
PSE_{tr} &= \int_{x_i}^{x_f} \int_{0}^{H} E_{PSE} \, dx \, dt = \int_{t_3}^{t_4} F(x_f,t) \, dt, \\
PSE_{ref} &= \int_{x_r}^{x_t} \int_{0}^{H} E_{PSE} \, dx \, dt = \int_{t_5}^{t_6} F(x_r,t) \, dt
\end{align*}
\]  

(23)

where \(t_i - t_j\) are the intervals of time when the wave passes the given cross-section.
Fig. 16. Pseudoenergy fluxes through the cross-section \( x_r/h_i = 355 \) for the incident and the reflected waves and through the cross section \( x_t/h_i = 330 \) for the transmitted wave.

The relative estimation of the energy loss is then given by

\[
\delta E_{\text{loss}} = \frac{PSE_{\text{in}} - PSE_{\text{tr}} - PSE_{\text{ref}}}{PSE_{\text{in}}} 
\]

and it is plotted versus the interaction parameter \( \mu = |a_z|/h_z \) in Fig. 17 for all three numerical runs. It seems that the three points lie almost on a straight line and the energy loss grows with the increase of the interaction parameter. For the first run the energy loss is about 5% and as
has been shown above the Gardner equation may be used for an approximate description of the transformation of an interfacial solitary wave at the step.

For the second and the third runs the energy losses are 18% and 48% respectively. In frame work of the weakly nonlinear theory the estimation of internal solitary wave energy for a two layer-system is given as

$$PSE = g \Delta \rho \int_{-\infty}^{\infty} \eta^2(x)dx = c_0^2 g \Delta \rho \int_{t_f}^{t_i} \eta^2(x_k, t)dt$$

(25)

where $x_k$ is the position of the cross-section. The estimates of $\delta E_{\text{loss}}$ using formula (25) are presented in Fig.17 also and these estimations lie low than estimates lie below the estimates from the fully nonlinear theory, while the difference between them increases with growth of the interaction parameter.

Fig.17. Loss of the energy due to mixing, turbulence and dissipation. The squares represent the estimation by (23) and the black circles correspond to formula (25).
Discussion

In this paper we continue the study of the transformation of an interfacial solitary wave over a bottom step by Grimshaw et al\textsuperscript{21} and Maderich et al\textsuperscript{22}, using here numerical simulations of the full system of the Navier-Stokes equations for waves of large amplitude. The new effect here is the polarity of the solitary wave which is now a wave of depression. We have also calculated the total energy of incident, reflected and transmitted waves. The results of this study can be summarized as follows:

1. As in study by Maderich et al\textsuperscript{22}, where interfacial solitary waves of elevation were studied, the simulations show the applicability of the weakly nonlinear model (the Gardner equation) to describe even a strongly nonlinear wave ($\varepsilon_{nl} = 4$) and its transformation over a relatively deep step (interaction parameter $\mu = 0.33$) if its amplitude does not exceed the limiting value (7) determined from the Gardner equation. In this case the shapes of the incident and transmitted solitary waves are well described by the Gardner and MCC models and the predictions of theory by Grimshaw et al\textsuperscript{21} agree well with simulation results as quite well with numerical simulations. The mixing at the step is weak (around 5% of the energy of the incident wave) which allows the use of ideal fluid theories. This case (Run 1) can be characterized as a weak interaction.

2. Next, the behavior of an interfacial solitary wave of amplitude larger than the Gardner equation limiting amplitude (7), but less then predicted by Miyata-Cammasa-Choi theory (12) differs from the previous case. The incident wave is stable and its shape is well described by MCC model well but it amplitude is larger than the limiting amplitude of an MCC soliton $A_{lim^+}$ at the step ($a/A_{lim^+} = 1.1$). The transformation of this incident wave is accompanied by shear instability and the formation of Kelvin-Helmholtz billows that is typical for the adjustment of large
amplitude interfacial solitary waves to a stable state\textsuperscript{11,23}. The parameters of the billows agree well with theoretical estimates\textsuperscript{30}. The shape of the transmitted (secondary) solitary wave is described by the MCC model and its amplitude, $a_+$, is less than the limiting value ($a_+/A_{lim+} = 0.8$). Thus, the Kelvin-Helmholtz instability is a mechanism for the formation of a stable secondary solitary wave from a large amplitude wave passing over the step through a decrease of wave mass by extraction of wave kinetic energy for mixing by billows and thickening of the interface layer. The wave reflection is small and the loss of energy of incident wave is 18\%. This case (Run 2) can be named by a moderate interaction that is characterized moderate value of the interaction parameter $\mu = 0.44$.

3. When height of the step is larger ($h_2/h_2 = 0.43$) an interfacial solitary wave of large amplitude (otherwise with the same parameters as in the previous run) can be characterized by a higher value of the interaction parameter $\mu = 0.73$. This case (Run 3) is classified as a strong interaction. The incident wave undergoes strong reflection and mixing between the layers. The interaction process can be divided on four stages: (i) the front of incident wave is deformed by flow forming in the lower layer; (ii) the flow becomes hydraulically critical and then supercritical; (iii) this flow entrains the wave trough into the bottom layer, with formation of a strong eddy below the step and a weaker eddy of opposite sign above the step; (iv) this pair of eddies leads to intensive mixing of the stratified water near the step. About 48\% of the energy of the incident wave was lost to dissipation and mixing, and 20\% of energy was reflected as a solitary wave. Thus the amplitude of the transmitted wave was less than in Run 2. Because of strong interaction the linear theory for wave reflection can not be applied to this case. Far from the step the shape of the secondary solitary wave is well described by both the Gardner and MCC models. However, in the transient phase it is
close to the table-top shape that again is described by the Gardner and MCC models. We suggest that this solitary wave formation is not a steady solitary wave. It provides an example, often observed in nature, of a “pseudo-solitary wave” that can be distinguished from a steady solitary wave only by repeated observations.

The authors thank the support of RFBR 08-05-91850-KO (T.T., E.P., R.G.), RFBR 09-05-00204 (T.T.), RFBR 09-05-90408-SFBR F28.6/010 (V.M., T.T., I.B., K.T.), RFBR-07-05-92310-HBO_a (T.T., E.P.)

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