Oblique spatial dark solitons in supersonic flow of Bose-Einstein condensate

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In framework of the Gross-Pitaevskii mean field approach it is shown that supersonic flow of Bose-Einstein condensate can support a new type of pattern—an oblique spatial dark soliton. Exact spatial dark soliton solution of the Gross-Pitaevskii equation is obtained. It is demonstrated by numerical simulations that spatial solitons can be generated by an obstacle inserted into the flow. Connection of the developed theory with available experimental data is discussed.

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Introduction. As is known, nonlinear and dispersive properties of Bose-Einstein condensate (BEC) provide a possibility of formation of various nonlinear structures (see, e.g., [1]). Until recently, main attention was drawn to experimentally observed vortices, bright, and dark solitons. Besides that, formation of dispersive shock waves in BECs with repulsive interactions between atoms was considered theoretically in [2, 3] and studied experimentally in rotating [4] and non-rotating [5] condensate, where it was shown that dispersive shocks are generated as a result of evolution of large disturbances in BEC. However, one more important type of nonlinear structures, spatial dark solitons, can also be realized in BEC, and first experimental evidences of their generation have recently appeared [6]. In fact, existence of oblique spatial solitons in BEC appears quite natural physically, if one considers Cherenkov generation of dispersive sound waves by an obstacle in a supersonic flow of BEC and asks oneself what happens if the amplitude of the waves is increased. Evidently, along with dispersion, nonlinear effects become equally important so that the Cherenkov cone unfolds into a spatial structure consisting of one or several dark solitons. Such a structure represents a dispersive analogue of well-known viscous spatial shock (an oblique compression jump) generated in supersonic flow of viscous compressible fluid past an obstacle. In this sense, we arrive at a spatial counterpart of the one-dimensional expanding dispersive shock [2]-[5] generated in the process of evolution of large disturbances in BEC. In the simplest case, the nonlinear wave structure consists of a single spatial dark soliton described by a steady solution of equations governing the BEC flow. Motivated by this physical consideration and the results of the experiment [6], we shall develop in this Letter the theory of spatial dark solitons in the framework of the Gross-Pitaevskii (GP) mean field approach.

Basic equations. Dynamics of BEC is described very well by the GP equation [1]

\[ \frac{i \hbar}{\partial t} \psi = - \frac{\hbar^2}{2m} \Delta \psi + V(r)\psi + g|\psi|^2\psi, \quad (1) \]

where \( \psi (r) \) is the condensate order parameter and \( g \) is an effective coupling constant, \( g = 4\pi \hbar^2 a_s/m, a_s \) being the \( s \)-wave scattering length and \( m \) the atomic mass. Here \( V(r) \) denotes the potential of the external forces acting on the condensate, as, e.g., the confining potential of the trap and/or the potential arising due to presence of an obstacle inside the BEC. When an “obstacle” is formed by a laser beam and the flow occurs due to free expansion of BEC after its release from the trap, the dependence of \( \psi \) on time and space coordinate along the beam is much slower in the soliton width scale than along two other spatial coordinates \( r = (x, y) \). Hence the trap potential can be put equal to zero in the free expansion of BEC and far enough from the obstacle we can neglect the obstacle potential as well. Also, we shall be interested in steady flows. To this end, we look for the solution of Eq. (1) with \( V(r) = 0 \), in the form

\[ \psi(r) = \sqrt{n(r)} \exp \left( \frac{i}{\hbar} \int u(r')dr' \right) \exp \left( - \frac{i\mu t}{\hbar} \right), \quad (2) \]

where \( n(r) \) is the density of atoms in BEC, \( u(r) \) denotes its velocity field and \( \mu \) is the chemical potential. It is convenient to introduce dimensionless variables \( \tilde{r} = r/\sqrt{2 \xi}, \tilde{\mu} = \mu/n_0, \tilde{u} = u/c_s \), where \( n_0 \) is a characteristic density of atoms in the problem under consideration, \( \xi = \hbar/\sqrt{2m \mu_0 \tilde{\mu}} \) is the healing length, and \( c_s = \hbar/\sqrt{2m \xi} \) is the sound velocity in BEC with the density \( n_0 \). Substituting Eq. (2) into (1) and separating real and imaginary parts we get the system of equations for the density \( n(x, y) \) and two components of the velocity field, \( \mathbf{u} = (u(x, y), v(x, y)) \),

\[ uu_x + vu_y + n_x + \left( \frac{n_x^2 + n_y^2}{8n^2} - \frac{n_{xx} + n_{yy}}{4n} \right)_x = 0, \quad (3) \]

\[ uv_x + vv_y + n_y + \left( \frac{n_x^2 + n_y^2}{8n^2} - \frac{n_{xx} + n_{yy}}{4n} \right)_y = 0, \]

where we have omitted tildes for convenience of the notation. If we restrict our consideration to potential flows...
using the condition
\[ u_y = v_x, \]  
then the second and third equations in (3) can be integrated once to give
\[ \frac{1}{2}(u^2 + v^2) + n + \frac{1}{8n^2}(n_x^2 + n_y^2) - \frac{1}{4n}(n_{xx} + n_{yy}) = \text{const.} \]  
This relationship may be considered as a generalization of the Bernoulli theorem to dispersive 2D hydrodynamics. Equations (4), (5) and the first equation (3) comprise the system of equations governing the potential BEC flow.

Our aim now is to find the solution of this system under condition that the BEC flow is uniform at infinity:
\[ n = 1, \quad u = M, \quad v = 0 \quad \text{at} \quad |x| \to \infty, \]  
where \( M \) denotes the ratio of the asymptotic velocity of the flow to the sound velocity, i.e., the Mach number.

**Oblique spatial dark soliton solution.** We look for the soliton solution in the form \( n = n(\theta), u = u(\theta), v = v(\theta) \), where \( \theta = x - ay \), and \( a \) denotes a slope of the soliton with respect to \( y \) axis. Substitution of this **ansatz** into (4) and the first equation (3), followed by a simple integration, yields expressions for the velocity components in terms of the density,
\[ u = \frac{M(1 + a^2)n}{(1 + a^2)n}, \quad v = -aM(1 - n)/(1 + a^2)n, \]  
where the integration constants are chosen according to the condition (6). Then substitution of (7) into (5) with a proper choice of the constant in the right-hand side leads to the equation
\[ \frac{1}{4}(1 + a^2)(n^2 - 2n\eta''') + 2n^3 - (2 + p)n^2 + p = 0, \]  
where
\[ p = M^2/(1 + a^2) \]  
and prime denotes differentiation with respect to \( \theta \). It is easy to check that Eq. (8) has the integral
\[ \frac{1}{4}(1 + a^2)n^2 = (1 - n^2)(n - p) \]  
where, again, the integration constant is chosen in accordance with the condition (6). Simple integration of this equation yields the desired soliton solution in the form
\[ n(\theta) = 1 - \frac{1}{\cosh^2[\sqrt{1 - p}\theta/\sqrt{1 + a^2}]} \]  
and the velocity components can be found by substitution of this \( n(\theta) \) into Eqs. (7). The inverse half-width of the soliton along \( x \)-direction can be introduced according to formula
\[ \kappa = 2\sqrt{1 - p}/1 + a^2. \]  
Formulae (11), (12) give the exact dark spatial soliton solution of the GP equation. We shall call it “oblique” because it is always inclined with respect to the direction of the supersonic flow.

It is instructive to consider the limiting cases of this oblique spatial dark soliton solution.

**Small amplitude Korteweg-de Vries (KdV) limit.** As is clear from (11), a small amplitude limit is achieved when \( 1 - p \ll 1 \). Then the parameters \( a \) and \( p \) can be expressed in terms of \( \kappa \) and \( M \) from Eqs. (9) and (12) as
\[ a \cong \sqrt{M^2 - 1} + \frac{M^4\kappa^2}{8\sqrt{M^2 - 1}}, \quad 1 - p \cong \frac{1}{4}\kappa^2M^2, \]  
that is we arrive at the density profile
\[ n \cong 1 - \frac{M^2\kappa^2}{4\cosh^2[\kappa(x - ay)/2]} \]  
where \( \kappa \ll 1/M \).

Note that the slope \( a = \sqrt{M^2 - 1} \) corresponds exactly to the Cherenkov cone, so that the small amplitude solitons are located inside the Cherenkov cone in near vicinity of it. This approximation corresponds to the KdV limit of the potential-free GP equations (3). Indeed, if we assume series expansions \( (\varepsilon \ll 1) \)
\[ n = 1 + \varepsilon n_1 + \ldots, \quad u = M + \varepsilon u_1 + \ldots, \quad v = \varepsilon v_1 + \ldots \]  
and introduce scaling of the dependent variables
\[ \xi = \varepsilon^{1/2}(x - ay), \quad \tau = \varepsilon^{3/2}t, \]  
then standard reductive perturbation theory leads to the KdV equation for \( n_1 \),
\[ n_{1,\tau} - \frac{3M^2}{2\sqrt{M^2 - 1}}n_1n_{1,\xi} + \frac{M^4}{8\sqrt{M^2 - 1}}n_{1,\xi \xi \xi} = 0 \]  
with well-known soliton solution equivalent to (14), (13).

**Nonlinear Schrödinger (NLS) equation limit.** Another important limit corresponds to large slopes \( a^2 \gg 1 \). Then we introduce \( \lambda \) according to the definition \( p \cong M^2/a^2 = \lambda^2 \), that is \( a = \pm M/\lambda \) and the soliton solution (11) can be represented in this approximation as
\[ n \cong 1 - \frac{1 - \lambda^2}{\cosh^2[\sqrt{1 - \lambda^2}(y + \lambda x/M)]}. \]  
This is exactly the solution \( n = |\Psi|^2 \) of the NLS equation
\[ i\Psi_T + \Psi_{YY} - 2|\Psi|^2\Psi = 0 \]  
for a complex field variable
\[ \Psi = \sqrt{\eta}\exp \left( i \int_Y v(Y',t)\,dY' \right) \].
where \( T = x/2M, Y = y \), derived in [7] from the GP equations (3) for description of highly supersonic \((M \gg 1)\) flow of BEC past a slender body.

**Generation of spatial solitons in BEC.** We consider a supersonic BEC flow past an obstacle. If the obstacle is small (impurity), then just linear sound waves are generated and they form the Cherenkov cone [8]. Large obstacles generate spatial dispersive shocks which can be viewed as trains of dark spatial solitons inside the Cherenkov cone. The theory of generation of spatial dispersive shocks has been developed in much detail for supersonic flows past a slender body when such a flow can be described by the KdV equation [9]. Analogous theory for the NLS equation case was developed in [7]. However, in real experiments the obstacles cannot be considered as slender bodies and the flow is not highly supersonic, hence the fully nonlinear solutions of the GP equation such as Eq. (11) should be used for the quantitative description of spatial dispersive shocks in BEC.

To make the process of the generation clearer, we have studied the time-dependent numerical solution of the GP equation (1) expressed in non-dimensional variables as

\[
i\psi_t = -\frac{1}{2}(\psi_{xx} + \psi_{yy}) + V(x,y)\psi + |\psi|^2, \tag{21}\]

where \( \tilde{\psi} = \psi/\sqrt{n_0}, \tilde{t} = (gm_0/\hbar)t \), the other variables defined as in Eq. (2), and, after this transformation, the tildes are omitted. The potential \( V(x,y) \) corresponds to interaction of the obstacle with the condensate. In our simulations the obstacle was modeled by an impenetrable disk with radius \( r = 1 \) in our non-dimensional units. At the initial moment \( t = 0 \) there is no disturbance in the condensate so it is described by the plane wave function

\[
\psi(x,y)\big|_{t=0} = \exp(iMx) \tag{22}
\]

corresponding to the uniform condensate flow. We took \( M = 5 \). Several stages of the BEC density evolution calculated numerically [10] are shown in Fig. 1.

Naturally, the pictures are symmetric with respect to the direction of the flow. It can be clearly seen how a pair of spatial solitons is gradually formed behind the obstacle. Their length grows with time and, except in the vicinity of the end points, the density distributions do not demonstrate any vorticity, which agrees with our supposition of the potentiality of the flow (4). However, near the end points the flow cannot be considered as stationary and potential. This is manifested by the presence of vortices behind the end points of the spatial solitons. One may interpret the described pattern as “vortex street” [11] radiation by spatial dark solitons. We have calculated the parameter \( p \) from Eq. (9) using the the value of the slope \( a \) inferred from the numerical solution. The comparison of the theoretical profile of the oblique spatial soliton given by Eq. (11) with the corresponding part of the density profile in the full numerical solution is shown in Fig. 2. Excellent agreement of these two distributions confirms that the line patterns in Fig. 1 are indeed the oblique spatial solitons generated by the obstacle.

In the above simulations, the parameters of the obstacle have been chosen so that only a single soliton is
FIG. 2: Cross sections of the density for \( x = 20 \) (dashed line), \( x = 60 \) (solid line) and \( y > 0 \) obtained by numerical solution of the GP equation \( (21) \). They are compared with soliton profiles \( (11) \) with the slope \( \alpha = 10 \) shown as functions of \( y \) at the same values of \( x \) \((x = 20 \) corresponds to “crosses” and \( x = 60 \) to “circles”). Oscillations at \( y \gtrsim 10 \) correspond to non-solitonic wave packet.

FIG. 3: Density plot at the moment \( t = 30 \) for the supersonic flow \((M = 5)\) past a disk-shaped impenetrable obstacle with radius \( r = 5 \) located at \((x = 0, y = 0)\).

generated at each side of the obstacle. However, with increase of the size \( r \) of the obstacle the number of solitons is also expected to increase. This effect is demonstrated in Fig. 3 where two symmetric fans of solitons can be seen behind the obstacle. As it should be, the depth of dark solitons grows with increase of the slope \( \alpha \) with respect to \( y \) axis. The whole structure represents a pair of dispersive shocks generated in a supersonic flow of BEC past an obstacle. Such a shock can be analytically represented as a modulated periodic solution of the system \((3)\). In the limiting case of highly supersonic flows \((M \gg 1)\) and slender obstacles \((\alpha \gg 1)\) such shocks have been considered in \([7]\). Our present numerical simulations show that spatial dispersive shocks represent a general phenomenon caused by interplay of dispersive and nonlinear effects described by the full GP equation without any approximations.

The developed here theory explains qualitatively the patterns observed in the experiment \([6]\) where the large amplitude nonlinear waves generated by the BEC flow past an obstacle are clearly seen. However, it is difficult to make a proper quantitative comparison since the flow in the experiment is not exactly uniform (rather it is cylindrically symmetric) and stationary.

In conclusion, we have found exact spatial dark soliton solution of the GP equation and demonstrated numerically that such solitons can be generated by obstacles inserted into the supersonic flow of BEC. The number of solitons increases with the size of the obstacle and the whole pattern can be considered as a stationary spatial dispersive shock.

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[10] Calculations were made in the stationary frame with respect to the obstacle after tranforming eq. \((21)\) with \( x' = x - Mt, \ y' = y, \ t' = t \) and \( \phi(x',y',t') = \psi(x,y,t) \exp(-it) \). Thus, the actual equation solved is \( i \frac{\partial \psi}{\partial t} = \frac{1}{2} (\frac{\partial^2 \psi}{\partial x'^2} + \frac{\partial^2 \psi}{\partial y'^2}) + iM \frac{\partial \psi}{\partial y'} - \phi + |\phi|^2 \phi + U(x') \phi \) with the boundary conditions at all four sides of the boundary \( \phi = 1 \). We used the split–step technique where the dispersion term was evolved by Crank-Nicolson algorithm and the advective part by the Beam-Warming algorithm \([12]\) with grids 1024 x 512 points and time step \( \Delta t = 0.91 \). Similar results were also obtained with the upwind algorithm (in the advective part) with larger grids and smaller time step.