Solutions of the Hyperbolic sine-Gordon Equations

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Abstract

We study soliton solutions of the Sinh-Gordon equation. It is also shown that the Cosh-Gordon equation, whilst being integrable, does not admit pure solitons.
1) Introduction

The sine-Gordon equation

\[(\partial_x^2 - \partial_t^2) \varphi = \partial_{uv} \varphi = \sin \varphi\]

(in space-time or double null coordinates) has been studied in many publications too numerous to list here, as one instance cf. [1]. Also, replacing the sine-function by the cosine-function does not yield anything new.

On the other hand, replacing the sine-function with its hyperbolic counterpart might lead to some new insights. In this short note we shall examine the hyperbolic counterparts - there are two of them, since there is no real transformation between the Sinh and the Cosh - of the sine-Gordon equation.

2) General Remarks

We study the equations

\[\partial_{uv} \varphi = \text{Sinh} \varphi \quad (1a)\]
\[\partial_{uv} \varphi = \text{Cosh} \varphi \quad (1b)\]

Both equations are known to be integrable and admit a pseudopotential, viz.

\[\partial_u \gamma = \partial_u \varphi - \frac{1}{\lambda} \text{Sinh} \gamma \quad (2a)\]
\[\partial_v \gamma = \lambda \text{Sinh} (\varphi - \gamma)\]
\[\partial_u \gamma = \partial_u \varphi + \frac{1}{\lambda} \text{Sinh} \gamma \quad (2b)\]
\[\partial_v \gamma = \lambda \text{Cosh} (\varphi - \gamma)\]

The integrability conditions for (2a) and (2b) yield the equations (1a) and (1b). With the help of the pseudopotential a new solution is given in both cases via a Bäcklund transformation as

\[\bar{\varphi} = \varphi - 2 \gamma \quad (3)\]

Each Bäcklund transformation generates a soliton. The difference between the two equations is that for the Sinh-Gordon equation one has a trivial solution, \(\varphi = 0\), to start the process whereas for the Cosh-Gordon equation this solution does not exist. This leads to the suspicion that the Cosh-Gordon equation, whilst being integrable, does not admit pure solitons.

Since there appears to be no generally accepted definition of a "soliton", we shall denote, in the present context, a localized "lump" of energy, to be described by the Hamiltonian of the system in question.
3) Some Solutions

We shall first present solutions for the Sinh-Gordon equation in space-time coordinates, viz.

\[ \partial_x^2 \varphi - \partial_t^2 \varphi = \sinh \varphi \quad (4) \]

We take this solution to be a travelling wave which, by the Lorentz invariance of the equation, can be taken to be in its rest frame, hence \( \varphi = \varphi(x) \). Multiplying (4) by \( \partial_x \varphi \) and integrating we get

\[ \frac{1}{2} \partial_x \varphi^2 = \cosh \varphi - E \quad (5) \]

the solution of which is given in terms of Jacobi elliptic functions (in the notation we follow [2]) as

\[ \varphi = 2 \ln \left( \sqrt{k} \, \text{sn} \left( \frac{x}{2 \sqrt{k}}, k \right) \right), \quad E = \frac{1}{2} \left( k + \frac{1}{k} \right) \quad (6) \]

In particular, for \( k = 1 \), \( E = 1 \), this solution degenerates into the 1-soliton solution which can be rewritten as

\[ \varphi = 4 \, \text{ArTanh} \, e^{-2x} \quad (7) \]

In analogy to the procedure for the sine-Gordon equation the 2-soliton solution in its rest frame is obtained from the Ansatz [3]

\[ \varphi = 4 \, \text{ArTanh} \left( a(x) \, b(t) \right) \]

\( a \) and \( b \) have to satisfy

\[ \partial_x a^2 = c_0 + c \, x^2 + c_4 \, x^4 \]

\[ \partial_t b^2 = c_4 + (c - 1) \, x^2 + c_0 \, x^4 \]

The 2-soliton solution is given by either \( c_0 = 0 \) or \( c_4 = 0 \). A typical solution, for \( c_0 = 0 \), reads

\[ a = \sqrt{\frac{c}{c_4}} \, \frac{1}{\sinh(\gamma c \, x)} \]

\[ b = \gamma \frac{c_4}{c - 1} \, \sinh(\gamma c - 1 \, t) \quad (8) \]

Generally, the behaviour of Sinh-Gordon solitons is rather similar to that of sine-Gordon solitons with the exception that Sinh-Gordon solitons are given by singularities rather than wave crests. The Lagrangian is given by

\[ L = \frac{1}{2} \left( \partial_x \varphi^2 - \partial_t \varphi^2 \right) + \cosh \varphi \]
and the Hamiltonian, the value of which, for a given solution, is the energy density, reads (the additive constant stems from the convention to have the lowest state at zero energy.)

$$H = \frac{1}{2} \left( \partial_x \phi^2 + \partial_t \phi^2 \right) + \cosh \phi - 1$$  \hspace{1cm} (9)

For the 1-soliton solution the energy density is

$$E = 8 \ e^{2\phi} \ (1 - e^{2\phi})^{-2}$$

While the integral over this energy density does not exist, its Cauchy principal value does exist and

$$\int_{-\infty}^{\infty} E \ dx = 4$$

We now turn to solutions of the Cosh-Gordon equation. In this case the equation reads

$$\partial_x^2 \phi - \partial_t^2 \phi = \cosh \phi$$  \hspace{1cm} (10)

and, looking for a travelling wave solution in its rest frame, the analogue of (7) becomes

$$\frac{1}{2} \partial_x \phi^2 = \sinh \phi - E$$

This right-hand side of this equation does not admit a double zero and thus there are no solutions in terms of elementary functions. A solution is given by

$$\phi = -2 \ \ln \left( \frac{1}{\sqrt{k}} \ \text{cn} \left( \frac{x}{2 \sqrt{k}}, \ k \right) \right), \ E = \frac{1}{2} \left( k + \frac{1}{k} \right)$$  \hspace{1cm} (11)

The Hamiltonian for the Cosh-Gordon equation, viz.

$$H = \frac{1}{2} \left( \partial_x \phi^2 + \partial_t \phi^2 \right) + \sinh \phi$$  \hspace{1cm} (12)

is not bounded from below and thus there is no ground state.

\section{Concluding remarks}

We have shown that integrability of an equation is not necessarily related to the existence of solitons. This is, in particular, the case if the equation in question does not admit a trivial solution and the Hamiltonian does not have a ground state.
References

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