Synchronization is a fundamental phenomenon in the physics of oscillations. It manifests not only in physics, but also in biology, chemistry, and many other branches of science and engineering. It can involve coupled or forced periodic [1–3], chaotic [4, 5], and even stochastic self-oscillators [6]. Synchronization implies an adjustment of the basic timescales of systems due to coupling between them. In the simplest case of periodic oscillations, synchronization can be attributed to certain bifurcations on an invariant ergodic torus. Two basic mechanisms for synchronization have been well-studied, allowing us to explain the phenomena mentioned above. It turns out that, together with an attracting torus, this structure contains a saddle torus whose manifolds separate coexisting synchronous regimes (phase multistability).

As a paradigm for a pair of coupled, periodic, self-sustained oscillators we consider two mutually coupled Van der Pol systems:

\[
\begin{align*}
\dot{x}_{1,2} &= y_{1,2}, \\
\dot{y}_{1,2} &= (\varepsilon_{1,2} - x_{1,2}^2)y_{1,2} - \omega_{1,2}^2 x_{1,2} + C_1(x_{2,1} - x_{1,2}) + C_2(x_{2,1} - x_{1,2})^2.
\end{align*}
\]

Here, \(\varepsilon_{1,2}\) are nonlinearity parameters, \(\omega_1 = \omega = \omega_2 = p\omega\) are eigenfrequencies, where \(p\) introduces detuning between the systems, and \(C_1, C_2\) are the strengths of the linear and of quadratic couplings, respectively.

We initiated our studies by considering the simplest linear coupling \((C_1 \neq 0, C_2 = 0)\) that leads to the coexisting limit sets and associated bifurcations of our interest. However, in order to estimate how a more complex coupling influences the coexisting regimes and their evolution we also considered nonlinear interaction between the systems when the coupling function included a quadratic term. Although in what follows we fix equal nonlinearity \(\varepsilon_{1} = \varepsilon_{2} = 0.5\), all results reported below were also reproduced for unequal nonlinearities \(\varepsilon_{1} = 0.5\) and \(\varepsilon_{2} = 0.52\). We choose \(\omega = 1\), and change \(p\) around 1 in order to study the basic 1:1 synchronization region. For these parameters values there are six cycles involved: two stable which we further refer to as \(S_{1,2}\), two saddle (with one unstable direction) \(S_{1,2}^{\ast}\), and two “double-saddle” (with two unstable directions, the most unstable of all cycles) \(U_{1,2}\). These cycles can undergo two types of bifurcation: saddle-node while merging in pairs \((S_{1,2}^{\ast} \cup S_{1,2}^{\ast} \cup U_{1,2})\), or torus birth \((S_{1,2} \cup U_{1,2})\). First, we study linear mutual coupling by setting \(C_2 = 0\). In the plane of parameters \(p\) and \(C_1\), the region of phase locking (main part of Fig. 1) [17] is outlined by solid lines on which saddle-node bifurcations between different pairs of cycles occur. Dashed lines mark torus birth bifurcation and enclose the region of suppression. At very small \(C_1\) the locking region looks like a classical Arnol’d tongue. However, at larger \(C_1\) another narrower “tongue” can be found, embedded within the first one.
FIG. 1: Central figure: bifurcation diagram showing synchronization region for linear coupling, on the detuning \( p \) vs. coupling strength \( C_1 \) plane. Solid lines are saddle-node bifurcations, and dashed lines are Neimark-Sacker bifurcations leading to torus birth. The dotted lines show the two routes \( A \) and \( B \) across the diagram that are illustrated in the insets. A black circle marks the parameter values for which the structure of phase space is shown in Fig. 4(b). Inset \( A \) shows how the positions of the maxima \( x_{\text{max}} \) of the stable \( S_{1,2} \) and single-saddle \( S_{1,2}^* \) cycles depend on \( p \) for \( C_1 = 0.15 \). Inset \( B \) shows how \( x_{\text{max}} \) of all six cycles depend on \( C_1 \) for \( p = 1.002 \). In insets solid line is \( S_{1,2} \), dashed line is \( S_{1,2}^* \), and grey line is double-saddle cycles \( U_{1,2} \).

Within the smaller tongue \( S_1 \) and \( S_2 \) coexist on the surface of the same stable torus, separated by \( S_1^* \) and \( S_2^* \), while \( U_{1,2} \) live in the neighbourhood. To illustrate how cycles appear/disappear within and at the boundaries of the phase locking region, we follow routes \( A \) and \( B \) in Fig. 1 as marked by the dotted lines. Insets show evolution of cycles maxima \( x_{\text{max}} \) along the routes. Along route \( A \) (left-hand inset of Fig. 1) \( S_{1,2} \) and \( S_{1,2}^* \) merge or appear in pairs, always on the same torus surface. \( U_{1,2} \) that do not undergo bifurcations exist throughout the route (not shown). Along route \( B \) (right-hand inset of Fig. 1) all six cycles undergo saddle-node bifurcations. Above the upper border of the phase locking region at \( C_1 \approx 0.192 \), there are only two cycles left, both stable.

Now let us take account of nonlinear coupling terms by increasing \( C_2 \) from zero. Fig. 2 shows bifurcation diagrams in the \( p-C_2 \) parameter plane, computed for four different values of \( C_1 \). Increase of \( C_2 \) evidently leads to the disappearance of one of the coexisting periodic regimes as a result of certain bifurcations, as illustrated.

Consider in more detail the lower part of the diagram for \( C_1 = 0.2 \) (Fig. 3). The left and right-hand borders of the synchronization region are formed by lines of saddle-node bifurcations of pairs of cycles \( S_{1,2} \) with \( S_{1,2}^* \), and of \( S_{1,2}^* \) with \( U_{1,2} \) as shown by solid lines. On dashed lines stable cycles \( S_{1,2} \) lose their stability via Neimark-Sacker bifurcation.

At \( C_2 = 0, C_1 = 0.2 \) two stable limit cycles coexist in phase space. However, as seen from Fig. 1, right-hand inset, there are no saddle cycles \( S_{1,2}^* \) with their three-dimensional stable manifolds that could separate the basins of attraction. So the question arises: What separates the basins of attraction now?

With increase of \( C_2 \) the stable cycle \( S_2 \) undergoes Neimark-Sacker bifurcation as a result of which torus \( T_2 \) is born, and the cycle becomes double-saddle (we mark its instability by a tilde: \( \tilde{S}_2 \)). An interesting component of this bifurcation diagram is the stable grey (green online) line on which \( T_2 \) disappears via a bifurcation. Since the latter is not of a cycle, but of torus, the corresponding line was obtained not by a continuation algorithm, but by observing phase portraits and marking the parameter values at which \( T_2 \) vanished. Remarkably, there is no sign of torus distortion or of the quasiperiodic tending to infinity before the bifurcation, as usually accompany homoclinic bifurcations; the torus vanishes while remaining smooth and of finite size. So, why does the torus disappear?

To answer these questions, we have to reveal the structure of phase space, and to show that it allows for all observed bifurcations of limit cycles and provides a separating surface for the coexisting stable cycles in the region of suppression. This structure should also explain the sudden disappearance of the smooth torus \( T_2 \). An important fact to be accounted for is: any single-saddle cycle \( S_{1,2}^* \) must be able to merge either with any stable cycle \( S_{1,2} \), or with any double-saddle cycle \( U_{1,2} \). We hypothesise that inside the central part of locking region in Fig. 1 that contains point 1, along with a saddle resonant torus formed by unstable manifolds of single-saddle cycles \( S_{1,2}^* \), there exists a saddle torus formed by unstable manifolds of double-saddle cycles \( U_{1,2} \), that intersects with the stable torus at the saddle cycles \( S_{1,2}^* \). On the latter structure the cycles \( U_{1,2} \) should lie in between \( S_{1,2}^* \).

Moreover, both torus should belong to the same closed
hypersurface ("sphere"). A sketch of the Poincaré section of this hypothesised structure in 3-dimensional space is shown in Fig. 4(a). Here, cycles are given by circles: black - $S_{1,2}$, white - $S_{1,2}^*$, grey (green online) - $U_{1,2}$. Solid closed curves show tori, surfaces show manifolds of the saddle torus: horizontal plane - stable, and sphere - unstable manifolds, respectively.

In this picture, the saddle torus results from an intersection of the globally attracting sphere and open manifold shown schematically by the horizontal plane in Fig. 4(a). The latter manifold is then the sought-after surface separating the basins of two coexisting cycles $S_{1,2}$. Although the proposed structure is self-consistent and would provide an elegant answer to the above questions, its existence needs to be verified. The best way to do this is to try visualising the structure numerically. However, at the moment there seem to be no numerical method available that would allow one to visualise a saddle torus.

We therefore require a way to plot the inferred saddle torus using what we know about the system and exploiting all necessary implications of the hypothesised structure’s existence, as follows.

First, we note that our hypothesis assumes two pairs of cycles $S_{1,2}$ and $S_{1,2}^*$ lying on the same closed curve (the resonant torus) $T$. An indication of this can be seen in Fig. 1 (inset A) showing that with the change in $p$ all four cycles move along the same closed curve. However, the assumption must also be verified for fixed system parameters. A resonant torus can be visualised numerically with the help of an existing technique [18] that involves surrounding the torus by a closed surface and following its evolution in time. Since $T$ is an attracting structure as a whole, and despite the real attractors being cycles $S_{1,2}$ lying on it, the closed surface will tend eventually to coincide with the torus. Performing this calculation for $C_2 = 0$, $p = 1.002$ and $C_1 = 0.15$ in Eq. (1) (point 1 in Fig. 1), we obtained a closed curve in the Poincaré section (black curve in Fig. 4(b)) representing the torus $T$ sought. As an independent test of validity of this torus, we plot the cycles $S_{1,2}$ and $S_{1,2}^*$ in the same figure to ensure that they do lie on the closed curve found.

Secondly, we note that, to enable trajectories from inside the sphere to be attracted to its surface, a repelling object should exist inside it; and, indeed, it can be easily shown that there is a repelling fixed point at the origin. If time is reversed, a repeller becomes an attractor, making the fixed point stable; and, if the hypothesised sphere exists, it will become the boundary between the basins of attraction of the fixed point and of infinity. We were thus able to compute the basin of attraction of the fixed point in reversed time for the above parameters, and estimated its boundary numerically. This indeed appeared to be a closed surface topologically equivalent to a sphere (not shown, to avoid overloading the figures).

Thirdly, if the hypothesised picture is true, both the stable torus $T$ and double-saddle cycles $U_{1,2}$ should lie on the sphere. $T$ should divide the sphere into two halves, each containing one of the double-saddle cycles, a conclusion that was verified within our numerical accuracy.

Fourthly, the manifold sketched in Fig. 4(a) as a plane separating the basins of attraction of coexisting limit cycles $S_1$ and $S_2$, can be calculated similarly to the sphere. The surface thus found (again, not shown) was indeed not closed, at least in the relevant part of phase space.
Fifthly, if the saddle torus exists, it should be a closed curve at which the surfaces found above intersect each other. We have established that the sphere and the plane do indeed intersect, and have estimated their points of intersection numerically, yielding the closed curve shown in Fig. 4(b) by grey (green online), that is supposedly the saddle torus $T^*$ being sought (cf. Fig. 4(a)).

Finally, the saddle torus $T^*$ should intersect with the stable torus $T$ at cycles $S^*_{1,2}$ and $U^*_{1,2}$, which should lie on its surface as well. This too appears to be true within numerical accuracy (compare Fig. 4(b) with Fig. 4(a)).

Our hypothesised structure of phase space in the central part of the locking region for two mutually coupled periodic oscillators is thus shown to be valid, and the question about the boundary of the basins of attraction of coexisting limit cycles has been answered: it is formed by the stable manifold of a saddle torus.

Now, we return to the question of the abrupt disappearance of the newly born torus under nonlinear coupling (Fig. 3). Given the structure of phase space that we have revealed inside the locking region, the newborn torus $T_2$ must also lie on the sphere. As parameter $C_2$ is gradually increased above the value of torus birth bifurcation, the torus diameter increases. However, being bounded to lie on the sphere, the only opportunity for it to grow is to move towards the equator, i.e. towards the saddle torus $T^*$. The inevitable consequence is that, eventually $T_2$ must collide with $T^*$ and disappear, which would imply a saddle-node bifurcation. After that, the only attractor of the system would be the stable cycle $S_1$.

This suggestion was tested by use of methods similar to those resulting in Fig. 4(b). For each of two sets of parameters inside the region where the cycle $S_1$ and torus $T_2$ coexist, marked as points 2 and 3 in Fig. 3, the two tori were obtained numerically. In Fig. 4(c) the phase space is illustrated for point 2, and in Fig. 4(d) for point 3. It is clearly seen that, immediately after the torus birth bifurcation (Fig. 4(c)), the new torus $T_2$ is small and is situated relatively far from the saddle torus $T^*$. Also, it does belong to the sphere which is not illustrated here. However, close to the parameter value at which $T_2$ disappears (Fig. 4(d)), both tori are very close to each other, almost coinciding in shape and size. These figures support our hypothesis of a saddle-node bifurcation of the two tori and thus provide the mechanism by which the stable torus disappears.

In conclusion, we have revealed the structure of the joint phase space of two mutually coupled periodic oscillators within the synchronization regime, thereby enabling us to answer the questions posed in the introduction. We have shown that, quite generally, there are two tori lying on the same closed manifold, intersecting to form an elegant structure. All evolutions of synchronous attractors are closely associated with bifurcations of solutions belonging to either saddle tori, or stable tori, or both, or of the tori themselves. That is, the transition from locking to suppression inside the synchronization region is determined by bifurcations of solutions on the tori, whereas the disappearance of multistability is determined by bifurcations of the tori themselves.

The work was supported by Leverhulme Trust, and U.S. Civilian Research Development Foundation (Grant No. REC 006)