Korteweg-de Vries Equation

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Historical Introduction

The Korteweg-de Vries (KdV) equation, given here in canonical form,
\[ u_t + 6uu_x + u_{xxx} = 0, \]  
(1)
is widely recognised as a paradigm for the description of weakly nonlinear long waves in many branches of physics and engineering. Here \( u(x, t) \) is an appropriate field variable, \( t \) is the time, and \( x \) is the space coordinate in the relevant direction. It describes how waves evolve under the competing but comparable effects of weak nonlinearity and weak dispersion. Indeed, if it is supposed that \( x \)-derivatives scale as \( \epsilon \) where \( \epsilon \) is the small parameter characterising long waves (i.e. typically the ratio of a relevant background length scale to a wavelength scale), then the amplitude scales as \( \epsilon^2 \) and the time evolution takes place on a scale of \( \epsilon^{-3} \). The KdV equation is characterised by its family of solitary wave solutions,
\[ u = \text{asech}^2(\gamma(x - Vt)) \quad \text{where} \quad V = 2a = 4\gamma^2. \]  
(2)
This solution describes a family of steady isolated wave pulses of positive polarity, characterised by the wavenumber \( \gamma \); note that the speed \( V \) is proportional to the wave amplitude \( a \), and also to the square of the wavenumber \( \gamma^2 \).

The KdV equation (1) owes its name to the famous paper of Korteweg and de Vries, published in 1895, in which they showed that small-amplitude long waves on the free surface of water could be described by the equation
\[ \zeta_t + c\zeta_x + \frac{3c}{2h}\zeta\zeta_x + \frac{ch^2}{6}\delta\zeta_{xxx} = 0. \]  
(3)
Here \( \zeta(x, t) \) is the elevation of the free surface relative to the undisturbed depth \( h \), \( c = (gh)^{1/2} \) is the linear long wave phase speed, and \( \delta = 1 - 3B \), where \( B = \sigma/gh^2 \) is the Bond number measuring the effects of surface tension (\( \rho \sigma \) is the coefficient of surface tension and \( \rho \) is the water density). Transformation to a reference frame moving with the speed \( c \) (i.e. \( (x, t) \) is replaced by \( (x - ct, t) \), and subsequent rescaling readily establishes the equivalence of (1) and (3). Although equation (1) now bears the name KdV, it was apparently first obtained by Boussinesq (1877) (see Miles (1970) and Pego and Weinstein (1997) for historical discussions on the KdV equation). Korteweg and de Vries found the solitary wave solutions (2) and, importantly, they showed that they are the limiting members of a two-parameter family of periodic travelling wave solutions, described by elliptic functions and commonly called cnoidal waves,
\[ u = b + a\text{cn}^2(\gamma(x - Vt)|m), \quad \text{where} \quad V = 6b + 4(2m - 1)\gamma^2, \quad a = 2m\gamma^2. \]  
(4)
Here \( \text{cn}(x|m) \) is the Jacobian elliptic function of modulus \( m \) (0 < \( m < 1 \)). As \( m \to 1 \), \( \text{cn}(x|m) \to \text{sech}(x) \) and then the cnoidal wave (4) becomes the solitary wave (2), now
riding on a background level $b$. On the other hand, as $m \to 0$, $\text{cn}(x|m) \to \cos 2x$ and so the cnoidal wave (4) collapses to a linear sinusoidal wave (note that in this limit $a \to 0$).

This solitary wave solution found by Korteweg and de Vries had earlier been obtained directly from the governing equations (in the absence of surface tension) independently by Boussinesq (1871, 1877) and Rayleigh (1876) who were motivated to explain the now very well-known observations and experiments of Russell (1844). Curiously, it was not until quite recently that it was recognised that the KdV equation is not strictly valid if surface tension is taken into account and $0 < B < 1/3$, as then there is a resonance between the solitary wave and very short capillary waves.

After this ground-breaking work of Korteweg and de Vries, interest in solitary water waves and the KdV equation declined until the dramatic discovery of the soliton by Zabusky and Kruskal in 1965. Through numerical integrations of the KdV equation they demonstrated that the solitary wave (2) could be generated from quite general initial conditions, and could survive intact collisions with other solitary waves, leading them to coin the term soliton. Their remarkable discovery, followed almost immediately by the theoretical work of Gardner, Greene, Kruskal and Miura (1967) showing that the KdV equation was integrable through an inverse scattering transform, led to many other startling discoveries and marked the birth of soliton theory as we know it today. The implication is that the solitary wave is the key component needed to describe the behaviour of long, weakly nonlinear waves.

An alternative to the KdV equation is the Benjamin-Bona-Mahony (BBM) equation in which the linear dispersive term $c \zeta_{xxx}$ in (3) is replaced by $-\zeta_{xxt}$. It has the same asymptotic validity as the KdV equation, and since it has rather better high wavenumber properties, is somewhat easier to solve numerically. However it is not integrable, and consequently has not attracted the same interest as the KdV equation.

Both the KdV and BBM equations are uni-directional. A two-dimensional version of the KdV equation is the KP equation (Kadomtsev and Petviashvili, 1970),
\begin{equation}
(u_t + 6uu_x + u_{xxx})_x \pm u_{yy} = 0.
\end{equation}
This equation includes the effects of weak diffraction in the $y$-direction, in that $y$-derivatives scale as $\epsilon^2$ whereas $x$-derivatives scale as $\epsilon$. Like the KdV equation it is an integrable equation. When the “+”-sign holds in (5), this is the KPII equation, and it can be shown that then the solitary wave (2) is stable to transverse disturbances. On the other hand if the “−”-sign holds, this is the KPI equation for which the solitary wave is unstable; instead this equation supports “lump” solitons. Both KPI and KPII are integrable equations. To take account of stronger transverse effects, and/or to allow for bi-directional propagation in the $x$-direction it is customary to replace the KdV equation with a Boussinesq system of equations; these combine the long wave approximation to the dispersion relation with the leading order nonlinear terms and occur in several asymptotically equivalent forms.

Although the KdV equation (1) is historically associated with water waves, it in fact occurs in many other physical contexts, where it arises as an asymptotic multi-scale
reduction from the relevant governing equations. Typically the outcome is

\[ A_t + cA_x + \mu AA_x + \lambda A_{xxx} = 0. \]  

(6)

Here \( c \) is the relevant linear long wave speed for the mode whose amplitude is \( A(x, t) \), while \( \mu \) and \( \lambda \), the coefficients of the quadratic nonlinear and linear dispersive terms respectively, are determined from the properties of this same linear long wave mode and, like \( c \) depend on the particular physical system being considered. Note that the linearization of (6) has the linear dispersion relation \( \omega = ck - \lambda k^3 \) for linear sinusoidal waves of frequency \( \omega \) and wavenumber \( k \); this expression is just the truncation of the full dispersion relation for the wave mode being considered, and immediately identifies the origin of the coefficient \( \lambda \). Similarly, the coefficient \( \mu \) can be identified with the an amplitude-dependent correction to the linear wave speed. Transformation to a reference frame moving with a speed \( c \) and subsequent rescaling shows that (6) can be transformed to the canonical form (1). Equations of the form (6) arise in the study of internal solitary waves in the atmosphere and ocean, mid-latitude and equatorial planetary waves, plasma waves, ion-acoustic waves, lattice waves, waves in elastic rods and in many other physical contexts (see, for instance, Ablowitz and Segur 1981, Dodd et al 1982, Drazin and Johnson 1989, and Grimshaw 2001).

In some physical situations, it is necessary to complement the KdV equation (6) with a higher-order cubic nonlinear term of the form \( \nu A^2 A_x \). After transformation and rescaling, the amended equation (6) can be transformed to the so-called Gardner equation

\[ u_t + 6uu_x + 6\delta u^2 u_x + u_{xxx} = 0. \]  

(7)

Like the KdV equation, the Gardner equation is integrable by the inverse scattering transform. Here the coefficient \( \delta \) can be either positive or negative, and the structure of the solutions depends crucially on which sign is appropriate. Again, in some physical situations, solitary waves propagate through a variable environment which means that the coefficients \( c, \mu \) and \( \lambda \) in (6) are functions of \( x \), while an additional term \( c(\sigma_x/2\sigma)A \) needs to be included, where \( \sigma(x) \) is a magnification factor. After transforming to new variables, \( \theta = (\int x \, dx/c) - t, x \) with \( U = \sigma^{1/2} u \), the variable- coefficient KdV equation is obtained,

\[ U_x + \alpha(x)UU_\theta + \beta(x)U_{\theta\theta\theta} = 0. \]  

(8)

Here \( \alpha = \mu/c\sigma^{1/2}, \beta = \lambda/c^3 \). In general, this is not an integrable equation and must be solved numerically, although we shall exhibit some asymptotic solutions below. Another modification of the KdV equation occurs when it is necessary to take account of background rotation, leading to the rotation-modified KP equation (see, for instance, Grimshaw 2001), in which a term \( -f^2u \) is added to the left-hand side of equation (5), where \( f \) is measure of the background rotation.
Solitons

The remarkable discovery of Gardner, Greene, Kruskal and Miura (1967) that the KdV equation was integrable through an inverse scattering transform marked the beginning of soliton theory. Their pioneering work was followed by the work of Zakharov and Shabat (1972) which showed that another well-known nonlinear wave equation, the nonlinear Schrödinger equation, was also integrable by an inverse scattering transform. Their demonstration that the integrability of the KdV equation was not an isolated result, was followed closely by analogous results for the modified KdV equation (Wadati, 1972) and the Sine-Gordon equation (Ablowitz et al, 1973). In 1974, Ablowitz, Kaup, Newell and Segur provided a generalisation and unification of these results in the AKNS scheme. From this point there has been an explosive and rapid development of soliton theory in many directions (see, for instance, Ablowitz and Segur 1981, Dodd et al 1982, Newell 1985 and Drazin and Johnson 1989).

For the KdV equation (1) the starting point is the Lax pair (Lax 1968) for an auxiliary function $\phi(x,t)$,

\[ L\phi \equiv -\phi_{xx} - u\phi = \lambda \phi, \quad (9) \]

\[ \phi_t = B\phi \equiv (u_x + C)\phi + (4\lambda - 2u)\phi_x. \quad (10) \]

Here $C(t)$ depends on the normalization of $\phi$. The first of these equations (9), with suitable boundary conditions at infinity (see below) defines a spectral problem for $\phi$ in the spatial variable $x$ with a spectral parameter $\lambda$, and with the time variable $t$ as a parameter. The second equation (10) then describes how the spectral function $\phi$ evolves in time. If it is now assumed that $\lambda$ is independent of time (i.e. $\lambda_t = 0$ then the KdV equation (1) is just the compatibility condition for these two equations (9, 10); that is, it emerges as a result of the condition that $(\phi_{xx})_t = (\phi_t)_{xx}$. In terms of the operators $L, B$ defined in the Lax pair (9,10) the KdV equation can be written in the symbolic form $L_t = BL - LB$ (Lax, 1968). This form indicates the path to further generalizations, in that other nonlinear wave equations can be obtained by choosing different operators $L, B$. The general strategy for integration of the KdV equation now consists of three steps. Here we will describe the process under the hypothesis that we seek solutions $u(x,t)$ of the KdV equation (1) which decay to zero sufficiently fast as $x \to \pm\infty$ and have the initial condition, $u(x,0) = u_0(x)$. First, we insert the initial condition into the spectral problem (9) to obtain the scattering data (these will be defined precisely below). Then (10) is used to move the scattering data forward in time; it transpires that is a very simple process, and note in particular that the spectral parameter $\lambda$ is independent of $t$ and hence is determined by the initial condition. The third step is to invert the scattering data at time $t > 0$ and so recover $u(x,t)$; this is the most difficult step, but for the KdV equation can be reduced to solution of a linear integral equation. Thus, the three steps constitute a linear algorithm for the solution of the KdV equation, and it is in this sense that it is said that the Lax pair (9,10) constitute integrability of the KdV equation.
The spectral problem (9) for the KdV equation consists of two parts. The discrete spectrum is found by seeking solutions such that \( \phi \to 0 \) as \( x \to \pm \infty \), which requires that \( \lambda < 0 \). It can be shown that there then exist a finite set of discrete eigenvalues \( \lambda = -\kappa_n^2, n = 1, 2, \ldots, N \), and corresponding real eigenfunctions \( \phi_n \) such that

\[
\phi_n \sim c_n \exp (-\kappa_n x) \quad \text{as} \quad x \to \infty.
\]

(11)

There is a similar condition as \( x \to -\infty \), namely that \( \phi_n \sim d_n \exp (\kappa_n x) \). The real constants \( c_n, d_n \) are determined once the normalization condition is satisfied, that is,

\[
\int_{-\infty}^{\infty} \phi_n^2 \, dx = 1.
\]

(12)

The continuous spectrum consists of all \( \lambda > 0 \), and so we set \( \lambda = k^2 \) where \( k \) is real. Then we define the scattering problem for solutions \( \phi(x; k) \) of (9) by the boundary conditions,

\[
\phi \sim \exp (-ikx) + R(k) \exp (ikx) \quad \text{as} \quad x \to \infty,
\]

(13)

\[
\phi \sim T(k) \exp (-ikx) \quad \text{as} \quad x \to -\infty.
\]

(14)

The scattering data then consists of the set \( (\kappa_n, c_n, n = 1, 2, \ldots, N) \) together with the reflection coefficient \( R(k) \). It is useful to note that \( R(k) \) may be continued into the upper half of the complex \( k \)-plane, has there a set of simple poles at \( k = i\kappa_n \), and \( R \to 1 \) as \( |k| \to \infty \).

The next step is to determine from (10) how the scattering data evolves in time (note that the dependence on time \( t \) has been suppressed in the preceding paragraph). First, we recall that the discrete eigenvalues \( \kappa_n \) are independent of \( t \). Next, we multiply (10) by \( \phi_n \) and integrate the result over all \( x \); on also using (9) it is readily found that

\[
\frac{d}{dt} \int_{-\infty}^{\infty} \phi_n^2 \, dx = C_n \int_{-\infty}^{\infty} \phi_n^2 \, dx,
\]

where here the constant \( C \) in (10) must be indexed with \( n \) to become \( C_n \). But then the normalization condition (12) implies that \( C_n = 0 \). Now substitute (11) into (10) to show that

\[
\frac{dc_n}{dt} = 4\kappa_n^3 c_n \quad \text{so that} \quad c_n(t) = c_n(0) \exp (\kappa_n^3 t).
\]

(15)

For the continuous spectrum, the asymptotic expressions (13, 14) are substituted into (10). Now it is found that the constant \( C(k) = 4ik^3 \), and that

\[
\frac{dR}{dt} = 8ik^3 R \quad \text{so that} \quad R(k; t) = R(0; k) \exp (8ik^3 t),
\]

(16)

Similarly, it can be shown that \( T(k; t) = T(k; 0) \).

The final step is the inversion of the scattering data at time \( t \) to recover the potential \( u(x, t) \) in (9). This is accomplished through the Marchenko integral equation for the function \( K(x, y) \)

\[
K(x, y) + F(x + y) + \int_{x}^{\infty} K(x, z) F(y + z) \, dz = 0.
\]

(17)
Here the function $F(x)$ is known in terms of the scattering data at time $t$,

$$F(x) = \sum_{n=1}^{N} c_n^2(t) \exp(-\kappa_n x) + \frac{1}{2\pi} \int_{-\infty}^{\infty} R(k; t) \exp(ikx) \, dk.$$  \hfill (18)

Here the $t$-dependence of $K, F$ has been suppressed as the linear integral equation (17) is solved with $t$ fixed. Then

$$u(x, t) = 2\frac{\partial}{\partial x} \{K(x, x; t)\},$$  \hfill (19)

where the $t$-dependence in $K$ has been restored.

The inverse scattering transform described by (17,18) enables one to find the solution of the KdV equation (1) for an arbitrary localised initial condition. The most important outcome is that as $t \to \infty$, the solution evolves into $N$ rank-ordered solitons propagating to the right ($x > 0$), and some decaying radiation propagating to the left ($x < 0$),

$$u \sim \sum_{n=1}^{N} 2\kappa_n^2 \text{sech}^2(\kappa_n(x - 4\kappa_n^2 t - x_n)) + \text{radiation}.$$  \hfill (20)

Here the $N$ solitons derive directly from the discrete spectrum, where each eigenvalue $-\kappa_n$ generates a soliton of amplitude $2\kappa_n^2$, while the phase shifts $x_n$ are determined from the constants $c_n(0)$. The continuous spectrum is responsible for the decaying radiation, which decays at each fixed $x < 0$ as $t^{-1/3}$.

The important special case when the reflection coefficient $R(k) \equiv 0$ leads to the $N$-soliton solution, for which there is no radiation. Indeed, the $N$-soliton solution can be obtained as an explicit solution of the Marchenko equation (17). We illustrate the procedure for $N = 1, 2$. First, for $N = 1$, $F(x) = c^2 \exp \kappa(x - 4\kappa^2 t)$ where we have omitted the subscript $n = 1$ for simplicity. Then seek a solution of (17) in the form $K(x, y, t) = L(x, t) \exp(-\kappa y)$, where $L$ can be found by simple algebra. The outcome is that

$$L(x, t) = \frac{-2\kappa c(0)^2 \exp(-\kappa x + 8\kappa^2 t)}{2\kappa + c(0)^2 \exp(-2\kappa x + 8\kappa^2 t)}.$$  \hfill (21)

Finally $u$ is found from (19),

$$u = 2\kappa^2 \text{sech}^2(\kappa(x - 4\kappa^2 t) - x_1).$$

This is just the solitary wave (2) of amplitude $2\kappa^2$; the phase shift $x_1$ is such that $c(0)^2 = 2\kappa \exp(2\kappa x_1)$. The procedure for $N = 2$ follows a similar course. Thus, with $R \equiv 0, N = 2$ in $F$ (18), seek a solution of the Marchenko equation (17) in the form $K(x, y, t) = L_1(x, t) \exp(-\kappa_1 y) + L_2(x, t) \exp(-\kappa_2 y)$, and again $L_{1,2}$ can be found by simple algebra. The outcome is the 2-soliton solution. For instance, with $\kappa_1 = 1, \kappa_2 = 2$, this is

$$u = 12 \frac{3 + 4 \cosh(2x - 8t) + \cosh(4x - 64t)}{[3\cosh(3x - 36t) + \cosh(3x - 28t)]^2}.$$  \hfill (21)

It can be readily shown that

$$u \sim 8 \text{sech}^2(2(x - 16t \mp x_2) + 2 \text{sech}^2(x - 4t \mp x_1) \quad \text{as} \quad t \to \pm \infty.$$  \hfill (22)
where the phase shifts $x_{1,2} = (-1/2, 1/4) \ln 3$. Thus the 2-soliton solution describes the elastic collision of two solitons, in which each survives the interaction intact, and the only memory of the collision are the phase shifts; note that $x_1 < 0, x_2 > 0$, so that the larger soliton is displaced forward and the smaller soliton is displaced backward. The general case of an $N$-soliton is analogous and is essentially a sequence of pair-wise 2-soliton interactions.

The integrability of the KdV equation (1) is also characterized by the existence of an infinite set of independent conservation laws. The most transparent conservation laws are

$$\int_{-\infty}^{\infty} u \, dx = \text{constant}, \quad (23)$$

$$\int_{-\infty}^{\infty} u^2 \, dx = \text{constant}, \quad (24)$$

$$\int_{-\infty}^{\infty} u^3 - \frac{1}{2} u_x^2 \, dx = \text{constant}, \quad (25)$$

which may be associated with the conservation of mass, momentum and energy respectively. Indeed (23) is obtained from the KdV equation (1) by integrating over $x$, while (24,25) are obtained in an analogous manner after first multiplying (1) by $u, u^2$ respectively. However, it transpires that these are just the first three conservation laws in an infinite set, where each successive conservation law contains a higher power of $u$ than the preceding one. This may be demonstrated using the inverse scattering transform (see Ablowitz and Segur 1981, Dodd et al 1982, and Newell 1985). However, here we use the original method based on the Miura transformation as adapted by Gardner. The Miura transformation is

$$u = v_x - v^2, \quad (26)$$

$$v_t + 6v^2 v_x + v_{xxx} = 0. \quad (27)$$

Here (27) is the modified KdV equation. Direct substitution of (26) into the KdV equation shows that if $v$ solves the modified KdV equation (27), then $u$ solves the KdV equation (1). This discovery was the starting point for the discovery of the inverse scattering transform, since if one considers (26) as an equation for $v$ and writes $v = \phi_x / \phi$, followed by a Galilean transformation for $u$ (i.e. $u \to u - \lambda, x \to x - 6\lambda t$), one obtains the spectral problem (9). Here we follow a different route, and write

$$v = \frac{1}{2\epsilon} - \epsilon w, \quad \text{which (after a shift } x \to x + 3t/2\epsilon^2) \text{ converts the mKdV equation (27) into the Gardner equation (7) with } \delta = -\epsilon^2. \quad \text{Apart from a constant, which may be removed by a Galilean transformation, the corresponding expression for } u \text{ is the Gardner transformation, }$$

$$u = w + \epsilon w_x - \epsilon^2 w^2. \quad (28)$$
Thus if \( w \) solves the Gardner equation
\[
 w_t + 6w w_x - 6\epsilon^2 w^2 w_x + w_{xxx} = 0, \tag{29}
\]
then \( u \) solves the KdV equation (1).

Next, we observe that the Gardner equation (29) has the conservation law
\[
 \int_{-\infty}^{\infty} w \, dx = \text{constant}. \tag{30}
\]
Since \( w \to u \) as \( \epsilon \to 0 \), we write the formal asymptotic expansion
\[
 w \sim \sum_{n=0}^{\infty} \epsilon^n w_n. \]

It follows from (30) that then
\[
 \int_{-\infty}^{\infty} w_n \, dx = \text{constant},
\]
for each \( n = 0, 1, 2, \ldots \). But substitution of this same asymptotic expansion for \( w \) into (28) generates a sequence of expressions for \( w_n \) in terms of \( u \), of which the first few are
\[
 w_0 = u, \quad w_1 = -u_x, \quad w_2 = -u^2 + u_{xx}.
\]
Thus we see that \( n = 0, 2 \) give the conservation laws (23,24) respectively, while \( n = 1 \) is an exact differential. It may now be shown that all even values of \( n \) yield non-trivial and independent conservation laws, while all odd values of \( n \) are exact differentials.

The KdV equation belongs to a class of nonlinear wave equations, which have Lax pairs and are integrable through an inverse scattering transform. It shares with these equations several other remarkable features, such as the Hirota bilinear form, Bäcklund transforms and the Painlevé property. Detailed descriptions of these and other properties of the KdV equation can be found in the referenced texts.

**Solitary waves in a variable environment**

In a variable environment, the governing equation which replaces (1) is the variable-coefficient KdV equation (8). In general, this is not an integrable equation, and is usually solved numerically. However, there are two distinct limiting situations in which some analytical progress can be made. First, let it be supposed that the coefficients \( \alpha(x), \beta(x) \) in (8) vary rapidly with respect to the wavelength of a solitary wave, and consider then the case when these coefficients make a rapid transition from the values \( \alpha_-, \beta_- \) in \( x < 0 \) to the values \( \alpha_+, \beta_+ \) in \( x > 0 \). Then a steady solitary wave can propagate in the region \( x < 0 \), given by
\[
 U = \text{asech}^2(\gamma(\theta - Wx)) \quad \text{where} \quad W = \frac{\alpha_- a}{3} = 4\beta_- \gamma^2. \tag{31}
\]
It will pass through the transition zone \( x \approx 0 \) essentially without change. However, on arrival into the region \( x > 0 \) it is no longer a permissible solution of (8), which now has constant coefficients \( \alpha_+, \beta_+ \). Instead, with \( x = 0 \), the expression (31) now forms
an effective initial condition for the new constant-coefficient KdV equation. Using the spectral problem (9) and the inverse scattering transform, the solution in $x > 0$ can now be constructed; indeed in this case the spectral problem (9) has an explicit solution (e.g. Drazin and Johnson, 1989). The outcome is that the initial solitary wave fissions into $N$ solitons, and some radiation. The number $N$ of solitons produced is determined by the ratio of coefficients $R = \alpha_+ \beta_- / \alpha_- \beta_+$. If $R > 0$ (i.e. there is no change in polarity for solitary waves), then $N = 1 + [(8R + 1)^{1/2} - 1]/2$ ([$\cdots$] denotes the integral part); as $R$ increases from 0, a new soliton (initially of zero amplitude) is produced as $R$ successively passes through the values $m(m + 1)/2$ for $m = 1, 2, \ldots$. But if $R < 0$ (i.e. there is a change in polarity) no solitons are produced and the solitary wave decays into radiation. For instance, for water waves, $c = (gh)^{1/2}$, $\mu = 3c/2h$, $\lambda = ch^2/6$, $\sigma = c$ and so $\alpha = 3/(2hc^{1/2})$, $\beta = h^2/(6c^2)$ where $h$ is the water depth. It can then be shown that a solitary water wave propagating from a depth $h_-$ to a depth $h_+$ will fission into $N$ solitons where $N$ is given as above with $R = (h_-/h_+)^{9/4}$; if $h_- > h_+$, $N \geq 2$, but if $h_- < h_+$ then $N = 1$ and no further solitons are produced (Johnson 1973).

Next, consider the opposite situation when the coefficients $\alpha(x), \beta(x)$ in (8) vary slowly with respect to the wavelength of a solitary wave. In this case a multi-scale perturbation technique (see Grimshaw 1979, or Grimshaw and Mitsudera 1993) can be used in which the leading term is

$$U \sim A \text{sech}^2 \gamma (\theta - \int_{x_0}^x W \, dx),$$

where

$$W = \frac{\alpha A}{3} = 4\beta \gamma^2.$$  

Here the wave amplitude $a(x)$, and hence also $W(x), \gamma(x)$, are slowly-varying functions of $x$. Their variation is most readily determined by noting that the variable-coefficient KdV equation (8) possesses a conservation law,

$$\int_{-\infty}^{\infty} U^2 \, d\theta = \text{constant}.$$  

which expresses conservation of wave-action flux. Substitution of (32) into (88 gives

$$\frac{2A^2}{3\gamma} = \text{constant}, \quad \text{so that } \quad A = \text{constant} \left(\frac{\beta}{\alpha}\right)^{1/3}.$$  

This is an explicit equation for the variation of the amplitude $A(x)$ in terms of $\alpha(x), \beta(x)$. However, the variable-coefficient KdV equation (8) also has a conservation law for mass,

$$\int_{-\infty}^{\infty} Ud\theta = \text{constant}.$$  

Thus, although the slowly-varying solitary wave conserves wave-action flux it cannot simultaneously conserve mass. Instead, it is accompanied by a trailing shelf of small amplitude but long length scale given by $U_s$, so that the conservation of mass gives

$$\int_{-\infty}^\phi U_s \, d\theta + \frac{2A}{\gamma} = \text{constant},$$
where \( \phi = \int_{x_0}^{x} W \, dx \) (\( \theta = \phi \) gives the location of the solitary wave) and the second term is the mass of the solitary wave (32). Differentiation then yields the amplitude \( U_- = U_s(\theta = \phi) \) of the shelf at the rear of the solitary wave,

\[
U_- = \frac{3\gamma_x}{\alpha \gamma^2}.
\]

(37)

This shows that if the wavelength \( \gamma^{-1} \) increases (decreases) as the solitary wave deforms, then the trailing shelf amplitude \( U_- \) has the opposite (same) polarity to the solitary wave. Once \( U_- \) is known the full shelf \( U_s(\theta, x) \) is found by solving (8) with the boundary condition that \( U_s(\theta = \phi) = U_- \) (see El and Grimshaw (2002), where it is shown that the trailing shelf may eventually generate secondary solitary waves).

For a solitary water wave propagating over a variable depth \( h(x) \) these results show that the amplitude varies as \( h^{-1} \), while the trailing shelf has positive (negative) polarity relative to the wave itself according as \( h_x < (>)0 \). A situation of particular interest occurs if the coefficient \( \alpha(x) \) changes sign at some particular location (note that in most physical systems the coefficient \( \beta \) of the linear dispersive term in (8) does not vanish for any \( x \)). This commonly arises for internal solitary waves in the coastal ocean, where typically in the deeper water, \( \alpha < 0, \beta > 0 \) so that internal solitary waves propagating shorewards are waves of depression. But in shallower water, \( \alpha > 0 \) and so only internal solitary waves of elevation can be supported. The issue then arises as to whether an internal solitary wave of depression can be converted into one or more solitary waves of elevation as the critical point, where \( \alpha \) changes sign, is traversed. This problem has been intensively studied (see, for instance, Grimshaw et al (1998) and the references therein), and the solution depends on how rapidly the coefficient \( \alpha \) changes sign. If \( \alpha \) passes through zero rapidly compared to the local width of the solitary wave, then the solitary wave is destroyed, and converted into a radiating wavetrain (see the discussion above in the first paragraph of this section). On the other hand, if \( \alpha \) changes sufficient slowly for the present theory to hold (i.e. (35) applies), we find that as that the as \( \alpha \to 0 \) then \( A \to 0 \) in proportion to \( |\alpha|^{1/3} \), while \( U_- \to \infty \) as \( |\alpha|^{-8/3} \). Thus, as the solitary wave amplitude decreases, the amplitude of the trailing shelf, which has the opposite polarity, grows indefinitely until a point is reached just prior to the critical point where the slowly-varying solitary wave asymptotic theory fails. A combination of this trailing shelf and the distortion of the solitary wave itself then provide the appropriate “initial” condition for one or more solitary waves of the opposite polarity to emerge as the critical point is traversed. However, it is clear that in situations, as here, where \( \alpha \approx 0 \), it will be necessary to include a cubic nonlinear term in (8), thus converting it into a variable-coefficient Gardner equation (cf. (7)). This case has been studied by Grimshaw, Pelinovsky and Talipova (1999), who showed that the outcome depends on the sign of the coefficient \( \nu \) of the cubic nonlinear term at the critical point. If \( \nu > 0 \) so that solitary waves of either polarity can exist when \( \alpha = 0 \), then the solitary wave preserves its polarity (i.e. remains a wave of depression) as the critical point is traversed. On the other hand if \( \nu < 0 \) so that no solitary wave can exist when \( \nu = 0 \) then the solitary wave of depression may be converted into one or more solitary
waves of elevation.

Roger Grimshaw

See also solitons, water waves, inverse scattering transform

Further Reading


