Finite-order solutions and the integrability of difference equations

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The existence of sufficiently many finite-order (in the sense of Nevanlinna) meromorphic solutions of a difference equation appears to be a good indicator of integrability. This criterion is used to single out \(dP_{II}\) from a natural class of second-order difference equations. The proof given uses an estimate related to singularity confinement.

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The Painlevé property has been used as a detector of integrability for differential equations since the nineteenth century. An ordinary differential equation is said to possess the Painlevé property if all solutions are single-valued about all movable singularities (see, e.g., [1]). In Ablowitz, Halburd, and Herbst [2] the idea of using the complex analytic structure of solutions as a detector of integrability was extended from differential to difference equations. There it was suggested that many difference equations that are considered to be of “Painlevé type” admit sufficiently many finite-order (in the sense of Nevanlinna theory) meromorphic solutions. We will show that this property singles out \(dP_{II}\) — Eq. (11) below, from a wide class of second-order difference equations. Eq. (11) has many properties associated with integrability and is known to possess a simple continuum limit to the second Painlevé (differential) equation, \(P_{II}\): 

\[
y'' = 2y^3 + zy + \sigma,\]

where \(\sigma\) is a constant.

The singularity confinement approach to integrability of Grammaticos, Ramani, and Papageorgiou [3] has proved to be an easy to implement and quite powerful detector of integrability. It has led to the discovery of many important discrete equations [4] which are widely believed to be integrable. The approach involves studying the behaviour of iterates of finite initial conditions which lead to some future iterate becoming infinite. On continuing through the singularity, one finds that generically future iterates oscillate between finite and infinite values. Roughly speaking, a singularity is “confined” if the iterates return to finite values and contain enough information about the initial conditions. See also [5].

The idea of singularity confinement also presents a number of problems. In particular, how do we decide whether a given singularity sequence is truly confined and what exactly is the property for which we are testing? Also, an example of a numerically chaotic discrete equation possessing the singularity confinement property was found by Hietarinta and Viallet [6]. They suggest that singularity confinement needs to be augmented by a condition that a sequence of iterates possesses zero algebraic entropy. This is related to a number of approaches to the integrability of discrete equations or maps in which one considers the growth of the degree of the \(n^{th}\) iterate as a rational function of the initial conditions [7–10]. Recently Costin and Kruskal [11] have introduced an idea of integrability related to whether the sequence of iterates of a solutions can be imbedded in the complex plane as an analyzable function.

In this letter we study an example in which, in general, singularities are not confined. We use a notion of confinement involving only five iterates. Non-confinement is used to
show that any corresponding meromorphic solution of the difference equation has infinite order. This helps to clarify a link between the singularity confinement approach [3] and the complex function-theoretic approach [2]. The fact that any transcendental meromorphic solution of the equation studied in [6] is of infinite order was shown in [2] and is apparently not related to confinement.

A wide class of rational difference equations, in particular, all difference equations in which $y(z + 1)$ is a rational function of $y(z)$, are known to admit non-constant meromorphic solutions [12–14]. Furthermore, Yanagihara [14] has shown that if the difference equation

$$y(z + 1) = R(z; y(z)),$$

where

$$R(z; y(z)) = \frac{a_0(z) + a_1(z)y(z) + \cdots + a_p(z) y^p(z)}{b_0(z) + b_1(z)y(z) + \cdots + b_q(z) y^q(z)}$$

and the $a$'s and $b$'s are polynomials in $z$, admits a transcendental finite-order meromorphic solution, then Eq. (1) becomes

$$\bar{y} = \frac{A + By}{C + Dy},$$

where $A, B, C, D$ are polynomials in $z$ and we have suppressed the $z$-dependence by writing $y \equiv y(z)$, and $\bar{y} \equiv y(z + 1)$. Eq. (3) is the difference Riccati equation whose general solution is given by $y(z) = \frac{R(z-1)}{R(z-1)} \left[ \frac{w(z)-w(z-1)}{w(z)} \right]$, where $w$ is the general solution of a certain second-order linear difference equation with polynomial coefficients.

Several authors have considered higher-order generalizations of this result. In particular, if the equation

$$\bar{y} + y = R(z; y),$$

where $\bar{y} = y(z - 1)$, admits a transcendental finite-order meromorphic solution, then the degree of $R$ as a function of $y$ is at most two [21]. Several other interesting generalizations appear in the literature [2, 15–17], however, the methods employed do not give any information on the explicit dependence of the equations on the independent variable $z$.

In [2], after finding some simple necessary conditions for certain difference equations to admit a transcendental (i.e., non-rational) finite-order meromorphic solution, an extra condition was added to the effect that certain series expansions of solutions should not contain di-gamma functions, which appear to play the role of logarithms in the series expansions of solutions of differential equations. This led to an effective algorithmic test. Here we show that the existence of sufficiently many finite-order meromorphic solutions alone appears to be a powerful detector of integrability.

We will now consider equations of the form

$$\bar{y} + y = \frac{a_0 + a_1y + a_2y^2}{1 - y^2},$$

where at least one of the $a_j$’s is non-zero and the degree of the right side as a function of $y$ is two (i.e., $a_0 \pm a_1 + a_2$ is not identically zero.) The main idea is to use Eq. (5) to deduce certain estimates about the relative number of times $y$ takes the special values 1, $-1$, and infinity in discs. To this end, we introduce some standard notation from Nevanlinna theory (the theory of meromorphic functions.) Let the counting function $\bar{n}(r, y)$ be the number
of poles of $y$ in $D_r := \{ z : |z| < r \}$, ignoring multiplicities. From the point of view of Nevanlinna theory, it is more natural to work with the integrated counting function, which is defined to be

$$\bar{N}(r, y) = \int_0^r \frac{\bar{n}(t, y) - \bar{n}(0, y)}{t} dt + \bar{n}(0, y) \log r.$$  

We now quote a lemma, a proof of which can be found at the end of this letter.

**Lemma 1** Let $y$ be a meromorphic solution of Eq. (5) such that

$$\bar{N} \left( r, \frac{1}{y - 1} \right) + \bar{N} \left( r, \frac{1}{y + 1} \right) \leq \alpha \bar{N} \left( r + 3, y \right),$$

where $\alpha < 2$. Then $y$ has infinite order.

We will say that the singularity at $z = z_0$ is confined if $y(z_0) = \infty$ but $y(z_0 \pm 1)$ and $y(z_0 \pm 2)$ (four points) are finite. It follows from shifting $z \mapsto z \pm 1$ in Eq. (5) that if the singularity at $z_0$ is confined then $y(z_0 \pm 1)^2 = 1$.

Suppose $y(z_0 - 1) = \varepsilon = \pm 1$ for some $z_0 \in \mathbb{C}$. Then we see from Eq. (5) that $y$ has a pole at either $z_0 - 2$ or $z_0$. Without loss of generality, we assume $y(z_0) = \infty$. Eq. (5) then shows that $y(z_0 + 1) = -\varepsilon - a_2(z_0)$.

Next, we determine the condition for $y(z_0 - 2)$ and $y(z_0 + 2)$ to be finite. This condition is essentially singularity confinement [3] re-interpreted for meromorphic functions. From Eq. (5), we see that if $y(z_0) = \infty$ and $y(z_0 + 2)$ is finite, then $y(z_0 + 1) = \pm 1$. So $a_2(z_0) = 0$ or $a_2(z_0) = -2\varepsilon$. It follows that if there are infinitely many points where $y$ has a confined singularity, then $a_2$ is identically either 0, 2 or $-2$ (since $a_2$ is a polynomial.)

Now we consider the case in which $y$ has only finitely many confined singularities but infinitely many unconfined singularities. We will obtain an inequality of the type (6). To this end we will consider singularity sequences containing as many $\pm 1$’s compared to the number of $\infty$’s as possible. Note that for unconfined singularities, this is achieved by a sequence of the form $(\ldots, \varepsilon_1, \infty, \varepsilon_2, \infty, \varepsilon_3, \ldots)$, where $\varepsilon_j^2 = 1$. It may be possible to have sequences of the form $(\ldots, \varepsilon_1, \infty, \varepsilon_2, \infty, \varepsilon_3, \ldots, \varepsilon_4, \infty, \varepsilon_5, \infty, \varepsilon_6, \ldots)$. Note that a sequence $(\ldots, \varepsilon_1, \varepsilon_2, \ldots)$ is necessarily part of the sequence $(\ldots, \infty, \varepsilon_1, \varepsilon_2, \infty, \ldots)$. Hence for any meromorphic solution of Eq. (5) with infinitely many unconfined singularities and no confined singularities, we have the estimate (6) with $\alpha = 3/2$. If there are finitely many confined singularities, then given $\varepsilon > 0$, we can take $\alpha = 3/2 + \varepsilon$ for $r$ sufficiently large. It follows from the Lemma 1 that $y$ has infinite order.

Finally, if $y$ has finite order and there are only finitely many points at which $y(z)^2 = 1$, then the following inequality (from Nevanlinna’s Second Main Theorem)

$$T(r, y) \leq \bar{N} \left( r, \frac{1}{y - 1} \right) + \bar{N} \left( r, \frac{1}{y + 1} \right) + \bar{N} (r, y)$$

$+ O(\log r)$, shows that

$$\bar{N} \left( r, \frac{1}{y - 1} \right) + \bar{N} \left( r, \frac{1}{y + 1} \right) = O(\log r) \leq \bar{N} (r, y),$$

which implies the inequality (6) with any positive $\alpha$. 

It follows that if \( y \) is a transcendental meromorphic solution of Eq. (5) then either \( y \) has infinite order or \( y \) has infinitely many confined singularities. The fact that \( a_2 \) is identically either 0, 2 or \(-2\) is a necessary but not a sufficient condition for confinement.

In the case \( a_2 \equiv 0 \), Eq. (5) implies

\[
(1 - y^2)(\bar{y} - y) = a_0 + \bar{a}_1 y - a_0 - a_1 y - (y + y) \left[ \frac{2y(a_0 + a_1 y)}{1 - y^2} - \left( \frac{a_0 + a_1 y}{1 - y^2} \right)^2 \right]. \tag{8}
\]

If \( y \) has finite order then for at least one choice of \( \varepsilon = \pm 1 \), there must be infinitely many points \( z_0 \) such that in the limit \( z \mapsto z_0 \), \( y \mapsto \varepsilon \), \( y \mapsto \infty \), \( \bar{y} \mapsto -\varepsilon \) and \( \bar{y} \) and \( y \) have finite limits. In this limit, Eq. (8) becomes

\[
a_1(z_0 + 1) - 2a_1(z_0) + a_1(z_0 - 1) - \varepsilon [a_0(z_0 + 1) - a_0(z_0 - 1)] = 0. \tag{9}
\]

Since Eq. (9) holds at infinitely many points and \( a_0 \) and \( a_1 \) are polynomials, it follows that (9) holds for all \( z_0 \), so

\[
a_1(z + 1) - a_1(z) = \varepsilon [a_0(z + 1) + a_0(z)] + \kappa_\varepsilon, \tag{10}
\]

where \( \kappa_\varepsilon \) is a constant.

Note that, if Eq. (10) holds for both \( \varepsilon = 1 \) and \( \varepsilon = -1 \), then Eq. (5) is precisely \( dP_H \),

\[
\bar{y} + y = \frac{(\lambda z + \mu)y + \nu}{1 - y^2}, \tag{11}
\]

where \( \lambda \), \( \mu \), and \( \nu \) are constants. However, the existence of a transcendental meromorphic solution of finite order is not sufficient to show that (10) holds for both choices of \( \varepsilon \). In fact, any solution of the Riccati Eq. (3) with \( B = C = \varepsilon \) and \( D = -1 \), satisfies Eq. (5) with \( a_0 = \varepsilon(A - \bar{A}) \) and \( a_1 = A + \bar{A} + 2 \). In this case, Eq. (10) is satisfied with \( \kappa_\varepsilon = 0 \).

Continuing with the case \( a_2 \equiv 0 \), we distinguish between two types of singularity. If \( y \) has a pole at \( z = z_0 \), we will say the singularity at \( z_0 \) is of type I if \( y(z_0 \pm 1) = \pm \varepsilon \) and of type II if \( y(z_0 \pm 1) = \mp \varepsilon \). Note that there may be poles which are neither type I nor type II, however, all points where \( y \) is \( \pm 1 \) will occur as part of one of these two types. Let \( \bar{n}_I(r, y) \) and \( \bar{n}_{II}(r, y) \) be the number of poles (not counting multiplicities) in \( \{ z : |z| < r \} \) of type I and type II respectively.

The finite-order Riccati solutions described above are degenerate in that they possess only type I singularities and no type II singularities. In order to avoid these solutions we further demand that there is a transcendental finite-order meromorphic solution with “comparably many” poles of type I and type II. In particular, we assume that there is a finite real constant \( c \geq 1 \), such that

\[
c^{-1}\bar{n}_I(r, y) \leq \bar{n}_{II}(r, y) \leq c\bar{n}_I(r, y), \tag{12}
\]

for sufficiently large \( r \). Let us assume that all but finitely many of the type II singularities are unconfined. In the “worst case scenario” (i.e., when there are as many \( \pm 1 \)-points per \( \infty \) in a sequence of iterates as possible) we can exhaust all the \( \pm 1 \)'s by associating at most two of them with each type I singularity and at most one with each type II singularity, with only finitely many exceptions. This is because confined type I singularities
look like \((\ldots, \varepsilon, \infty, -\varepsilon, \ldots)\) and, in the “worst case”, the unconfined singularities look like \((k_1, \varepsilon, \infty, -\varepsilon, \varepsilon, k_2)\), where \(k_1\) and \(k_2\) are finite. So for sufficiently large \(r\),

\[
\bar{n}(r, \frac{1}{y-1}) + \bar{n}(r, \frac{1}{y+1}) \leq 2(1+\varepsilon)\bar{n}_I(r+3, y) + (1+\varepsilon)\bar{n}_II(r+3, y) \\
\leq (1+\varepsilon)\left(2 - (\varepsilon + 1)^{-1}\right)\bar{n}(r+3, y),
\]

which implies the inequality (6) with \(\alpha < 2\), so \(y\) has infinite order. Hence, if a solution \(y\) has finite order then it must have infinitely many confined singularities of both types and hence Eq. (5) is \(dP_{II}\), Eq. (11).

In the case \(a_2 \equiv -2\varepsilon\), the only singularities \(y(z_0) = \infty\) involving \(\pm 1\) again fall into two types; \(y(z_0 \pm 1) = \varepsilon\) (type I singularities) and \(y(z_0 \pm 1) = -\varepsilon\) (type II singularities). The assumption (12) once again implies that if \(y\) is a transcendental finite order meromorphic solution then there must be infinitely many singularities of both types which are confined. This leads to a contradiction (since \(a_2\) would be both 2 and \(-2\)).

In this letter we have used concepts related to singularity confinement to show that all transcendental meromorphic solutions of a class of difference equations are of infinite order. In their study of the discrete Painlevé I equation, Hietarinta and Viallet [18] suggested that, although in a sequence of iterates there are infinitely many points at which a singularity could be confined, it is the first such point that is the decisive one (their paper was concerned with a class of equations related to \(dP_I\) only.) This is again the case that we have found. It is not necessary to know whether the singularity is “infinitely long” in order to show that the order of the solution is infinite.

The existence of a finite-order solution with comparably many singularities of both allowable types is enough to single out \(dP_{II}\) and appears to be a strong indicator of integrability. Similar methods to those described here have been used for the difference third and fourth Painlevé equations (see [22]).

We conjecture that if \(y\) is a finite-order transcendental meromorphic solution of equation (5) then either this equation is \(dP_{II}\) (Eq. 11) or \(y\) satisfies a difference Riccati or linear equation. In the case \(\lambda = 0\), Eq. (11) is the McMillan map, which has a two-parameter family of finite-order meromorphic solutions given in terms of elliptic functions. It remains to be shown that Eq. (11) admits a non-trivial class of meromorphic solutions when \(\lambda \neq 0\).

We conclude this letter with a proof of Lemma 1. The proof uses some basic properties of the Nevanlinna characteristic \(T(r, y)\). The reader is referred to Hayman [19] for standard notation in Nevanlinna theory.

**Proof of Lemma 1**

Assume that \(y\) is a finite-order meromorphic solution of Eq. (5). From the inequality (6) and Nevanlinna’s Second Main Theorem (7), we have

\[
T(r, y) - \bar{N}(r, y) \leq \alpha \bar{N}(r + 3, y) + O(\log r). \quad (13)
\]

The Valiron-Mohon’ko Theorem (see, e.g., [20]) allows us to approximate the Nevanlinna characteristic of a rational function of a meromorphic function such as \(R\) given by Eq. (2). It says that if \(y\) is a transcendental meromorphic function and the coefficients \(a_j\) and \(b_k\) are rational functions, then \(T(r, R) = dT(r, y) + O(\log r)\), where \(d\) is the degree of the rational
function. Equating the Nevanlinna characteristic of both sides of Eq. (5) gives
\[
2T(r, y) = T(r, \bar{y} + y) + O(\log r)
\]
\[
= T(r, \bar{y} + y) - \bar{\mathcal{N}}(r, \bar{y} + y) + \bar{\mathcal{N}}(r, \bar{y} + y) + O(\log r)
\]
\[
\leq T(r, \bar{y}) - \bar{\mathcal{N}}(r, \bar{y}) + T(r, y) - \bar{\mathcal{N}}(r, y) + \alpha \bar{\mathcal{N}}(r + 3, y) + O(\log r).
\]  
(14)

Using the inequalities (13) and (14), we have
\[
T(r, y) - \bar{\mathcal{N}}(r, y) \leq \frac{3\alpha}{2} \bar{\mathcal{N}}(r + 4, y) - \bar{\mathcal{N}}(r, y) + O(\log r),
\]
where we have used the fact that \(\bar{\mathcal{N}}(r + 3, \bar{y})\) and \(\bar{\mathcal{N}}(r + 3, y)\) are bounded by \(\bar{\mathcal{N}}(r + 4, y)\).

We will now use induction to show that
\[
T(r, y) - \bar{\mathcal{N}}(r, y) \leq \frac{n + 2}{2} \bar{\mathcal{N}}(r + 2n + 2, y) - n \bar{\mathcal{N}}(r, y) + O(\log r),
\]  
(15)
for all \(n \in \mathbb{N}\). We have just proved that Eq. (15) is true for \(n = 1\). If Eq. (15) is true for some \(n \in \mathbb{N}\), then from Eq. (14),
\[
2T(r, y) \leq \alpha \frac{n + 2}{2} \bar{\mathcal{N}}(r + 2n + 2, \bar{y}) - n \bar{\mathcal{N}}(r, \bar{y}) + \alpha \bar{\mathcal{N}}(r + 3, y) + O(\log r)
\]
\[
\leq \alpha (n + 3) \bar{\mathcal{N}}(r + 2n + 3, y) - 2n \bar{\mathcal{N}}(r - 1, y) + O(\log r).
\]
Using the fact that \(T(r, y) \leq T(r + 1, y)\), we see that (15) holds with \(n\) replaced by \(n + 1\). So (15) holds for all \(n \in \mathbb{N}\). Hence,
\[
\bar{\mathcal{N}}(r, y) \leq \alpha \frac{n + 2}{2n} \bar{\mathcal{N}}(r + 2n + 2, y) + O(\log r),
\]  
(16)
for all \(n \in \mathbb{N}\). Now, since \(\alpha < 2\), we can choose \(n\) such that \(\alpha (n + 2)/(2n) < 1\). It follows from (16) that \(\bar{\mathcal{N}}(r, y)\) grows exponentially with \(r\) and therefore so does \(T(r, y)\). Therefore the order of \(y\) is infinite. \(\square\)

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[21] See [2]. Note there is a typographical error in the statement of the theorem there in which the degree is said to be equal to two, rather than less than or equal to two.
[22] In preparation.