Homoclinic Orbits for Perturbed Lattice modified KdV equation

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Abstract

We establish the splitting of homoclinic orbits for a near-integrable lattice modified KdV equation with periodic boundary conditions. The Bäcklund transformation is successfully utilized to construct homoclinic orbits of lattice mKdV equation. The Melnikov function is built with the gradient of invariant defined through the discrete Floquet discriminant evaluated at critical points. The criteria for the persistence of homoclinic solutions for the perturbed lattice mKdV are established.

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1 Introduction

Nonlinear dispersive wave equations provide excellent examples of infinite dimensional dynamical systems which possess diverse and fascinating solutions including solitary waves, pattern formation, singularity formation, localized time-periodic structures, dispersive turbulence, and spatio-temporal chaos, [7]. There is a class of nonlinear wave equations (‘soliton equations’) that have the special property of complete integrability and for which there is now a mature literature. Examples include the sine-Gordon (SG) equation [3], the modified Korteweg-de Vries (mKdV) [12], and the cubic nonlinear Schrödinger (NLS) equation [6]. These equations under periodic boundary conditions in space admit particular solution which localized in space and homoclinic in time.

In order to extract quantitative information from the models described by PDEs, it is often necessary to solve numerically with the help of various discretization schemes. By investigating the long-time dynamics, their qualitative features become of first rank importance. In particular, we are interested to investigate the behavior of particular solutions of discrete models once integrability is broken by perturbation (dissipative or Hamiltonian) of the equation. Specifically, for the study of global behavior of near-integrable lattice equations, we advocate implementing intuition from the theory of discrete dynamical systems with methods natural for the lattices.

In recent years there has been remarkable progress on discrete integrable systems and associated difference equations. Discrete space-time model in soliton theory have acquired a prominent role of mathematical physics. Subsequent development was carried on mostly by the Dutch group (see Nijhoff and Capel [9] and references therein).

This paper is concerned with the existence of homoclinic orbits of the near–integrable lattice mKdV equation,

\[
(p - r) \frac{u_{n+1}^{m+1} - 2u_n^{m+1}}{u_n^{m+1}} - (q - r) \frac{u_{n+1}^{m} - 2u_n^{m}}{u_n^{m}} = (p + r) \frac{u_{n+1}^{m+1} - 2u_n^{m+1}}{u_n^{m+1}} - (q + r) \frac{u_{n+1}^{m} - 2u_n^{m}}{u_n^{m}}
\]  

(1.1)

we mention that \( u = u_n^m \) is the dynamical (field) variable at site \((n, m) \in \mathbb{Z}, p, q \in \mathbb{C} \) are lattice parameters. Equation (1.1) was derived from the direct linearization approach (see [9] and references therein). A starting point is an integrable lattice version of the mKdV equation, which is a nonlinear partial difference equation (PΔE), i.e a system in which both the spatial–as well as the time–variable is discrete. We show that the equation with periodic boundary conditions in the discrete space admits solutions which are homoclinic in the “discrete” time to the hyperbolic fixed point. We consider here the lattice mKdV as a discrete evolution equation with respect to index \( n \) (in the continuum limit approaches the time variable \( t \)) and we assume that \( u_n^m \) satisfies the periodic and even boundary conditions

\[
\mathcal{L} = \{ u_{n+M}^m = u_n^m, \quad u_{M-n}^m = u_n^m, \quad M = 2\kappa, \ n \in \mathbb{Z}, \ m = 0, 1, \ldots, M - 1, \ h = 2\pi/M \} 
\]  

(1.2)

We present necessary criteria for the persistence of homoclinic structure of the lattice mKdV equation with small dissipative perturbations. With the method being developed here we exploit two facts: first we investigate the existence of homoclinic orbits associated to a hyperbolic fixed point of the unperturbed equation through a Bäcklund transformation and second we establish the persistence of these orbits under small dissipative perturbations. The Mel’nikov method [8, 11] is one of the techniques that has proved to be very useful in dynamical systems. The method establishes a way to measure the distance between the stable and unstable manifolds of a saddle–type invariant set for a perturbed integrable system that originally has homoclinic structure.

The paper has the following structure. In Section 2, we recall some definitions related to iterations maps and review some basic concepts of the lattice mKdV as an integrable nonlinear PΔE. In section 3, we find an analytic expression of the homoclinic orbits in discrete time \((n)\) through the Bäcklund transformation and we obtain the formula for the gradient of the individual constant of motion evaluated on the homoclinic orbits. In Section 4, we derive necessary conditions for the persistence of homoclinic
solutions of lattice mKdV equation under dissipative perturbations based on the Mel’nikov analysis and geometric arguments of our problem.

2 Integrable lattice mKdV equation

In this section, we present some definitions related to finite dimensional maps and characteristic properties of lattice mKdV equation.

2.1 Saddle Points and Separatrices

Let \( F \) be a diffeomorphism. A point \( x_0 = F(x_0) \) is called fixed point of the map \( F \). Fixed points may be classified according to the spectrum of the matrix \( DF(x_0) \). Hyperbolic fixed points are special interests. A fixed point \( x_0 \) is called hyperbolic, if the eigenvalues of the matrix \( DF(x_0) \) do not belong to the unit circle. With the hyperbolic fixed point we can associate invariant objects, which play an important role in the dynamics.

Let \( \delta > 0 \) be a small positive parameter. The set of points, which never leave a \( \delta \)-neighborhood of \( x_0 \), is called a local stable manifold (or local stable separatrix),

\[
W_{\text{loc}}^s(x_0) = \left\{ x : \forall n \geq 0 \| F^n - x_0 \| \leq \delta \right\} \tag{2.1}
\]

A local unstable manifold (or local unstable separatrix) is the local stable manifold of the inverse map \( F^{-1} \),

\[
W_{\text{loc}}^u(x_0) = \left\{ x : \forall n \geq 0 \| F^{-n} - x_0 \| \leq \delta \right\} \tag{2.2}
\]

If \( \delta \) is sufficiently small, a local manifold is the image of a unit disk under a smooth (analytic) immersion. The dimension of the disk coincides with the number of eigenvalues of \( DF(x_0) \) outside (for local unstable manifold) or inside (for local stable manifold) the unit circle.

Iterations of a point on the local stable separatrix converge to the fixed point. This permits to define the global stable separatrix as the set of all points, whose iterations converge to the fixed point:

\[
W^s(x_0) = \left\{ x : \lim_{n \to +\infty} F^n = x_0 \right\}
\]

A (global) unstable manifold is the stable manifold of the inverse map \( F^{-1} \),

\[
W^u(x_0) = \left\{ x : \lim_{n \to -\infty} F^n = x_0 \right\}
\]

The global (un)stable manifold may be obtained by iterations of the local manifolds.

A local separatrix is the embedding of the unit disk. Consequently, it has no self-intersections. The map \( F \) is a diffeomorphism, and the corresponding global separatrix can not intersect itself. It clear to see that if the map has two hyperbolic fixed points, their unstable (or their stable) manifolds do not intersect.

A point of intersection of two invariant manifolds associated with one fixed point is called homoclinic. As both manifolds contain the fixed point \( x_0 \), this point is excluded when we speak about homoclinic points. A point is heteroclinic, if it belongs to the intersection of separatrices associated with two different fixed points.
2.2 Integrable Background of lattice mKdV

The lattice mKdV (1.1) with periodic boundary condition arises from a discrete action principle. The action for the lattice mKdV reads [2]:

\[
S = \sum_{n \in \mathbb{Z}} \sum_{m=0}^{M-1} \left\{ y^{n+1}_{m+1}(y^{n+1}_{m+1} + y^{n}_{m}) + F'(y^{n}_{m} - y^{n+1}_{m+1} + \sigma_2) - F(y^{n+1}_{m} - y^{n+1}_{m+1} + \sigma_1) \right\}
\]

\[
\equiv \sum_{n \in \mathbb{Z}} \sum_{m=0}^{M-1} L(u^{n}_{m}, u^{n+1}_{m+1}, u^{n+1}_{m+1}, u^{n+1}_{m+1})
\]

(2.3)

in which \(u^{n}_{m} = e^{\theta^{n}_{m}}\) and the function \(F\) is related to the dilogarithm function

\[
F(x) = \int_{-\infty}^{x} \ln(1 + e^{i\xi}) \, d\xi
\]

and where \(\sigma_1, \sigma_2\) are parameters related to the lattice parameters \(p, q\):

\[
e^{\sigma_1} = \frac{q}{p}, \quad e^{\sigma_2} = \frac{p}{q}
\]

The Euler-Lagrange equation for (2.3) which are obtained by variation of \(S\) with respect to the variables \(u^{n}_{m}\), i.e.

\[
\frac{\delta S}{\delta u^{n}_{m}} = 0
\]

Another important aspect of the lattice mKdV, directly related to the integrability of this system, is that arises as the compatibility condition of an over-determined linear system, Lax pair [2]:

\[
\phi^{n}_{m+1}(k) = V_{n,m}(k)\phi^{n}_{m}
\]

(2.4a)

\[
\phi^{n+1}_{m+1}(k) = U_{n,m}(k)\phi^{n}_{m}
\]

(2.4b)

where \(k\) denotes the spectral parameter and \(V_{n,m}\) is given by

\[
V_{n,m}(k) = \begin{pmatrix}
\frac{q}{q-k} & \frac{v^{n+1}_{m}}{q-k} \\
\frac{k^2}{q-k} \frac{1}{v^{n}_{m}} & \frac{q}{q-k} \frac{v^{n+1}_{m}}{v^{n}_{m}}
\end{pmatrix}
\]

(2.5)

and where \(U_{n,m}\) is given by a similar matrix obtained from (2.5) by making the replace \(q \rightarrow p\) and \((n, m+1) \rightarrow (n+1, m)\).

Let \(M_{n,m}(k, u^{n}_{m})\) be the fundamental matrix to the Lax pair, i.e the matrix solution to (2.4a, 2.4b) with initial conditions \(M_0\) being a \(2 \times 2\) identity matrix. The characteristic polynomial

\[
\det(M_{n,m}(k, u^{n}_{m}) - \lambda I) = 0
\]

(2.6)

defines an invariant algebraic curve. The substitution

\[
\mathcal{M} = \lambda - \frac{\text{trace}M_{n,m}(k)}{2}
\]

reduces this curve to the hyperelliptic form:

\[
\mathcal{M}^2 = \left(\frac{\text{trace}M_{n,m}(k)}{2}\right)^2 - \det M_{n,m}(k)
\]

(2.7)
We define the function $\Delta$ as:

$$
\Delta : \mathbb{C} \times \mathcal{L} \rightarrow \mathbb{C} \\
\Delta(k, u^n_m) = \text{trace} M_{n,m}(M; k, u^n_m)
$$

(2.8)

The spectral parameter $k \in \mathbb{C}$ fulfills the following condition

$$
-2D \leq \Delta(k, u^n_m) \leq 2D
$$

(2.9)

with $D$ is constant and a critical point $k_c$ of $\Delta$ satisfies

$$
\frac{d\Delta}{dk} \bigg|_{(k_c, u^n_m)} = 0
$$

(2.10)

An important sequence of constant of motion $\mathcal{F}_j : \mathcal{L} \rightarrow \mathbb{C}$ is defined by

$$
\mathcal{F}_j(u^n_m) = \frac{1}{D} \Delta(k_{c,j}; u^n_m)
$$

3 Stability analysis and Homoclinic Orbits

3.1 Linearized analysis

It is clear that the fixed solutions are $\tilde{u}^n_m = e^{2(\pm 1)^{m+1}}$. Assume that the small perturbation $\hat{u}^n_m(\varepsilon)$ are exponentially fast growing, as a consequence of the linearly unstable modes of the continuum mKdV equation in space interval $(0,1)$

$$
u^n_m = \hat{u}^n_m + \eta \varphi^n_m, \quad 0 < \eta \ll 1
$$

(3.1a)

with

$$
\varphi^n_m = \Omega^n_k \exp[iA_km], \quad A_k = \frac{2\pi k}{M}
$$

(3.1b)

where $\hat{u}^n_m$ is a solution of (1.1), $M = 2\kappa$ the total number of grid points in the space interval $(0,1)$ and $k = -\frac{M}{2}, \ldots, \frac{M}{2} - 1$.

Linearizing the equation (1.1) we obtain

$$
p(\varphi^n_{m+1} + \varphi^n_m) + q(\varphi^n_{m+1} + \varphi^n_m) = p(\varphi^{n+1}_{m+1} + \varphi^n_m) + q(\varphi^{n+1}_m + \varphi^n_m)
$$

(3.2)

After straightforward calculations, we take the following equation for eigenvalues $\Omega_k$:

$$
\Omega_k = \left( \frac{B + \cos A_k}{1 + B \cos A_k} \right) + i \left( \frac{-pq(p^2 + q^2)^{-1} \sin A_k}{1 + B \cos A_k} \right)
$$

(3.3)

with

$$
B = \frac{p^2 - q^2}{p^2 + q^2}, \quad h = 2\pi / M \ll 1
$$

(3.4)

The stability condition $|\Omega_k| \leq 1$ becomes:

$$
\cos A_k \leq 1
$$

(3.5)

The condition (3.5) is violated by the wave number $A_k \simeq 0$ for $k = 0$. The unstable wavenumbers satisfy $|\Omega_k| \geq 1$. 

5
3.2 Analytic expression of homoclinic orbits

We construct the homoclinic orbits to hyperbolic fixed points for the lattice mKdV through the Bäcklund transformation. For more complete studies on Bäcklund transformation see the book [13]. The unperturbed lattice mKdV equation has an unstable hyperbolic fixed point at \( \hat{u}_n^m \). We utilize the Bäcklund transformation for a concrete example, to generate homoclinic orbits through an iteration of the transformation to incorporate all the unstable and stable modes.

Let \( \phi_m^n \) denote a fix solution of the Lax pair at \((u_m^n, k_d)\), where \( k_d \) is the double critical point of the function \( \Delta \), i.e. satisfies the condition (2.10) and \( \Delta(k_d; u_m^n) = \pm 2D \). We define the 2 x 2 transformation matrix \( \Gamma_{n,m} \) by

\[
\Gamma_{n,m} = \begin{pmatrix}
-k - \alpha_{n,m} & \beta_{n,m} \\
\gamma_{n,m} & -k - \delta_{n,m}
\end{pmatrix}
\]

(3.6)

where

\[
\alpha_{n,m} = \frac{k}{D} (|\phi_{m,n}^{1,1}|^2 + \bar{k}^2 |\phi_{m,n}^{2,1}|^2), \quad \delta_{n,m} = \bar{\alpha}_{n,m}
\]

\[
\beta_{n,m} = \frac{\phi_{m,n}^{1,1} \phi_{m,n}^{2,2}}{D} (\bar{k}^2 - |k|^2), \quad \gamma_{n,m} = -\bar{\beta}_{n,m}, \quad D = |\phi_{m,n}^{1,1}|^2 + |k|^2 |\phi_{m,n}^{2,1}|^2
\]

We define

\[
U_m^n = p \left( k + \alpha_{n-1,m} + \left( 1 - \frac{k^2}{p^2} \right) + \frac{p-k}{v_m^n} \beta_{n,m} \right) \left( -\beta_{n-1,m} + \frac{p-k}{p} \frac{\bar{k} + \bar{\alpha}_{n-1,m}}{v_m^{n-1}} \right)^{-1}
\]

(3.7)

and

\[
\Phi_m^n (k) = \Gamma_{n,m}(k_d) \phi_m^n (k)
\]

(3.8)

**Proposition 1** Let \( u_m^n \) denote a solution of lattice mKdV with double critical point \( k_d \). We denote \( \phi_m^n \) the general solution of the Lax pair at \((u_m^n, k_d)\). We define \( U_m^n \) and \( \Phi_m^n (k) \) by (3.7) and (3.8).

i. \( U_m^n \) is a solution of lattice mKdV,

ii. \( \Phi_m^n (k) \) solves the Lax pair (2.4a, 2.4b) at \((U_m^n, k)\),

iii. \( U_m^n \) is homoclinic to fixed solution \( \hat{u}_n^m \), such that \( \lim_{n \to \pm \infty} U_m^n = e^{2i\pi[-1]^n} \).

**Proof.** We show that \( \Phi_m^n (k) \) solves the Lax pair at \((U_m^n, k)\) provided that \( \phi_m^n \) solves (2.4a, 2.4b) at \((u_m^n, k)\). From the equations of Lax pair we get

\[
\Phi_{m+1}^n - V_{n,m}(U_m^n) \Phi_m^n = \left[ \Gamma_{n,m+1} V_{n,m}(u_m^n) - V_{n,m}(U_m^n) \Gamma_{n,m} \right] \Phi_m^n
\]

(3.9a)

\[
\Phi_{m+1}^n - U_{n,m}(U_m^n) \Phi_m^n = \left[ \Gamma_{n+1,m} U_{n,m}(u_m^n) - U_{n,m}(U_m^n) \Gamma_{n,m} \right] \Phi_m^n
\]

(3.9b)

Hence if

\[
\Gamma_{n,m+1} V_{n,m}(u_m^n) - V_{n,m}(U_m^n) \Gamma_{n,m} = 0
\]

(3.9a)

\[
\Gamma_{n+1,m} U_{n,m}(u_m^n) - U_{n,m}(U_m^n) \Gamma_{n,m} = 0
\]

(3.9b)

the proposition is established. In the component form equations (3.9a, 3.9b) together the definitions (3.6)–(3.7) are equivalent to the eight algebraic equations. Indeed, the system

\[
\Gamma_{n,m+1} = V_{n,m}(U_m^n) \Gamma_{n,m} V_{n,m}^{-1}(u_m^n)
\]

\[
\Gamma_{n+1,m} = U_{n,m}(U_m^n) \Gamma_{n,m} U_{n,m}^{-1}(u_m^n)
\]
yields the following set of algebraic equations:

\[ P(\tilde{k} + \alpha_{n+1,m}) = \left( \tilde{k} + \alpha_{n,m} + \frac{p - k}{p} \frac{1}{\nu_{n+1,m}} \beta_{n,m} \right) - \frac{U_{m}^{n+1}}{p} \left( \gamma_{n,m} + (p - k) \frac{(\tilde{k} + \delta_{n,m})}{p\nu_{n+1,m}} \right) \]

\[ P\beta_{n+1,m} = \left( \tilde{k} + \alpha_{n,m} \right) \frac{\nu_{m}}{p} + \beta_{n,m} \frac{\nu_{m}}{\nu_{n+1,m}} - \frac{U_{m}^{n+1}}{p} \left( \frac{\beta_{n,m}}{\nu_{m}} + (\tilde{k} + \alpha_{n,m}) \frac{\nu_{m}}{\nu_{n+1,m}} \right) \]

\[ P\gamma_{n+1,m} = -\frac{k^{2}}{pU_{m}^{n}} \left( \tilde{k} + \alpha_{n,m} + \frac{1}{q} \right) \frac{\nu_{m}}{\nu_{n+1,m}} + \frac{U_{m}^{n+1}}{p} \left( \gamma_{n,m} + \frac{k^{2}}{p\nu_{n+1,m}} \right) \]

\[ Q(\tilde{k} + \alpha_{n,m+1}) = \left( \tilde{k} + \alpha_{n,m} + \frac{q}{q} \right) \frac{\nu_{m}}{\nu_{n+1,m}} - \frac{U_{m}^{n+1}}{q} \left( \gamma_{n,m} + \frac{k^{2}}{q\nu_{n+1,m}} \right) \]

\[ Q\beta_{n,m+1} = \left( \tilde{k} + \alpha_{n,m} \right) \frac{\nu_{m}}{q} + \beta_{n,m} \frac{\nu_{m}}{\nu_{n+1,m}} - \frac{U_{m}^{n+1}}{q} \left( \gamma_{n,m} \frac{\nu_{m}}{q} + (\tilde{k} + \delta_{n,m}) \frac{\nu_{m}}{\nu_{n+1,m}} \right) \]

\[ Q\gamma_{n,m+1} = -\frac{k^{2}}{qU_{m}^{n}} \left( \tilde{k} + \alpha_{n,m} + \frac{1}{q} \right) \frac{\nu_{m}}{\nu_{n+1,m}} + \frac{U_{m}^{n+1}}{q} \left( \gamma_{n,m} + \frac{k^{2}}{q\nu_{n+1,m}} \right) \] \hspace{1cm} (3.10)

where

\[ P = \left( 1 - \frac{k^{2}}{p^{2}} \right)^{-1}, \quad Q = \left( 1 - \frac{k^{2}}{q^{2}} \right)^{-1} \]

The problem is reduced to checking these eight equations. From representation of \( \alpha_{n,m}, \beta_{n,m}, \gamma_{n,m}, \delta_{n,m} \) and \( U_{m}^{n} \) this is done explicitly. \( \square \)

We consider as potential function on the Lax pair \( u_{m}^{n} = \alpha_{m}^{n} \) (the fixed point) , thus the solutions of (2.4a, 2.4b) takes the form:

\[ \phi_{m}^{n+} = \left( \frac{1}{k} \right)^{r_{m}} \lambda_{m}^{r_{m}}, \quad \phi_{m}^{n+} = \left( \frac{1}{k} \right)^{b_{m}} \lambda_{m}^{b_{m}} \] \hspace{1cm} (3.11)

where

\[ \lambda_{+} = \alpha_{+} \frac{k_{a}}{q}, \quad a := \frac{q}{q - k}, \quad b := \frac{p + k}{p - k} \] \hspace{1cm} (3.12)

and \( k = k_{d} \) is a double critical point of \( \Delta \) this means that satisfies the conditions:

\[ \Delta(k) = 2D, \quad \frac{d}{dk} \Delta(k) = 0 \]

with

\[ \Delta(k) = \left( a - \frac{k_{a}}{q} \right)^{m} + b^{2}k \left( a + \frac{k_{a}}{q} \right)^{m}, \quad |M_{n,m}| := 2b^{2}k \left( a^{2} - \left( \frac{k_{a}}{q} \right)^{2} \right)^{m} \]

and

\[ \text{det}M_{n,M}(\phi^{-}, \phi^{+}) = D^{2} \text{det}M_{n,0}, \quad \text{with} \quad D^{2} := \prod_{m=0}^{M-1} (\lambda_{-} \lambda_{+})^{m} \] \hspace{1cm} (3.13)
Insertion (3.11) to (3.7), we obtain the following analytic expression of homoclinic orbits:

\[
U_n^m = \left\{ \frac{1 - P^2 \cos^2(mhw_1 + x) \sech^2(mhw_2 + \tau) + 2iP \cos(mhw_1 + x) \sech(mhw_2 + \tau)}{1 + P^2 \cos^2(mhw_1 + x) \sech^2(mhw_2 + \tau)} \right\} e^{2(1)^m i \pi}
\]

with

\[
P = \frac{\sinh_2 h}{\sinh_1 h} \quad \text{and} \quad h \ll 1
\]

\( w_{1,2} \) are constants and \( \lim_{n \to \pm \infty} U_n^m = \hat{u}_n^m \).

### 3.3 The gradient of \( \Delta(k, u_n^m) \)

In this section, we obtain an explicit formula for the gradient of an important invariant of motion which we will use to measure the distance function between the invariant manifolds.

The trace of the fundamental matrix of Lax pair is an important invariant of motion for the sine-Gordon equation. The corresponding invariants \( \mathcal{F}_j(u_n^m) \) for the lattice mKdV system are defined as:

\[
\mathcal{F}_j(u_n^m) = \frac{1}{D} \Delta(k^e_j(u_n^m), u_n^m, u_{m+1}^m)
\]

We shall use the invariants \( \mathcal{F}_j \) to build the Mel’nikov functions, in Section 4.

**Lemma 1** Let \( k^e_j(u_n^m, u_{m+1}^m) \) be a simple critical point of \( \Delta \):

\[
\frac{\delta \mathcal{F}_j}{\delta u_n^m} (u_m^m, u_{m+1}^m) = \frac{1}{D} \frac{\delta \Delta}{\delta u_n^m} (k^e_j(u_n^m), u_n^m, u_{m+1}^m)
\]

\[
\frac{\delta \mathcal{F}_j}{\delta u_{m+1}^n} (u_m^m, u_{m+1}^m) = \frac{1}{D} \frac{\delta \Delta}{\delta u_{m+1}^n} (k^e_j(u_n^m), u_n^m, u_{m+1}^m)
\]

where

\[
\frac{\delta \Delta}{\delta u_n^m} (k, u_n^m) = \frac{2i}{q-k} \text{trace} \left[ M_{n,m+1}^{-1} \begin{pmatrix} 0 & 0 \\ k^2 - q \nu_{m+1}^n & -q \nu_m^n \end{pmatrix} M_{n,m} \right]
\]

\[
\frac{\delta \Delta}{\delta u_{m+1}^n} (k, u_n^m) = \frac{2i}{q-k} \text{trace} \left[ M_{n,m+1}^{-1} \begin{pmatrix} 0 & \nu_{m+1}^n \\ 0 & -q \nu_m^n \end{pmatrix} M_{n,m} \right]
\]

\( \nu_n^m = e^{2i u_n^m} \) and \( M_{n,m} = \begin{bmatrix} \phi_{m-n}^- & \phi_{m-n}^+ \\ \phi_{m+n}^- & \phi_{m+n}^+ \end{bmatrix} \), \( \phi_{m-n}^+ = (\phi_{m+n}^{(-)}, \phi_{m+n}^{(+)}) \) are two Bloch functions of the Lax pair (2.4a, 2.4b) and \( |M_{n,m}| = \det M_{n,m} \).

**Proof.** Since \( k^e_j \) be a critical point, \( \Delta'(k^e_j(u_n^m), u_n^m) = 0 \) and

\[
\frac{\delta k^e_j}{\delta u_n^m} = -\frac{1}{\Delta' \frac{\delta \Delta}{\delta u_n^m}}
\]

then \( k^e_j \) is a differential function

\[
D \frac{\delta \mathcal{F}_j}{\delta u_n^m} (k, u_n^m) = \left. \frac{\delta \Delta}{\delta u_n^m} \right|_{k=k^e_j} + \left. \frac{\delta \Delta}{\delta k} \right|_{k=k^e_j} \frac{\delta k^e_j}{\delta u_n^m} = \left. \frac{\delta \Delta}{\delta u_n^m} \right|_{k=k^e_j}
\]
Let $M_{n,m}$ be the fundamental matrix of the Lax pair, for the potential function $v^n$. Variation of the $v^n_m$ leads to the variational equation for the variation of $M_{n,m}$ at fixed $k$:

\[
M_{n,m+1} = V_{n,m} M_{n,m}, \quad \delta M_{n,m+1} = V_{n,m} \delta M_{n,m} + \delta V_{n,m} M_{n,m},
\]

\[
\delta V_{n,m} = \begin{pmatrix}
0 & \frac{1}{q-k} \delta v_{n,m+1}^n \\
\frac{k^2}{q-k} \delta \left( \frac{1}{v^n_m} \right) & \frac{q}{q-k} \delta \left( \frac{v_{n,m+1}^n}{v^n_m} \right)
\end{pmatrix}
\]

Let $\delta M_{n,m} = M_{n,m} \delta A_{n,m}$, where $A_{n,m}$ is a $2 \times 2$ matrix to be determined.

\[
\delta M_{n,m+1} = M_{n,m+1} A_{n,m+1} = V_{n,m} M_{n,m} A_{n,m} + \delta V_{n,m} M_{n,m},
\]

\[
= M_{n,m+1} A_{n,m} + \delta V_{n,m} M_{n,m}
\]

thus,

\[
A_{n,m+1} - A_{n,m} = M_{n,m+1}^{-1} \delta V_{n,m} M_{n,m},
\]

\[
A_{n,0} = 0
\]

(3.18)

Solving the system (3.18) we have,

\[
\delta M_{n,m} = M_{n,M} \left[ \sum_{j=1}^{M-1} M_{n,j}^{-1} \delta V_{n,j-1} M_{n,j-1} \right], \quad \delta M_{n,0} = 0
\]

Then,

\[
\delta \Delta(k, u^n_m) = \text{trace} \left\{ M_{n,M} \left[ \sum_{j=1}^{M-1} M_{n,j}^{-1} \delta V_{n,j-1} M_{n,j-1} \right] \right\}
\]

thus, we obtain the equations (3.17). Substitution the representation of $M_{n,m}$ into (3.17) entails:

\[
\frac{\delta \Delta}{\delta u^n_m}(k, u^n_m) = \frac{2i}{q-k} \frac{1}{|M_{n,m+1}|} \left\{ v^n_m \left[ \phi^n_{m+1} \phi^n_{M}^{-1} + \phi^n_{m} \phi^n_{M}^{-1} \right] + q v^n_{m+1} \left[ \phi^n_{m+1} \phi^n_{M}^{-1} \phi^n_{M}^{+2} \right] \right\}
\]

(3.19a)

\[
\frac{\delta \Delta}{\delta u^n_{m+1}}(k, u^n_m) = \frac{2i}{q-k} \frac{1}{|M_{n,m+1}|} \left\{ v^n_{m+1} \left[ \phi^n_{m} \phi^n_{M}^{-1} + \phi^n_{m} \phi^n_{M}^{-1} \right] + q v^n_{m+1} \left[ \phi^n_{m} \phi^n_{M}^{-1} \phi^n_{M}^{+2} \right] \right\}
\]

(3.19b)

Substitute (3.11) and (3.14) into (3.19a) and (3.19b), we obtain the expressions of $\delta \Delta/\delta u^n_m$ evaluated on the homoclinic orbits. This completes the proof of the Lemma.

\[ \square \]

9
4 Homoclinic Orbits for perturbed lattice mKdV equation

Next, we will establish the persistence of homoclinic orbits (3.14) for the perturbed lattice mKdV equation (4.1) (see below).

We consider the perturbed lattice mKdV equation in the form:

\[
u_{m}^{n+1} = F(u_{m}^{n-1}, u_{m}^{n}) + \varepsilon G(u_{m}^{n}, u_{m+1}^{n})
\]  

(4.1)

\(G\) denotes the perturbation:

\[
G := b_1 u_{m}^{n} + c_2 (u_{m+1}^{n})
\]

With the geometric structures constructed in the previous sections, we start to construct orbits for the perturbed lattice mKdV (4.1) homoclinic to the hyperbolic point \(\hat{u}_{m}^{n}(\varepsilon)\). The construction is composed of two steps. We start with an initial point in the unstable manifold of \(\hat{u}_{m}^{n}(\varepsilon)\), and first we show that the forward orbit enters the center-stable manifold of the neighborhood \(\mathcal{M}_{\varepsilon}\), if the parameters \(b, c\) are properly chosen. Second, we prove that the perturbed orbits approach the fixed point \(\tilde{u}_{m}^{n}(\varepsilon)\) in forward time \(n \to +\infty\). The Mel’nikov method is the main tool.

The perturbed system has a saddle fixed point \(\hat{u}_{m}^{n}(\varepsilon)\) and in the full phase space linear stability analysis shows that \(\hat{u}_{m}^{n}(\varepsilon)\) is a saddle point and by the invariant manifold theory \(W^{u}(\hat{u}_{m}^{n}(\varepsilon))\) exists and is two-dimensional. \(W^{s}(\hat{u}_{m}^{n}(\varepsilon))\) also exists and has codimension 2 and \(W_{\varepsilon}^{s}\) has codimension 1. The intersection \(W^{u}(\hat{u}_{m}^{n}(\varepsilon)) \cap W^{s}_{\varepsilon}\) will be one-dimensional.

Thus, the unstable, stable and center manifolds of the map \(F\) (c.f lattice mKdV) associated to \(\hat{u}_{m}^{n}\):

\[
W^{u} = \left\{ u_{m}^{n} \in \mathcal{L} : \lim_{n \to -\infty} F^{n}(u_{m}^{n}) = \hat{u}_{m}^{n} \right\}
\]

\[
W^{s} = \left\{ u_{m}^{n} \in \mathcal{L} : \lim_{n \to +\infty} F^{n}(u_{m}^{n}) = \hat{u}_{m}^{n} \right\}
\]

(4.2)

\(W^{s}(\hat{u}_{m}^{n}(\varepsilon)) \subset W^{s}_{\varepsilon}\) and \(\mathcal{M}_{\varepsilon}\).

For the perturbed system, we have the local stable and unstable manifolds \(W^{s,u}_{\varepsilon,loc}\) of the perturbed point \(\hat{u}_{m}^{n}(\varepsilon)\) are “\(\varepsilon\)-close” to those of the unperturbed point \(\hat{u}_{m}^{n} = e^{2i(-1)^{m}\pi}\).

Consider the saddle fixed point \(\hat{u}_{m}^{n}(\varepsilon)\) on the invariant plane \(\Pi\) and we seek an orbit, not on \(\Pi\), which is homoclinic to \(\hat{u}_{m}^{n}(\varepsilon)\). We present a geometrical mean to establish the persistence of homoclinic orbits for the near-integrable lattice mKdV equation. Our method based on the Mel’nikov measurement [8] through which we answer the question: Is there any intersection between \(W_{\varepsilon,loc}^{u}(\hat{u}_{m}^{n}(\varepsilon))\) and \(W^{s}(\mathcal{M}_{\varepsilon}) \subset W^{s}_{\varepsilon}\) ?

The essence of Mel’nikov method is as follows: we establish the intersection of \(W^{s}_{\varepsilon}\) and \(W^{u}_{\varepsilon,loc}(\hat{u}_{m}^{n}(\varepsilon))\) of the perturbed lattice mKdV equation, so that for external parameters in a fixed open set, there are orbits which tend to \(\tilde{u}_{m}^{n}(\varepsilon)\) when \(n \to \pm\infty\).

Then the distance between stable and unstable manifolds along the tangent vector \(\nu_{m}^{n}\) [11] is given by

\[
d = \varepsilon(D^{u}(0) - D^{s}(0)) + O(\varepsilon^{2}) = \varepsilon M_{\mathcal{F}} + O(\varepsilon^{2})
\]

(4.3)

where

\[
M_{\mathcal{F}} := \sum_{n \in \mathbb{Z}} \sum_{m=0}^{M-1} \nu_{m}^{n}(U_{m}^{n}) \cdot G(U_{m}^{n-1})
\]

(4.4)

with \(\nu_{m}^{n}\) defined in (3.19a, 3.19b), evaluated on the homoclinic orbit \(U_{m}^{n}\).

The Mel’nikov function (4.4) becomes:

\[
M_{\mathcal{F}}(x, \tau; h, M) = bS_{1} + cS_{2}
\]

(4.5a)
with

\[
S_1(x, \tau; h, M) = \sum_{n \in \mathbb{Z}} \sum_{m=0}^{M-1} \frac{\delta \mathcal{F}}{\delta u^m_n}(u^m_n \cdot g_1(u^m_n)) = \sum_{n \in \mathbb{Z}} s_n^{(1)}
\]

\[
S_2(x, \tau; h, M) = \sum_{n \in \mathbb{Z}} \sum_{m=0}^{M-1} \frac{\delta \mathcal{F}}{\delta u^m_n}(u^m_n \cdot g_2(u^m_n)) = \sum_{n \in \mathbb{Z}} s_n^{(2)}
\]  (4.5b)

Setting \( M_\mathcal{F} = 0 \) we obtain an algebraic equation which describe a two-dimensional surface on the external parameter space \((b, c)\):

\[
bS_1 + cS_2 = 0 \quad (4.6)
\]

We make the following observation, the only singularities of the homoclinic orbits \( U^n_m \) are simple poles at any point \( \tau_0 \in \frac{1}{h} + i\pi \mathbb{Z} \) and therefore \( U^n_m \) is analytic at \( \tau_0 + hk \) for \( k \in \mathbb{Z} \setminus \{0\} \). The above sum reduces to compute the residues of the function \( s^{(i)} \), \( i = 1, 2 \). In particular, the function that plays a significant role in the computation of the infinite sums in (4.5a) is a complex function \( \chi \) satisfying the following properties: \( \chi \) is meromorphic in \( \mathbb{C} \), is \( T \)-periodic and \( w \)-periodic and the set of poles of \( \chi \) is \( \mathbb{N} + T \mathbb{Z} \), being all of them simple and of residue 1. Also, the functions \( s^{(i)} \) appear in formula (4.5a) verifying:

(a) are analytic in \( \mathbb{R} \) and has only isolated singularity in \( \mathbb{C} \);

(b) are \( T \)-periodic for \( T > 0 \),

(c) and fulfills

\[
|s^{(i)}(\tau)| \leq A_i \exp[-c_i |\text{Re}\tau|]
\]

when \( |\text{Re}\tau| \to \infty \) for some constants \( A_i, c_i > 0 \).

Then, the Mel’nikov function \( M_\mathcal{F} \) is analytic in \( \mathbb{R} \), has only isolated singularities in \( \mathbb{C} \) and is doubly periodic with periods \( h, T \). Moreover \( M_\mathcal{F}(\tau) \) can be expressed by the following sum:

\[
M_\mathcal{F}(\tau; h) = - \sum_{z \in \mathcal{J}} b \text{ res}(\chi_r s^{(1)}; z) + c \text{ res}(\chi_r s^{(2)}; z) \quad (4.7)
\]

where \( \tau \) parameterizes the homoclinic solutions and

\[
\mathcal{J} := \left\{ z \in \mathbb{C} : 0 < \text{Im} z < T \right\}
\]

We can now state the result.

**Proposition 2** Consider the near-integrable lattice mKdV equation (4.1) with periodic boundary conditions (1.2). Let \( M \) be an even fixed positive integer and \( h := 2\pi/M < 1 \) be a constant real number. Then, there exists \( \varepsilon_0 > 0 \) such that, for any fixed parameters \( \{ w_1, w_2, b, c, \varepsilon \} \) there is isolated value \( \tau_0 \) of \( \tau \) at the Melnikov functions \( M_\mathcal{F} \) satisfy the following conditions

\[
M_\mathcal{F} = 0, \quad \partial_\tau M_\mathcal{F} \neq 0
\]

The perturbed lattice mKdV admits a homoclinic orbit \( u^n_m(\varepsilon) \) that is doubly asymptotic to the fixed point \( \bar{u}^n_m(\varepsilon) = e^{2i(-1)^m} + O(\varepsilon) \).
5 Conclusion

In this paper, we have developed a geometric method of studying the behavior of homoclinic solutions of lattice mKdV equation with periodic boundary conditions under small dissipative perturbations. The Bäcklund transformation is successfully utilized to construct homoclinic orbits of lattice mKdV equation through an elegant iteration of the transformation. We have shown that these orbits persist under dissipative perturbations through the Mel'nikov criteria for discrete dynamical systems. The Melnikov function is defined with the gradients of the invariant $\mathcal{F}$ defined through the discrete Floquet discriminant evaluated at critical points.

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