Higher-order Korteweg-de Vries models for internal solitary waves in a stratified shear flow with a free surface

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Abstract

A higher-order extension of the familiar Korteweg-de Vries equation is produced for internal solitary waves in a density- and current-stratified shear flow with a free surface. All coefficients of this extended Korteweg-de Vries equation are expressed via integrals of the modal function for the linear theory of long internal waves. An illustrative example of a two-layer shear flow is considered, and the parameter dependence on the coefficients is discussed.

1 Introduction

The Korteweg-de Vries (KdV) equation is a well-known model for the description of nonlinear long internal waves in a fluid stratified by both density and current. The steady-state version of this equation was produced by Long (1956), while Benney (1966) gave the integral expressions for calculation of the coefficients of the Korteweg-de Vries equation for waves in a fluid with arbitrary stratification in the density and current. The next step was due to Lee & Beardsley (1974) who indicated the asymptotic procedure needed to produce higher-order Korteweg-de Vries equations based on two small parameters representing dispersion and nonlinearity. More detailed information was obtained for interfacial waves in a two-layer fluid, and in particular, Kakanan & Yamasaki (1978) found the coefficient of the cubic nonlinear term in an implicit form, and showed its importance for the certain conditions (i.e. the pycnocline lies in the middle of the fluid), in which case the quadratic and cubic nonlinear terms are of the same order. Due to the negative sign of the coefficient of the cubic nonlinear term, this situation leads to an upper limit for the solitary wave amplitude. Then all nonlinear-dispersive coefficients for all second order terms were found for a two-layer fluid (Koop & Butler, 1981), and the extended Korteweg-de Vries equation was compared with results of laboratory experiments of internal solitary waves. A more detailed analysis of the properties of the steady-state solitary waves in a fluid with arbitrary density and current stratification, valid to the second order of an asymptotic expansion was reported by Gear & Grimshaw (1983). They calculated the shape of the internal solitary wave and its speed for different models of the fluid stratification.

The necessity to explain observed data of the internal wave field in the ocean, and a lot of such data has been obtained in the last twenty years by remote sensing and in situ measurements, has induced an interest to develop
the models for unsteady internal solitary waves in the ocean with the realistic stratification both in density and current, which may also vary horizontally. The first step was the introduction of a variable-coefficient Korteweg-de Vries equation. (Pelinovsky et al, 1994; Grimshaw, 1997). Some calculations of the coefficients of this equation showed that the coefficient of the quadratic nonlinear term will typically change its sign in the coastal zone (see, for instance, Holloway et al, 1997), and therefore, the contribution of higher-order terms (in particular the cubic nonlinear term) should be important. General integral expressions for the coefficients of these higher-order terms for the case of a continuous density-stratification (in the Boussinesq approximation) were produced by Lamb & Yan (1996), and then by Pelinovsky, Poloukhina, Lamb (2000) with the addition of a shear flow, but also in the Boussinesq approximation. These expressions are quite complicated, and their signs are not evident without extensive calculation. Consequently, the simpler expressions for a two-layer fluid (Kakutani & Yamasaki, 1978; Koop & Butler, 1981) have usually been used to estimate the signs and the value of the coefficients in the evolution equation. Grimshaw et al (1997) and Talipova et al (1999) calculated the coefficient of the cubic nonlinear term for a three-layer fluid with a constant buoyancy frequency in an each layer (but in the Boussinesq) and showed that it may have either sign depending on the layer locations. Such an variation in the sign of the coefficient of the cubic nonlinear term was also obtained for the real stratification of a shelf zone (Holloway et al, 1999).

The goal of this paper is to obtain a higher-order Korteweg-de Vries equation for the internal waves in arbitrary density- and current-stratified fluid without using the Boussinesq approximation, and also taking into account the free surface. An explicit example of a two-layer shear flow will be considered to obtain the coefficients explicitly, and so then analyse them for different parameter settings.

2 Governing equations

The governing equations are those for a two-dimensional flow in an inviscid, incompressible stratified fluid. In Eulerian coordinates they are,

$$\rho \frac{du}{dt} + \frac{\partial p}{\partial x} = 0,$$  \hspace{1cm} (2.1)

$$\rho \frac{dw}{dt} + \frac{\partial p}{\partial z} + \rho g = 0,$$  \hspace{1cm} (2.2)

$$\frac{d\rho}{dt} = 0,$$  \hspace{1cm} (2.3)
\[ u_x + w_z = 0, \]  
\[ (2.4) \]

where \( \{u, w\} \) is the fluid velocity, \( \rho \) is the fluid density, \( p \) is the pressure, \( g \) is gravitational acceleration, \( \{x, z\} \) are the spatial coordinates (horizontal and vertical) and \( d/dt \) is the convective time derivative.

The fluid is bounded below by the rigid boundary \( z = -h \) and above by the free surface whose equilibrium position is at \( z = 0 \). Thus, boundary conditions for equations (2.1)- (2.4) have the form,

\[ w = 0 \quad \text{at} \quad z = -h \]  
\[ (2.5) \]

\[ p = 0 \quad \text{at} \quad z = \xi(x, t), \]  
\[ (2.6) \]

\[ w = \frac{\partial \xi}{\partial t} + u \frac{\partial \xi}{\partial x}, \quad \text{at} \quad z = \xi \]  
\[ (2.7) \]

where \( \xi(x, t) \) is the vertical displacement of the free surface.

Let us express these equations in non-dimensional form based on the dimensionless variables,

\[ (x, z, t) = (h_0 \tilde{x}, h_0 \tilde{z}, \frac{1}{N_0} \tilde{t}), \]  
\[ (2.8) \]

\[ (u, w) = h_0 N_0 (\tilde{u}, \tilde{w}), \]  
\[ (2.9) \]

\[ (\rho, p) = (\tilde{\rho}, \tilde{\rho} h_0 \tilde{p}), \]  
\[ (2.10) \]

where \( h_0 \) is a typical length scale, \( N_0 \) is a typical value of the buoyancy frequency, and \( \tilde{\rho} \) is typical density value.

After dropping the tilde superscripts equations (2.3), (2.4) are unchanged, as are the boundary conditions, while equations (2.1), (2.2) take the form,

\[ \rho \frac{du}{dt} + \frac{1}{\sigma} \frac{\partial p}{\partial x} = 0, \]  
\[ (2.11) \]

\[ \rho \frac{dw}{dt} + \frac{1}{\sigma} \left( \frac{\partial p}{\partial z} + \rho \right) = 0, \]  
\[ (2.12) \]

where \( \sigma = N_0^2 h_0 / g \). This parameter is small for oceanic conditions, and \( \sigma \to 0 \) defines the Boussinesq approximation.

Thus we shall use equations (2.1), (2.2), (2.3), (2.4) and kinematic condition (2.7) with the boundary conditions (2.5), (2.6) to study internal wave dynamics in a stratified fluid.
3 Semi-Lagrangian form of the governing equations

First we introduce $\zeta(x, z, t)$ as the vertical displacement of a fluid particle from its rest position, so that

$$w = d\zeta/dt$$

We suppose that the density of the fluid in the rest state is given by $\rho_0(z)$. Then in the disturbed state, $\rho(x, z, t) = \rho_0(z - \zeta(x, z, t))$, so that (2.3) is now satisfied. Also it is convenient to express the pressure in the form,

$$p(x, z, t) = -\int_0^z \rho_0(z')dz' + \sigma q(x, z, t).$$

Now introduce the isopycnal (Lagrangian) coordinate,

$$y = z - \zeta(x, z, t)$$

Thus density $\rho(x, z, t) = \rho_0(y)$ - is fixed in this representation. To determine how the equations transform, when we change the $(x, z, t)$ coordinates to $(x, y, t)$, let

$$f(x, z, t) = F(x, y, t).$$

Therefore we have the relations,

$$\frac{\partial f}{\partial x} = \frac{\partial F}{\partial x} - \frac{\partial F}{\partial y} \frac{\partial \zeta}{\partial x},$$

$$\frac{\partial f}{\partial t} = \frac{\partial F}{\partial t} - \frac{\partial F}{\partial y} \frac{\partial \zeta}{\partial t},$$

$$\frac{\partial f}{\partial z} = \frac{\partial F}{\partial y} \left(1 - \frac{\partial \zeta}{\partial z}\right).$$

Consequently

$$\frac{df}{dt} = \frac{\partial F}{\partial y} + u \frac{\partial F}{\partial x}.$$  

In particular, if we let

$$\zeta(x, z, t) = \eta(x, y, t),$$

then

$$\zeta_z = \frac{\eta_y}{1 + \eta_y}, \quad \zeta_x = \frac{\eta_x}{1 + \eta_y}, \quad \zeta_t = \frac{\eta_t}{1 + \eta_y}.$$  

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where the indices denote partial derivatives. Then, using (3.4), (3.5), (3.6), (3.7), we can rewrite equations (2.11), (2.12), (2.4), so that, on omitting the formal distinction between $f$ and $F$ we get,

$$
\rho_0(y) \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} \right) + \frac{\partial q}{\partial x} - \frac{1}{1 + \frac{\partial \eta}{\partial y}} \frac{\partial q}{\partial \eta} = 0 \quad (3.8)
$$

$$
\rho_0(y) \left( \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} \right) + \frac{1}{1 + \frac{\partial \eta}{\partial y}} \frac{\partial q}{\partial \eta} + \frac{1}{\sigma} [\rho_0(y) - \rho_0(y + \eta)] = 0 \quad (3.9)
$$

$$
\frac{\partial u}{\partial x} + \frac{\partial w}{\partial y} - \frac{1}{1 + \frac{\partial \eta}{\partial y}} \left( \frac{\partial u}{\partial y} \frac{\partial \eta}{\partial x} + \frac{\partial w}{\partial y} \frac{\partial \eta}{\partial y} \right) = 0. \quad (3.10)
$$

Also, the kinematic condition (2.7) becomes,

$$
w = \frac{\partial \eta}{\partial t} + u \frac{\partial \eta}{\partial x} \quad (3.11)
$$

In these new coordinates the boundary conditions (2.5), (2.6) become respectively,

$$
\eta = 0 \quad \text{at} \quad y = -h, \quad (3.12)
$$

$$
\int_{0}^{\eta} \rho_0(y') dy' = \sigma Q \quad \text{at} \quad y = 0. \quad (3.13)
$$

The set of equations (3.8), (3.9), (3.10) with kinematic condition (3.11) can be reduced to two equations for $u(x, y, t)$, $\eta(x, y, t)$. The first is

$$
\frac{\partial}{\partial y} \left\{ \rho_0(y) \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} \right) \right\} - \rho_0(y) N^2(y) \frac{\partial \eta}{\partial x} - \frac{\partial}{\partial x} \left\{ \rho_0(y) \left( \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} \right)^2 \eta \right\} \left( 1 + \frac{\partial \eta}{\partial y} \right) + \frac{\partial}{\partial y} \left\{ \rho_0(y) \left( \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} \right)^2 \eta \right\} \frac{\partial \eta}{\partial x} = 0, \quad (3.14)
$$

and the second is,

$$
\frac{\partial u}{\partial x} (1 + \eta_y) + \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial x} \eta \right) \eta_y = 0, \quad (3.15)
$$
where
\[ N^2(y) = -\frac{1}{\sigma \rho_0(y)} \frac{d\rho_0}{dy}, \]  
(3.16)

The boundary conditions for these two equations are
\[ \eta = 0 \quad \text{at} \quad y = -h \]  
(3.17)
\[ \frac{\partial \eta}{\partial x} = -\sigma \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} \right) - \sigma \frac{\partial \eta}{\partial x} \left( \frac{\partial \eta}{\partial t} + u \frac{\partial \eta}{\partial x} \right)^2 \eta \quad \text{at} \quad y = 0 \]  
(3.18)

Thus, the basic equations in this semi-Lagrangian formulation are equations (3.14), (3.15) with boundary conditions at the surface and bottom (3.17), (3.18). We will use these to describe internal waves. It is interesting to note that this semi-Lagrangian approach leads to an increase in the order of nonlinearity to four, whereas the original governing equations had only second order nonlinearity.

4 Derivation of the nonlinear evolution equation

We shall suppose that the waves are long, their amplitude is small, but finite, and that the basic horizontal shear flow is stable. We introduce the small parameter \( \epsilon \) to describe long waves, and hence define the slow variables,
\[ X = \epsilon x, \quad T = \epsilon t, \]  
(4.1)

Then we decompose the horizontal velocity field into the basic component and a perturbation,
\[ u(x, y, t) = U(y) + u'(x, y, t) \]  
(4.2)

Equations (3.14) and (3.15) then transform to
\[
\frac{\partial}{\partial y} \left\{ \rho_0(y) \left( \frac{\partial u'}{\partial T} + (U(y) + u') \frac{\partial u'}{\partial X} \right) \right\} - \rho_0(y) N^2(y) \frac{\partial \eta}{\partial X} - \\
- \epsilon^2 \frac{\partial}{\partial X} \left\{ \rho_0(y) \left( \frac{\partial}{\partial T} + (U(y) + u') \frac{\partial}{\partial X} \right)^2 \eta \right\} \left( 1 + \frac{\partial \eta}{\partial y} \right) + \\
+ \epsilon^2 \frac{\partial}{\partial y} \left\{ \rho_0(y) \left( \frac{\partial}{\partial T} + (U(y) + u') \frac{\partial}{\partial X} \right)^2 \eta \right\} \frac{\partial \eta}{\partial X} = 0, 
\]  
(4.3)
\[
\frac{\partial u'}{\partial X} + \frac{\partial^2 \eta}{\partial T \partial y} + (U(y) + u') \frac{\partial^2 \eta}{\partial X \partial y} + \frac{\partial u'}{\partial X} \frac{\partial \eta}{\partial y} = 0. \tag{4.4}
\]

The boundary condition at the surface becomes,
\[
\frac{\partial \eta}{\partial X} = -\sigma \left( \frac{\partial u'}{\partial T} + (U(y) + u') \frac{\partial u'}{\partial X} \right) -
\]
\[
-\sigma^2 \frac{\partial \eta}{\partial X} \left( \frac{\partial}{\partial T} + (U(y) + u') \frac{\partial}{\partial X} \right)^2 \eta \quad \text{at } y = 0, \tag{4.5}
\]

while the bottom boundary condition (3.17) remains unchanged. We let the nonlinear parameter be $\mu$, and anticipate the KdV scaling $\mu = c^2$. If $c$ is the speed of a linear long wave (yet to be determined) we introduce the new variables,
\[
\xi = X - cT, \quad \text{and} \quad \tau_1 = \mu T, \quad \tau_2 = \mu^2 T, \quad \ldots . \tag{4.6}
\]

It follows that
\[
\frac{\partial}{\partial T} = -c \frac{\partial}{\partial \xi} + \mu \frac{\partial}{\partial \tau_1} + \mu^2 \frac{\partial}{\partial \tau_2} + \ldots , \quad \frac{\partial}{\partial X} = \frac{\partial}{\partial \xi}. \tag{4.7}
\]

Then on substituting (4.7) into (4.3), (4.4), and (4.5) we find that
\[
\frac{\partial}{\partial y} \left\{ \rho_0 (U - c) \frac{\partial u'}{\partial \xi} \right\} - \rho_0 N^2 \frac{\partial \eta}{\partial \xi} = F, \tag{4.8}
\]
\[
\frac{\partial u'}{\partial \xi} + (U - c) \frac{\partial^2 \eta}{\partial \xi \partial y} = G, \tag{4.9}
\]
\[
\frac{\partial \eta}{\partial \xi} + \sigma (U - c) \frac{\partial u'}{\partial \xi} = -\sigma \left( \mu \frac{\partial u'}{\partial \tau_1} + \mu^2 \frac{\partial u'}{\partial \tau_2} + u' \frac{\partial u'}{\partial \xi} + \frac{\partial \eta}{\partial \xi} H \right) \quad \text{at } y = 0, \tag{4.10}
\]

where \( F = -\frac{\partial}{\partial y} \left\{ \rho_0 \left( \mu \frac{\partial u'}{\partial \tau_1} + \mu^2 \frac{\partial u'}{\partial \tau_2} + u' \frac{\partial u'}{\partial \xi} \right) \right\} + \)
\[
+ \mu \left( 1 + \frac{\partial \eta}{\partial y} \right) \frac{\partial}{\partial \xi} (\rho_0 H) - \mu \frac{\partial \eta}{\partial \xi} \frac{\partial}{\partial y} (\rho_0 H), \tag{4.11}
\]
\[
G = -\mu \frac{\partial^2 \eta}{\partial y \partial \tau_1} - \mu^2 \frac{\partial^2 \eta}{\partial y \partial \tau_2} - \frac{\partial}{\partial \xi} \left( u' \frac{\partial \eta}{\partial y} \right), \tag{4.12}
\]
\[
H = \left( (U - c) \frac{\partial}{\partial \xi} + \mu \frac{\partial}{\partial \tau_1} + \mu^2 \frac{\partial}{\partial \tau_2} + u' \frac{\partial}{\partial \xi} \right)^2 \eta. \tag{4.13}
\]
Here the left-hand side of these equations, when equated to zero, describe the linear long-wave theory, and thus form the basis of our asymptotic expansion. Equations (4.8), (4.9) can be reduced to one equation containing $\eta$ only,

$$\frac{\partial}{\partial y} \left\{ \rho_0 (U - c)^2 \frac{\partial \eta}{\partial \xi} \right\} + \rho_0 \lambda^2 \frac{\partial \eta}{\partial \xi} = M, \quad (4.14)$$

where $M = \frac{\partial}{\partial y} \left\{ \rho_0 (U - c)G \right\} - F \quad (4.15)$

with the boundary conditions,

$$\frac{\partial \eta}{\partial \xi} = \sigma (U - c)^2 \frac{\partial^2 \eta}{\partial \xi \partial y} - \sigma (U - c)G + \sigma H_1 \quad \text{at} \; y = 0, \quad (4.16)$$

where $H_1 = - \left( \mu \frac{\partial u'}{\partial \tau_1} + \mu^2 \frac{\partial u'}{\partial \tau_2} + u' \frac{\partial u'}{\partial \xi} + \mu \frac{\partial \eta}{\partial \xi} H \right). \quad (4.17)$

Again (3.17) holds at the bottom.

Next we assume that our internal wave field (i.e. the vertical displacement and the horizontal component of velocity) has the asymptotic expansion,

$$\eta(\xi, y, \tau) = \mu A(\xi, \tau) \Phi(y) + \mu^2 \eta_1(\xi, y, \tau) + \mu^3 \eta_2(\xi, y, \tau) + \ldots, \quad (4.18)$$

$$u'(\xi, y, \tau) = \mu u_0(\xi, y, \tau) + \mu^2 u_1(\xi, y, \tau) + \mu^3 u_2(\xi, y, \tau) + \ldots \quad (4.19)$$

After substitution of (4.18), (4.19) into equation (4.14) and the boundary conditions (3.17), (4.16) and collecting terms of the same order in $\mu$, we obtain at the lowest order the equation determining the modal function $\Phi(y)$,

$$\frac{d}{dy} \left[ \rho_0 (c - U)^2 \frac{d\Phi}{dy} \right] + \rho_0 \lambda^2 \Phi = 0 \quad (4.20)$$

$$\Phi = 0 \quad \text{at} \; y = -h, \quad (4.21)$$

$$\Phi = \sigma (c - U)^2 \frac{d\Phi}{dy} \quad \text{at} \; y = 0. \quad (4.22)$$

It is well known that this eigenvalue problem (4.20), (4.21) and (4.22) has, in general, an infinite sequence of modes $\Phi_n^\pm$ with corresponding speeds $c_n^\pm$, for $n = 0, 1, 2, \ldots$. Here we consider only stable waves, so that $c_n^+ > U_M = \max U(y)$ and $c_n^- < U_m = \min U(y)$. Note that a sufficient condition to exclude any unstable waves (where $u_m < Re(c) < U_M$) is that the Richardson number $Ri = N^2 / U_M^2 > 1 / 4$. The theory we shall develop is valid for any of these modes, but usually it is relevant to consider only the first internal mode, which has the greatest phase speed of all the internal modes and has
just a single extremum for the modal function in the interior of the fluid. Also it is important to note, that each mode is determined only to within a multiplicative constant. We choose this constant in such a way that modal function is normalized at it’s extreme value, i.e.,

\[ \Phi_{\text{max}} = 1. \]  

(4.23)

This choice of modal function has an obvious physical sense, in that at the first order of approximation, the function \( A(\xi, \tau) \) is the displacement of the isopycnal surface at the point where \( \Phi = \Phi_{\text{max}} \), denoted by \( y = y_{\text{max}} \).

From equation (4.9), substituting (4.18), (4.19), at the lowest order we find

\[ u_0(\xi, y, \tau) = -A(\xi, \tau)(U - c) \frac{d\Phi}{dy}. \]  

(4.24)

Equation (4.14) can be written in operator form

\[ L \frac{\partial \eta}{\partial \xi} = M. \]  

(4.25)

Here \( L \) is the linear operator,

\[ L = \frac{\partial}{\partial y} \left[ \rho_0 (c - U)^2 \frac{\partial}{\partial y} \right] + \rho_0 N^2 \]

The compatibility condition, that is, the condition for solvability of the inhomogeneous problem (4.25) with the boundary conditions (3.17), (4.16), is

\[ \int_{-h}^{0} M \Phi dy = \sigma \left[ \rho_0 (U - c)^2 \frac{d\Phi}{dy} \right] (U - c)G - H_1 \bigg|_{y=0}, \]  

(4.26)

or, using (4.15):

\[ \int_{-h}^{0} F \Phi dy + \int_{-h}^{0} \rho_0 (U - c)G \frac{d\Phi}{dy} dy - \sigma \left[ \rho_0 (U - c)^2 H_1 \frac{d\Phi}{dy} \right] y=0 = 0. \]  

(4.27)

It is useful to note, that

\[ F = \frac{\partial F_1}{\partial y} + \frac{\partial F_2}{\partial \xi}, \]  

(4.28)

where, from (4.11), (4.17):

\[ F_1 = \rho_0 H_1, \]  

(4.29)

\[ F_2 = \mu \rho_0 \left( 1 + \frac{\partial \eta}{\partial y} \right) H. \]  

(4.30)
Hence the compatibility condition (4.27) now becomes:

\[
\int_{-h}^{0} \frac{\partial F_2}{\partial \xi} \Phi dy - \int_{-h}^{0} F_1 \frac{d\Phi}{dy} dy + \int_{-h}^{0} \mu_0 (U - c) G \frac{d\Phi}{dy} dy = 0, \tag{4.31}
\]

and so it does not contain the boundary terms.

We’ll consider the expression (4.31) at each order of \(\mu\). At \(O(\mu^2)\) we obtain:

\[
\frac{\partial A}{\partial r_1} + \beta \frac{\partial^3 A}{\partial \xi^3} + \alpha A \frac{\partial A}{\partial \xi} = 0, \tag{4.32}
\]

where

\[
\alpha = \frac{3}{2} \frac{\int_{-h}^{0} \rho_0 (c - U)^2 \left( \frac{d\Phi}{dy} \right)^3 dy}{\int_{-h}^{0} \rho_0 (c - U) \left( \frac{d\Phi}{dy} \right)^2 dy}, \tag{4.33}
\]

\[
\beta = \frac{1}{2} \frac{\int_{-h}^{0} \rho_0 (c - U)^2 \Phi^2 dy}{\int_{-h}^{0} \rho_0 (c - U) \left( \frac{d\Phi}{dy} \right)^2 dy}. \tag{4.34}
\]

Equation (4.32) is the well-known KdV equation for internal solitary waves. Note that (4.32) is expressed in a moving coordinate system, and when (4.7) is used we get the same equation but in the fixed coordinate system,

\[
\frac{\partial A}{\partial t} + c \frac{\partial A}{\partial x} + \mu \left( \beta \frac{\partial^3 A}{\partial x^3} + \alpha A \frac{\partial A}{\partial x} \right) = 0. \tag{4.35}
\]

The coefficients \(\alpha\) and \(\beta\) of this KdV equation were first produced by Benney (1966).

Equation (4.14) at \(O(\mu^2)\) is,

\[
L \frac{\partial \eta_1}{\partial \xi} = \frac{\partial^3 A}{\partial \xi^3} \left[ -2 \beta \frac{\partial}{\partial y} \left\{ \rho_0 (c - U) \frac{d\Phi}{dy} \right\} - \rho_0 (c - U)^2 \Phi \right] + \frac{\partial A}{\partial \xi} \left[ -2 \alpha \frac{\partial}{\partial y} \left\{ \rho_0 (c - U) \frac{d\Phi}{dy} \right\} + 3 \frac{\partial}{\partial y} \left\{ \rho_0 (c - U)^2 \left( \frac{d\Phi}{dy} \right)^2 \right\} \right]. \tag{4.36}
\]

Therefore

\[
\eta_1 = \frac{\partial^2 A}{\partial \xi^2} T_d(y) + A^2 T_n(y), \tag{4.37}
\]

where \(T_n(y)\) is the first nonlinear correction to the modal structure of internal wave; it is solution of

\[
LT_n = -\alpha \frac{d}{dy} \left\{ \rho_0 (c - U) \frac{d\Phi}{dy} \right\} + \frac{3}{2} \frac{d}{dy} \left\{ \rho_0 (c - U)^2 \left( \frac{d\Phi}{dy} \right)^2 \right\} \tag{4.38}
\]
with boundary conditions
\[ T_n = 0 \quad \text{at } y = -h, \]
\[ T_n = \sigma \left[ (c - U)^2 \frac{dT_n}{dy} + \alpha (c - U) \frac{d\Phi}{dy} - \frac{3}{2} (c - U)^2 \left( \frac{d\Phi}{dy} \right)^2 \right] \quad \text{at } y = 0, \]
\[ (4.39) \]
while \( T_d(y) \) is the first dispersion correction to the modal structure of internal wave; it is solution of
\[ LT_d = -2 \beta \frac{d}{dy} \left\{ \rho_0 (c - U) \frac{d\Phi}{dy} \right\} - \rho_0 (c - U)^2 \Phi \]
\[ (4.40) \]
with boundary conditions
\[ T_d = 0 \quad \text{at } y = -h, \]
\[ T_d = \sigma \left[ (c - U)^2 \frac{dT_d}{dy} + 2 \beta (c - U) \frac{d\Phi}{dy} \right] \quad \text{at } y = 0. \]
\[ (4.41) \]
It is important to note that solutions of the boundary-value problems (4.38), (4.39) and (4.40), (4.41) are unique only up to additive multiples of \( \Phi \). This problem was discussed in Lamb & Yan (1996), Lamb (1999) and Holloway et al. (1999). It is convenient to let \( A(\xi), \tau - 1 \) represent the isopycnal displacement at the level \( y_{\text{max}} \) where there is a maximum in the linear mode \( \Phi(y) \). Hence we choose the auxiliary conditions
\[ T_n(y_{\text{max}}) = 0, \quad T_d(y_{\text{max}}) = 0 \]
\[ (4.42) \]
In this case the series (4.18), using (4.37) and (4.23), at the point \( y_{\text{max}} \) is
\[ \eta(\xi, y_{\text{max}}, \tau) = \mu A(\xi, \tau) + O(\mu^3). \]
\[ (4.43) \]
Also, equation (4.9) at \( O(\mu^1) \) and (4.37) give
\[ u_1 = \frac{\partial^2 A}{\partial \xi^2} \left[ \beta \frac{d\Phi}{dy} - (U - c) \frac{dT_d}{dy} \right] + \]
\[ + A^2 \left[ \frac{1}{2} \alpha \frac{d\Phi}{dy} + (U - c)^2 \left( \frac{d\Phi}{dy} \right)^2 - (U - c) \frac{dT_n}{dy} \right]. \]
\[ (4.44) \]
The compatibility condition (4.31) at \( O(\mu^3) \) then generates the equation,
\[ \frac{\partial A}{\partial \tau_2} + \beta \frac{\partial^3 A}{\partial \xi^3} + \alpha_1 A^2 \frac{\partial A}{\partial \xi} + \gamma_1 A \frac{\partial^3 A}{\partial \xi^3} + \gamma_2 \frac{\partial A}{\partial \xi} \frac{\partial^2 A}{\partial \xi^2} = 0, \]
\[ (4.45) \]
where

\[
\beta_1 = \frac{1}{I} \int_{-h}^{0} \rho \, dy \left\{ (c - U)^2 \Phi T_d - \beta^2 (d\Phi/dy)^2 + 
+ 2\beta (c - U) \left[ \Phi^2 - (d\Phi/dy) (dT_d/dy) \right] \right\},
\] (4.46)

\[
\alpha_1 = \frac{1}{I} \int_{-h}^{0} \rho \, dy \left\{ 3(c - U)^2 [3(dT_n/dy) - 
- 2(d\Phi/dy)^2] (d\Phi/dy)^2 - \alpha^2 (d\Phi/dy)^2 + 
+ \alpha (c - U) \left[ 5(d\Phi/dy)^2 - 4(dT_n/dy) (d\Phi/dy) \right] \right\},
\] (4.47)

\[
\gamma_1 = \frac{1}{I} \int_{-h}^{0} \rho \, dy \left\{ 2\alpha \beta (d\Phi/dy)^2 - 2\alpha (c - U) \Phi^2 + 
+ (c - U)^2 \Phi^2 (d\Phi/dy) - (c - U)^2 [2T_n \Phi + 
+ 3(dT_d/dy) (d\Phi/dy)^2] + 2(c - U) [\alpha (dT_d/dy) + 
+ 2\beta (dT_n/dy)] (d\Phi/dy) - 4\beta (c - U) (d\Phi/dy)^3 \right\},
\] (4.48)

\[
\gamma_2 = \frac{1}{I} \int_{-h}^{0} \rho \, dy \left\{ (c - U) [2\beta (d\Phi/dy)^3 + 6\alpha \Phi^2] - 
- 3\alpha \beta (d\Phi/dy)^2 - 2(c - U)^2 [\Phi^2 (d\Phi/dy) - 3T_n \Phi] - 
- 6\alpha (c - U) (dT_d/dy) (d\Phi/dy) + 
+ 3(c - U)^2 (dT_d/dy) (d\Phi/dy)^2 \right\},
\] (4.49)

\[
I = 2 \int_{-h}^{0} \rho \, (c - U) \, (d\Phi/dy)^2 \, dy.
\] (4.50)

Then, on again using (4.7), and the KdV equation (4.32), and neglecting terms of \( O(\mu^3) \) we obtain the second-order KdV equation, or extended KdV
equation,
\[
\frac{\partial A}{\partial T} + c \frac{\partial A}{\partial X} + \mu \left( \beta \frac{\partial^3 A}{\partial X^3} + \alpha A \frac{\partial A}{\partial X} \right) + \\
+ \mu^2 \left( \beta_1 \frac{\partial^5 A}{\partial X^5} + \alpha_1 A^2 \frac{\partial A}{\partial X} + \gamma_1 \frac{\partial^3 A}{\partial X^3} + \gamma_2 \frac{\partial A}{\partial X} \frac{\partial^2 A}{\partial X^2} \right) = 0. 
\]

(4.51)

This equation was produced by Koop & Butler (1981) for a two-layer system, and then by Lamb & Yan (1996) for a continuous density stratification (but in the Boussinesq approximation and with no free surface), and without a basic shear flow. Recently this last result was extended to include a basic shear flow (but again in the Boussinesq approximation and with no free surface) by Pelinovsky, Poloukhina, Lamb (2000). It is important to note that our derivation is completely general, and in particular, also includes the case of a surface wave, in which case one should choose \( y_{\text{max}} = 0 \). Further, the procedure described here allows us, in principle, to obtain the extension of the Korteweg-de Vries equation to any order. However, as shown by Prasad and Akylas (1997) for the case when the upper boundary is rigid and there is no basic shear flow, we would expect the generation of both upstream and downstream shelves at \( O(\mu^4) \), which are associated with the necessity for the total asymptotic expansion to conserve mass.

5 Interfacial waves in a two-layer shear flow

Consider a two-layer system, bounded below by a rigid flat bottom and above by a free surface. The lower and upper layer densities are \( \rho_1 \) and \( \rho_2 \) (\( \rho_1 > \rho_2 \)), and the corresponding layer depths are \( h \) and \( H - h \), i.e. \( H \) is the undisturbed fluid depth; note the change of notation from the general theory of the preceding sections. This case was analyzed by Koop & Butler (1981) without any shear flow. Here we include a shear flow by including a constant velocity \( U_0 \) in the upper layer.

The parameter \( \sigma \) (Boussinesq parameter) for this two-layer case is
\[
\sigma = \frac{2\rho_1 - \rho_2}{\rho_1 + \rho_2}. 
\]

(5.1)

The vertical structure of the modal function corresponding to the internal mode can be found from the eigenvalue problem (4.20), (4.21) and (4.22),
and has the form

\[ \Phi(y) = \begin{cases} 
  y/h & , \ 0 \leq y < h, \\
  (y + m - h)/m & , \ h \leq y \leq H, 
\end{cases} \]  

(5.2)

where \( m = \sigma(c - U_0)^2 + h - H. \) \( \) (5.3)

Note that the fluid domain is now given by \( 0 < y < H, \) but all the formulae of the preceding sections are readily altered accordingly. The linear long-wave phase speed \( c \) is a solution of

\[ -c^2(c - U_0)^2\sigma(2 + \sigma) + c^2(2 + \sigma)(H - h) + 
\]  

\[ +(c - U_0)^2h(2 + \sigma) - 2h(H - h) = 0. \]  

(5.4)

The phase speed can be expressed in semi-explicit form

\[ c^2 = \frac{(2 + \sigma)(H - h + (1 - u)^2h) - \sqrt{D}}{2(1 - u)^2\sigma(2 + \sigma)}, \]  

(5.5)

where \( D = (2+\sigma)\left[(2 + \sigma)(H - h + (1 - u)^2h)^2 - 8(1 - u)^2\sigma h(H - h)\right], \) and \( u = U_0/c \) is the relative shear flow velocity. An increase \( \sigma \) causes a decrease in the phase speed \( c. \)

From (4.38), (4.39), (4.40), (4.41), (4.42) we can find the nonlinear and dispersion corrections to the modal function (5.2). The nonlinear correction is given by,

\[ T_n(y) = \begin{cases} 
  0 & , \ 0 \leq y < h \\
  a_1(h - y) & , \ h \leq y \leq H 
\end{cases} \]  

(5.6)

where \( a_1 = \frac{m + H - h}{m^2(c - U_0)} \left( \frac{\alpha - 3(c - U_0)}{2} \right), \)

while the dispersion correction has the form,

\[ T_d(y) = \begin{cases} 
  -\frac{y^3}{6h} + \frac{hy}{6} & , \ 0 \leq y < h \\
  \frac{(y - h + m)^3}{6m} + a_2(y - h) + \frac{m^2}{6} & , \ h \leq y \leq H 
\end{cases} \]  

(5.7)

where \( a_2 = -\frac{2\beta(m + H - h)}{(c - U_0)m^2} + \frac{(m + H - h)^3}{3m^2} + \frac{m}{6} \)}
Next, from (4.33), (4.34), (4.46), (4.47), (4.48), (4.49) we can evaluate the coefficients of the extended Korteweg - de Vries equation (4.51) written here for the interfacial vertical displacement. Formulas for these quantities are complicated and hence are given in the Appendix.

In figures 1-3 we display graphs of the non-dimensionalised quantities $\alpha$, $\beta$, $\alpha_1$, $\beta_1$, $\gamma_1$, $\gamma_2$ as functions of $l = h/H$ for different values of the relative shear flow velocity $u = U_0/c$, and the relative density $r = \rho_1/\rho_2$. Note that as $l$ and $r$ vary, so does $c$ and hence so does $U_0$ when we keep $u$ fixed.

First we note that nonlinear coefficients $\alpha$ and $\alpha_1$ have infinite values when $l = 0$, $l = 1$, i.e. when the thickness of either the lower layer, or that of the upper layer, tends to zero. The coefficient $\gamma_2$ of the nonlinear dispersion term also has an infinite value at $l = 1$, when $r > 1$. Of course, our results are not valid in the vicinity of the such points.

The coefficient $\alpha$ of the quadratic nonlinear term is positive when the pycnocline near the bottom and negative when pycnocline is near to the surface. For $u = 0$ and $\sigma \rightarrow 0$ $\alpha > 0(< 0)$ according as $h < (>)H/2$. The effect of increasing the shear flow velocity is to put nearer to surface the location where $\alpha$ is equal to zero. Varying the relative density $r$ has a very weak effect in the region of positive $\alpha$, but its effect is more noticeable in the region where $\alpha$ is negative; increasing $r$ causes an increase in the absolute value of $\alpha$.

The coefficient $\alpha_1$ of the cubic nonlinear term is negative for any layer depth ratio, and takes smaller (in modulus) values as the relative shear flow velocity $u$ increases. When the lower layer is thin enough, the density ratio $r$ has almost no influence on the cubic nonlinear coefficient, but when the thickness of the lower layer is not so small, we get larger absolute values of $\alpha_1$ for larger values of $r$.

The coefficient $\beta$ of the first-order dispersion term is zero at $l = 0$ and $l = 1$ and positive for any other layer depth ratio, and hence it has a maximum value whose magnitude and location depend on the values of the parameters $u$ and $r$. When there is no shear flow ($u = 0$), $\beta(1)$ is symmetric function, and its maximum is at $l = 1/2$ (i.e equal layer depths), while the effect of increasing $r$ is to decrease the maximum of $\beta$. The presence of a shear flow destroys the symmetry of $\beta(l)$, and the position of its maximum moves to larger l with an increase in $u$.

The coefficient $\beta_1$ of the second order dispersion term has a behavior similar to that of $\beta$ for $r$ close to 1. However, when the difference in layer densities is significant, $\beta_1$ can take negative values if the upper layer is thinner than the lower. The qualitative behavior of the coefficients $\gamma_{1,2}$ of the nonlinear dispersion terms is similar to that of $\alpha$, except when the pycnocline is near to the bottom (when $\gamma_1$ and $\gamma_2$ take finite positive values), or when
the pycnocline is near to surface (when $\gamma_1$ has finite negative values, and $\gamma_2$ has infinite positive values for $r > 1$ and finite negative values for $r = 1$).

In the case of the Boussinesq approximation, when the density jump is small (i.e $\sigma \approx 0$, or $\rho_1 \approx \rho_2$), so that the influence of the free surface is then also small and upper boundary is effectively rigid, all coefficients coincide with those calculated by Pelinovsky, Poloukhina, Lamb (2000).

6 Conclusion

We have presented an evolution equation, the extended Korteweg de-Vries equation, to describe solitary waves in an arbitrary density- and current-stratified flow, without using the Boussinesq approximation and with a free surface, valid to the second order of perturbation theory. All the coefficients of this equation are given explicitly as integrals of the modal function and its nonlinear and dispersion corrections.

The special case of a two-layer fluid with a shear flow due to a constant velocity in the upper layer is discussed in detail. For this special case the coefficients are obtained explicitly form as functions of the parameters of the model (layer depths, density ratio and the relative shear flow velocity), and are analysed as functions of these parameters. It is shown that the influence of the density ratio, and the shear flow velocity, is significant for all the coefficients, and should not be neglected.

Next, we must point out again that the higher-order Korteweg-de Vries equation (4.51) is, strictly speaking, an asymptotic result valid when $\mu$ is sufficiently small, and is most likely to be useful when the coefficient $\alpha$ of the quadratic nonlinear term is small (e.g. $O(\epsilon)$ where we recall that $\mu = \epsilon^2$). However, because observed internal solitary waves are often quite large, we suggest that in practice it may be useful to use (4.51) as the model equation even when $\mu$ is not small.

In conclusion we note that, although the higher-order Korteweg-de Vries equation (4.51), and the expressions defining the associated nonlinear and dispersion corrections (4.38,4.40) to the modal functions, are all uniquely defined with our choice of normalisation (see (4.23, 4.42), a different choice for the normalisation would produce a different set of coefficients. In particular, we recall that the nonlinear and dispersion corrections to the modal function, $T_n(y)$ and $T_d(y)$ respectively, are only defined by (4.38) and (4.40) to within an arbitrary multiple of the modal function $\Phi(y)$ itself; it is only the normalisation conditions (4.42) which then uniquely determine them. Omitting these normalisation conditions has the effect of allowing us to make a
near-identity transformation

\[ B = A + \mu \left\{ \frac{1}{2} a A^2 + b A_{XX} \right\}, \quad (6.1) \]

where a specific choice of the coefficients \(a, b\) represents a specific normalisation of \(T_n(y), T_d(y)\) respectively. It can now be readily verified that substitution of (6.1) into (4.51) allows us to generate asymptotically a higher-order Korteweg-de Vries equation for \(B\) of the same form as (4.51), but with the coefficients \(\alpha_1, \beta_1, \gamma_1\) and \(\gamma_2\) replaced by \(\beta_1, \alpha_1 - a \alpha / 2, \gamma_1\) and \(\gamma_2 - 3a \beta + 2b \alpha\) respectively. Note that the coefficients \(\alpha, \beta\) in the first-order Korteweg-de Vries equation are not changed, and nor are the coefficients \(\beta_1\) and \(\gamma_1\) of the higher-order dispersive term. Indeed it is easily seen from the integral expressions (4.46), and (4.48) for \(\beta_1\) and \(\gamma_1\) respectively, that adding a multiple of \(\phi\) to \(T_d\) and/or \(T_n\) leaves \(\beta_1\) and \(\gamma_1\) unchanged.

In particular, we can now choose \(a\) so that the higher-order Korteweg-de Vries equation for \(B\) is Hamiltonian. For (4.51) to be exactly Hamiltonian (as opposed to being only asymptotically Hamiltonian), it is necessary that \(\gamma_2 = 2\gamma_1\) which is generally not the case (see (4.48, 4.49). However the near-identity transformation (6.1) with the choice \(3a \beta = \gamma_2 = 2\gamma_1\) and \(b = 0\) will produce a Hamiltonian form. With \(\gamma_2 = 2\gamma_1\) the Hamiltonian form for (4.51) is,

\[ A_T = - \frac{\partial}{\partial X} \frac{\delta H}{\delta A}, \quad (6.2) \]

where the Hamiltonian \(H\) has the density

\[ \frac{1}{2} \epsilon A^2 + \mu \left( \frac{1}{2} \beta A_X^2 + \frac{1}{6} \alpha A^3 \right) + \mu^2 \left( \frac{1}{2} \beta_1 A_{XXX}^2 + \frac{1}{12} \alpha_1 A^4 - \frac{1}{2} \gamma_1 A A_X^2 \right). \quad (6.3) \]

When (4.51) is Hamiltonian then it conserves not only the mass (i.e. the integral of \(A\)), but also the momentum whose density is \(A^2\), as well as the Hamiltonian itself, whose density is \(H\). For numerical purposes it is perhaps desirable that the evolution equation (4.51) should be Hamiltonian, and hence the renormalisation implied by (6.1) is generally recommended.

Further, more is possible when the near-identity transformation (6.1) is enhanced to

\[ B = A + \mu \left\{ \frac{1}{2} a A^2 + b A_{XX} + a' A_X + a'' A_{XX} \int A \, dX + b' X A_T \right\}. \quad (6.4) \]

It can then be shown that, provided only that \(\alpha, \beta\) are not zero, it is possible to choose the available constants \(a, b, a', b'\) so that the resulting equation for \(B\) is just the Korteweg-de Vries equation (i.e. has the form (4.32), or alternatively, the coefficients \(\beta_1, \alpha_1, \gamma_1, \gamma_2\) in (4.51) are all zero) with an error
of $O(\mu^3)$ (see, e.g. Kodama (1985), Fokas & Liu (1996) and Fokas et al (1996)). Thus, in general the (4.32) equation is asymptotically reducible to the integrable Korteweg-de Vries equation. However, we hasten to point out that although this is an intriguing result, its use in practice in this context may well be very limited because the amplitude $A$ is not necessarily so small that the transformation (6.1) is applicable, and also there are circumstances when the coefficient $\alpha$ is zero, in which case this reduction is not possible. 

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Appendix. Coefficients of the extended Korteweg-de Vries equation for a two-layer shear flow

$$\alpha = \frac{3}{2m} \frac{\rho_1 c^2 m^3 + \rho_2 (c - U_0)^2 h^2 (H - h)}{\rho_1 cm^2 + \rho_2 (c - U_0) h (H - h)}, \quad (7.1)$$

$$\beta = \frac{h \rho_1 c^2 h m^2 + \rho_2 (c - U_0)^2 ((H - h + m)^3 - m^3)}{6 \rho_1 cm^2 + \rho_2 (c - U_0) h (H - h)}, \quad (7.2)$$

$$\alpha_1 I = \rho_1 \left\{ \frac{9 ac^2}{h} - \frac{6c^2}{h^3} - \frac{\alpha^2}{h} + \frac{5ac}{h^2} - 4\alpha c \right\} +$$

$$+ \rho_2 (H - h) \left\{ \frac{3(c - U_0)^2}{m^2} \left[ \frac{3ah}{m} - 3a_1 - \frac{2}{m^2} \right] - \frac{\alpha^2}{m^2} + \frac{\alpha(c - U_0)}{m} \left[ \frac{5}{m^2} - \frac{4ah}{m} + 4a_1 \right] \right\}, \quad (7.3)$$

$$\beta_1 I = \rho_1 \left\{ -\frac{c^2 h^3}{30} + \frac{2c^2 h^3}{18} + \frac{2\beta ch}{3} - \frac{\beta^2}{h} \right\} +$$

$$+ \rho_2 \left\{ -\frac{(c - U_0)^2}{30m^2} \left[ (m + H - h)^5 - m^5 \right] + \right.$$  

$$+ \frac{(c - U_0)}{3m} \left[ a_2 (c - U_0) + \frac{3\beta}{m} \right] \left[ (m + H - h)^3 - m^3 \right] - \frac{(c - U_0)^2}{2} \left[ -a_2 + \frac{m}{6} \right] \left[ (m + H - h)^2 - m^2 \right] -$$

$$- (H - h) \left[ \frac{\beta^2}{m^2} + 2\beta (c - U_0) a_2 \right] \right\}. \quad (7.4)$$
\[ \gamma_1 I = -\rho_1 \left\{ \frac{-2}{3} \alpha ch + \frac{1}{3} c^2 - 2 \alpha c^2 h^2 + \frac{2 \alpha \beta}{h} + 
\right. \\
\left. + 4 \beta ac - \frac{4 \beta c}{h^2} \right\} - \\
- \rho_2 \left\{ \frac{(c - U_0)}{3m} \left[ \frac{5(c - U_0)}{2m^2} - \frac{3\alpha}{m} \right] \\
- 2(c - U_0) \left( \frac{ah}{m} - a_1 \right) \left[ (m + H - h)^3 - m^3 \right] - \\
- (c - U_0)^2 a_1 \left[ (m + H - h)^2 - m^2 \right] + \\
+ \frac{(H - h)}{m} \left[ \frac{2\alpha \beta}{m} - 3(c - U_0)^2 \frac{a_2}{m} + 2\alpha (c - U_0)a_2 + 
\right. \\
\left. + 4 \beta (c - U_0) \left( \frac{ah}{m} - a_1 - \frac{1}{m^2} \right) \right] \left. \right\}, \quad (7.5) \\
\]

\[ \gamma_2 I = \rho_1 \left\{ \frac{2\beta c}{h^2} + 2\alpha ch + 2ac^2 h^2 - \frac{3\alpha \beta}{h} - \frac{2c^2}{3} \right\} + \\
+ \rho_2 \left\{ \frac{(c - U_0)}{m} \left[ \frac{3\alpha}{m} - \frac{7(c - U_0)}{6m^2} + 2(c - U_0) \left( \frac{ah}{m} - a_1 \right) \right] \\
\left[ (m + H - h)^3 - m^3 \right] + 3a_1 (c - U_0)^2 \left[ (m + H - h)^2 - m^2 \right] + \\
+ \frac{(H - h)}{m} \left[ \frac{2\beta (c - U_0)}{m^2} - \frac{3\alpha \beta}{m} - 6\alpha (c - U_0)a_2 + 
\right. \\
\left. + \frac{3(c - U_0)^2 a_2}{m} \right] \left. \right\}, \quad (7.6) \\
\]

where \[ I = 2\rho_1 cm^2 + \rho_2 (c - U_0)h(H - h), \quad a = a_1 \frac{H - h}{h}. \quad (7.7) \]
References


Figure captions

**Figure 1** Coefficients of the extended Korteweg-de Vries equation for a two-layer fluid with no current.

**Figure 2** Coefficients of the extended Korteweg-de Vries equation for a two-layer fluid with a current in upper layer \( u = \frac{U_0}{c} = 0.25 \).

**Figure 3** Coefficients of the extended Korteweg-de Vries equation for a two-layer fluid with a current in upper layer \( u = \frac{U_0}{c} = 0.5 \).
Figure 1: Coefficients of the extended Korteweg-de Vries equation for a two-layer fluid with no current.
Figure 2: Coefficients of the extended Korteweg - de Vries equation for a two-layer fluid with a current in upper layer, $(u = U_0/c = 0.25)$. 
Figure 3: Coefficients of the extended Korteweg-de Vries equation for a two-layer fluid with a current in upper layer ($u = U_0/c = 0.5$).