Decomposition and Reduction Methods for Analysing Systems of Systems

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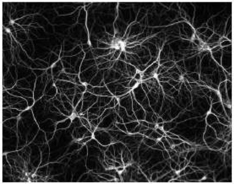
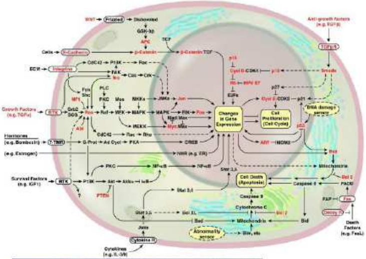
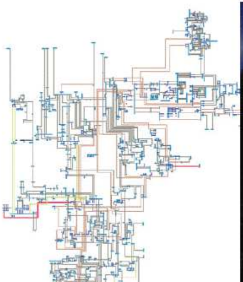
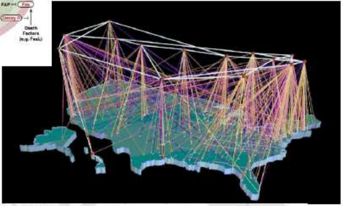
Motivation









A System of Systems

Many names for the same thing:

- Networked Systems
- Cyber Physical Systems
- Systems of Systems

These systems are typically characterised by

- Spatial structure (who interacts with who?)
- Dynamics of the subsystems

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Many possible objectives, some of the simplest (to state) include:

- Stability analysis
- Performance and *robustness* analysis
- Control, i.e. modify the behaviour/improve the performance
- Scale up

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- The dynamic models of such systems are *huge*!
- Cannot hope to design a centralised controller for the whole system

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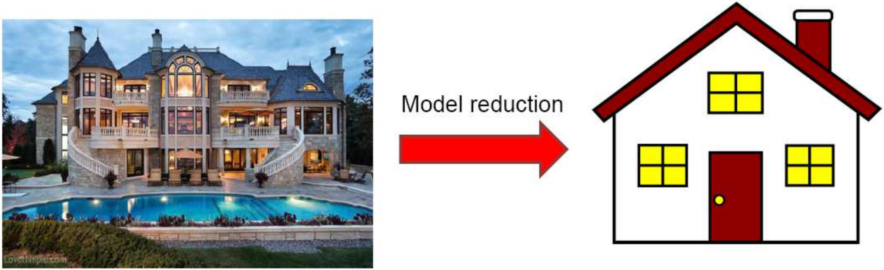
Model Reduction



Model Reduction

Assume that a model of the system is given, and that the model has n state variables. Fix a value $k < n$, find a model with k state variables that optimally approximates the original model.

Is it possible to preserve some properties of the original system?

Model Reduction

- Consider the nonlinear dynamical system

$$\begin{aligned} \dot{x} &= g(x, u) \\ y &= h(x) \end{aligned} \quad (1)$$

where $x \in \mathbb{R}^n$ is the state vector and $x_{ss} = 0$ is a stable steady state.

- The goal is to construct a system

$$\begin{aligned} \dot{\tilde{x}} &= \tilde{g}(\tilde{x}, u) \\ \tilde{y} &= \tilde{h}(\tilde{x}) \end{aligned}$$

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LTI System

Linearising (1) about x_{ss} with a constant input u_{ss} we obtain

$$A = \frac{\partial g(x,u)}{\partial x}\Big|_{x=x_{ss},u=u_{ss}}, \quad B = \frac{\partial g(x,u)}{\partial u}\Big|_{x=x_{ss},u=u_{ss}},$$

giving

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

$$G = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right].$$

$$G(s) = C(sI - A)^{-1}B$$

Measuring the size of a system

We will use the $\mathcal{H}_\infty$-norm of G as a measure of *size*.

$$\|G(s)\|_{\mathcal{H}_\infty} := \sup_{\omega \in \mathbb{R}^+} \bar{\sigma}[G(j\omega)]$$

$$\|G\|_{\mathcal{H}_\infty} := \sup_{u(\cdot)} \frac{\|y(t)\|_{\mathcal{L}_2}}{\|u(t)\|_{\mathcal{L}_2}}$$

This leads to an intuitive way to describe the error between two systems:

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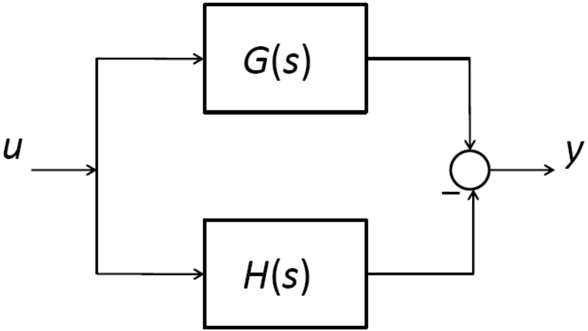
Block Diagram View



Block Diagram Interpretation

Truncation

Assume the system has been partitioned as follows

$$G = \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & 0 \end{array}\right], \quad x_1 \in \mathbb{R}^{n-k}, x_2 \in \mathbb{R}^{k}.$$

Then a simple reduced order model is:

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Balanced Realization

1. Assume have a stable LTI system:

$$\widehat{G} = \left[\begin{array}{c|c} \widehat{A} & \widehat{B} \\ \hline \widehat{C} & 0 \end{array}\right].$$

2. Compute the gramians, P and Q

$$\begin{aligned} \widehat{A}P + P\widehat{A}^T + \widehat{B}\widehat{B}^T &= 0, \quad P \succeq 0, \\ \widehat{A}^T Q + Q\widehat{A} + \widehat{C}^T\widehat{C} &= 0, \quad Q \succeq 0. \end{aligned}$$

3. Find R such that $P = R^T R$ and set $RQR^T = U\Sigma^2 U$.
4. Let $T^{-1} = R^T U \Sigma^{1/2}$, then

$$G = \left[\begin{array}{c|c} T\widehat{A}T^{-1} & T\widehat{B} \\ \hline \widehat{C}T^{-1} & 0 \end{array}\right]$$

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Balanced Realization

- The system G is balanced:

$$G = \left[\begin{array}{c|c} T\widehat{A}T^{-1} & T\widehat{B} \\ \hline \widehat{C}T^{-1} & 0 \end{array}\right] =: \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]$$

thus

$$\begin{aligned} A\Sigma + \Sigma A^T + BB^T &= 0, \\ A^T\Sigma + \Sigma A + C^T C &= 0, \end{aligned}$$

where

$$\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n),$$

and $\sigma_1 \geq \sigma_2 \geq \ldots, \sigma_n \geq 0$.

$$\begin{aligned} \Sigma_1 &= \mathrm{diag}(\sigma_1, \ldots, \sigma_k), \\ \Sigma_2 &= \mathrm{diag}(\sigma_{k+1}, \ldots, \sigma_n). \end{aligned}$$

Balanced Truncation

1. Partition G according to

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2. Then the k-dimensional reduced system is

$$G_r = \left[\begin{array}{c|c} A_{11} & B_1 \\ \hline C_1 & 0 \end{array}\right], \quad \|G - G_r\|_{\mathcal{H}_\infty} \leq 2 \sum_{i=k+1}^{n} \sigma_i.$$

3. The *projectors* that map $(A, B, C) \to (A_{11}, B_1, C_1)$ are

$$V = T^{-T}[\; I_k \quad 0_{k\times n-k} \;], \quad W = T[\; I_k \quad 0_{k\times n-k} \;]$$

which gives $(A_{11}, B_1, C_1) = (V^T AW, V^T B, CW)$.

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Structured Projection

- Constructing the balanced realization destroys the structure of G.
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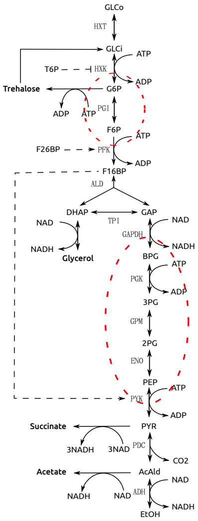
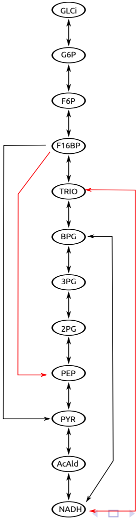
Yeast Glycolysis Pathway





Structured Projection

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(Note the switch in dimensions)

2. Define the *generalised gramians*

$$\mathcal{P} = \begin{bmatrix} \mathcal{P}_{11} & 0_{n-k,k} \\ 0_{k,n-k} & \mathcal{P}_{22} \end{bmatrix} \succeq 0, \quad \mathcal{Q} = \begin{bmatrix} \mathcal{Q}_{11} & 0_{n-k,k} \\ 0_{k,n-k} & \mathcal{Q}_{22} \end{bmatrix} \succeq 0,$$

which are obtained from the solutions of

$$\begin{aligned} A\mathcal{P} + \mathcal{P}A^T + BB^T &\preceq 0 \\ A^T\mathcal{Q} + \mathcal{Q}A + C^TC &\preceq 0 \end{aligned}$$

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Structured Projectors Contd.

1. Assume the states of x_2 are reduction candidates. Define

$$\mathcal{T} := \begin{bmatrix} I_{n-k} & 0_{n-k,k} \\ 0_{n-k,k} & \mathcal{T}_{22} \end{bmatrix},$$

where $\mathcal{T}_{22}$ satisfies

$$\mathcal{T}_{22}^{-1}\mathcal{P}_{22}\mathcal{T}_{22}^{-T} = \mathcal{T}_{22}^{T}\mathcal{Q}_{22}\mathcal{T}_{22} = \Sigma_{22},$$

where Σ_{22} is diagonal.

2. Assume we will truncate r states from x_2, then

$$\mathcal{W} = \begin{bmatrix} I_{n-k} & 0_{n-k,k-r} \\ 0_{k-r,n-k} & \mathcal{T}_{22}^{\diamond} \end{bmatrix}, \quad \mathcal{W}_r = \begin{bmatrix} 0_{n-k,r} \\ \mathcal{T}_{22}^{r} \end{bmatrix},$$

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Monotone systems

Definition 1 (Metzler Matrices)

A matrix $M \in \mathbb{R}^{n\times n} = \{m_{ij}\}$ is said to be Metzler if $m_{ij} \geq 0$ for all $i \neq j$.

Definition 2 (Monotone dynamical system)

Consider the system $\dot{x} = g(x)$, with g locally Lipschitz, $g : \mathbb{R}^n_{\geq 0} \to \mathbb{R}^n$ and $g(0) = 0$. The associated flow map is $\phi : \mathbb{R}_{\geq 0} \times \mathbb{R}^n_{\geq 0} \to \mathbb{R}^n$. The systems is said to be monotone (w.r.t $\mathbb{R}_{\geq 0}$) if $x \leq y \implies \phi(t,x) \leq \phi(t,y)$ for all $t \geq 0$.

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Linking Metzler and Monotone

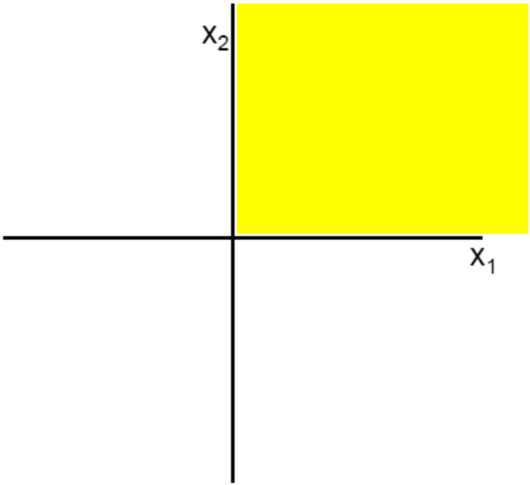
Proposition 1

A system $\dot{x} = g(x)$ is monotone with respect to the positive orthant if and only if

$$\frac{\partial(g_i(x))}{\partial x_j} \geq 0 \quad \forall i \neq j, \quad \forall x$$

Or simply put, the Jacobian of $g(x)$ is a Metzler matrix for all x in $\mathbb{R}^n_{\geq 0}$.

Monotone Systems



Existence of Structured Gramians

- Assume we have the system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \qquad (2)$$

1. A is Metzler and stable, B and C are *not* required to be positive
2. The system is partitioned as shown earlier

Lemma 1

Under the assumptions above, the generalised structured Gramians $\mathcal{P}$ and $\mathcal{Q}$ satisfying the Lyapunov inequalities always exist, thus the transformation matrix $\mathcal{T}$ exists.

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Reduction Algorithm

1. Solve the Lyapunov inequalities for $\mathcal{P}$ and $\mathcal{Q}$
2. Construct $\mathcal{T}_{22}$ using $\mathcal{P}_{22}$ and $\mathcal{Q}_{22}$
3. Set $r = k - 1$ and construct the projectors $\mathcal{V}$ and $\mathcal{W}$. Define $w = \mathcal{W}_{22}$, $v = \mathcal{V}_{22}$
4. Apply the projectors to get

$$A_t = \begin{bmatrix} A_{11} & A_{12}w \\ v^T A_{21} & v^T A_{22} w \end{bmatrix}, B_t = \begin{bmatrix} B_1 \\ v^T B_2 \end{bmatrix},$$

$$C_t^T = \begin{bmatrix} C_1^T \\ w^T C_2^T \end{bmatrix}.$$

Reduction Algorithm cont.

Lemma 2

Let $\mathcal{P}$ and $\mathcal{Q}$ be block-diagonal structured Gramians. Assume the matrix $\mathcal{P}_{22}\mathcal{Q}_{22}$ is irreducible. Let $\mathcal{T}_{22}$ be a transformation such that

$$\mathcal{T}_{22}^{-1}\mathcal{P}_{22}\mathcal{T}_{22}^{-T} = \mathcal{T}_{22}^{T}\mathcal{Q}_{22}\mathcal{T}_{22} = \Sigma,$$

with $\Sigma_{11} \geq \Sigma_{22} \geq \cdots \geq \Sigma_{kk}$. Let w and v be as defined previously. Then, there exists such a balancing transformation $\mathcal{T}_{22}$ such that:

1. *The vectors w and v are nonnegative.*
2. *A_t is Metzler.*
3. *Let $G_r = (A_t, B_t, C_t)$ Then $\|G - G_r\|_{\mathcal{H}_\infty} \leq 2\sum_{i=2}^{k}\Sigma_{ii}$.*

The Nonlinear System

- Using the projections we've computed, the nonlinear system

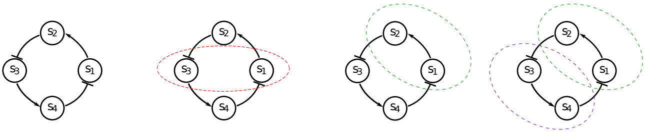
$$\begin{aligned} \dot{x} &= g(x, u) \\ y &= h(x) \end{aligned}$$

is transformed into

$$\begin{aligned} \dot{z}_m &= \mathcal{V}^T g(\mathcal{W} z_m, \mathcal{W}_r z_r, u) \\ \dot{z}_r &= \mathcal{V}_r^T g(\mathcal{W} z_m, \mathcal{W}_r z_r, u) = 0 \\ y_r^d &= \Omega C(\mathcal{W} z_m + \mathcal{W}_r z_r) \end{aligned}$$

via the state transformation $z = \mathcal{T} x$.

Toy Example



$$\dot{m}_i = \frac{c_{i1}}{1+p_j^2} - c_{i2} m_i + c_{i5} u_i, \quad i,j \in \{1,2\}, i \neq j$$

$$\dot{p}_i = c_{i3} m_i - c_{i4} p_i$$

$x = [p_1, m_1, p_2, m_2]^T$, monotone w.r.t diag$(1,1,-1,-1)\mathbb{R}^4_{\geq 0}$.

Toy Example

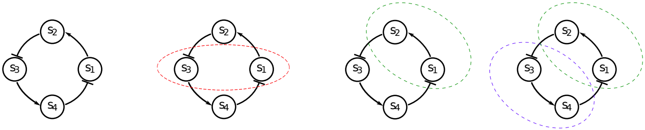
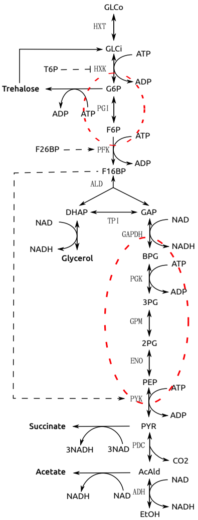
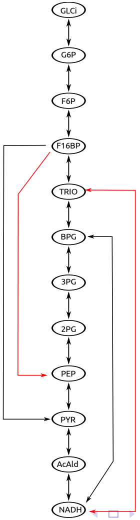


Table : The error in the macroscopic concentrations.

Method \ Error	L_1	L_2	L_∞
QSSA	67.3	11.9	3.2
Left Configuration	61.0	8.1	2.2
Middle Configuration	1.9	0.59	1.1
Right Configuration	13.8	2.3	0.79

Yeast Glycolysis Model





Yeast Glycolysis Model

QSSA

States \ Error	L_1	L_2	L_∞	$t(s)$
F6P, 2PG, PEP	1.21	0.75	0.98	163
G6P, F6P, 3PG, 2PG, PEP	2.05	1.16	1.59	214

REDUCTION BY $\{k_1, k_2\}$ STATES IN EVERY REGION

Lumped Region(s)	$\{k_1, k_2\}$	L_1	L_2	L_∞	$t(s)$
{G6P, F6P}, {2PG-PEP}	$\{1, 2\}$	1.18	0.79	1.03	161
{GLCi-F6P}, {BPG-PEP}	$\{2, 3\}$	1.05	0.57	0.78	260
{GLCi-F6P}, {3PG-PEP}	$\{2, 1\}$	0.47	0.3	0.4	137
{GLCi-F6P}, {3PG-PEP}	$\{1, 1\}$	0.14	0.07	0.09	116

TRUNCATION BY $\{k_1, k_2\}$ STATES IN EVERY REGION

Lumped Region(s)	$\{k_1, k_2\}$	L_1	L_2	L_∞	$t(s)$
{G6P, F6P}, {2PG-PEP}	$\{1, 2\}$	15.1	3.2	6.1	14
{GLCi-F6P}, {BPG-PEP}	$\{2, 3\}$	5.9	2.8	2.9	14
{GLCi-F6P}, {3PG-PEP}	$\{2, 1\}$	4.1	1.9	1.9	14
{GLCi-F6P}, {3PG-PEP}	$\{1, 1\}$	4.0	1.8	1.6	15

Conclusion

- Derived structured projection-based reduction method
- Applicable to biological systems (monotone, almost monotone)
- Paves the way for the stochastic problem...

Thank you!