A model of investment subject to financing constraints

Jukka Isohätälä∗, Alistair Milne†, and Donald Robertson‡†

August 23, 2012

Abstract

This paper unifies the recent analysis of macroeconomic fluctuations of [Brunnermeier and Sannikov(2012)], in which firms subject to financing constraints can sell capital in order to avoid liquidation, with the earlier related paper of [Milne and Robertson(1996)], in which firms manage cash flow, output and investment decisions in order to lower the probability of liquidation. This yields more tractable solution, elucidates the underlying economic mechanisms and reveals how the key result reported by Brunnermeier and Sannikov (a bimodal ergodic distribution of firm net worth) varies with parameter choice, arising in some situations but not others. [97 words]

Journal of Economic Literature number: E44

Keywords: Liquidation, asset sales, financial accelerator, cash flow management, continuous time dynamic stochastic optimisation

∗ Department of Physics, Loughborough University, Loughborough, LE11 3TU, UK.
† Corresponding author, School of Business and Economics, Loughborough University, Epinal Way, Loughborough LE11 3TU, UK, email: a.k.l.milne@lboro.ac.uk
‡ Faculty of Economics, University of Cambridge, Cambridge, UK
1 Introduction

This paper analyses the dynamics of investment subject to convex costs of adjustment and constraints in the supply of finance. This model developed here is a version of the earlier work of [Milne and Robertson(1996)], extending one version of their analysis (Section 5, pages 1440-1441) to the case where output is a linear function of capital stock. This set up is remarkably close to that analysed in recent working paper by [Brunnermeier and Sannikov(2012)], where firm investment is also a subject to convex costs of adjustment and constraints on the supply of finance.

There are differences between these two models. In Brunnermeier and Sannikov (BS) the capital is held by ‘experts’ who are never liquidated; instead they sell this capital in a secondary market and shrink their activities to reduce risk and maintain debt payments; in the model presented by Milne and Robertson (MR) firms cannot sell any assets and are liquidated when net worth falls to a lower bound. Another difference is that firms are subject to exogenous shocks to the productivity of capital in BS and to cash flow in MR. Still these models are closely related and it is appropriate to consider, as here, an extended model that combines the assumptions of the two and clarifies the relationship between them.

Our central finding is a greater variety of behaviour than that reported by BS. Their key result, the emergence of a bi-modal ergodic distribution of firm net worth, turns out to be highly parameter dependent arising only for example with sufficiently large exogenous uncertainty or a high degree of equity investor impatience. Even in these cases bimodality does not arise if firms are unable to avoid risk by lowering their asset holdings or if the costs of recapitalisation through new equity issue are sufficiently low. Thus while our analysis supports the theoretical model of macroeconomic downturns proposed by BS – a potential ‘ergodic instability’ in which following a large aggregate shock financial constraints firms can be trapped in a region of reduced net worth where output falls and from which the probability of escape is low – this is by no means a necessary consequence of the presence of external financing constraints.

‘Ergodic stability’ can also arise, consistent with the predictions of the standard model of the financial accelerator (of e.g. [Bernanke and Gertler(1989)], [Gertler(1992)] and [Ben Bernanke Mark Gertler and Gilchrist(1999)]). In these models output and investment is also a negative function of net worth but after even a large aggregate shock net worth subsequently increase relatively rapidly and the impact on investment and output is temporary rather than long-lived. Determining the dynamic macroeconomic consequences of external financing constraints cannot be based on a theoretical model alone,
but require a much more detailed examination of how these constraints affect
individual firms and households and ensuring that the resulting behaviour is
aggregated appropriately in a general equilibrium setting.

Our modelling also clarifies the distinction, not always properly made
in the literature, between ‘endogenous risk aversion’ and ‘endogenous risk’. In
our synthesis of BS and MR the impact of net worth on output and in-
vestment emerges because constraints on access to external finance limit the
ability of firms to cope with risk. Even when, as we assume, shareholders
and providers of debt finance are risk-neutral, financing constraints mean
that investment, output and other firm decisions must always take risk into
account, and the extent to which firms seek to avoid risk increases as their
net worth declines. This endogenous risk aversion is not the same thing
as endogenous risk, in which the volatility of asset prices and of incomes
increases as a consequence of feedback loops operating between economic
agents and markets. In this setting firm decisions depend upon only a single
state variable (net worth expressed as a ratio of the capital stock) and hence
there is no feedback from the decisions of other firms (through goods, labour
or asset markets). Thus this modelling framework (and also the closely re-
lated financial accelerator) cannot recreate the endogenous risk interactions
that emerged during the crisis of 2007-2009.

Our paper also makes two technical contributions. We follow MR by
using dynamic programming tools, showing that these can be applied to the
setting of BS and result in a more direct and easily interpretable solution than
the asset pricing methods they employ. In addition we utilise asymptotic
expansion of the ergodic density functions at the lower new worth boundary,
revealing power-law relationships that facilitate solution when (as sometimes
occurs) there is a singularity or where the ergodic density diverges at the
lower boundary.

The remainder of the paper is set out as follows. Section 2 comments on
the underlying microeconomics of our own and other related models and pro-
vides a brief review of other related literature (there is a large body of related
work using similar continuous time methods as MR and BS, most of it in the
mathematical insurance literature). Section 3 then sets out a first version of
our model, combining assumptions of both BS and MR. The analysis of MR,
Section 5, was limited by their assumption that output is a concave function
of capital. As a result full analysis required the solution of a two-state partial
differential equation (this PDE is stated but not solved in MR; subsequent
unpublished work (Milne and Robertson (1999)) presents a numerical solu-
tion). By introducing instead the linear production assumption of BS the
model simplifies, requiring solution of only an ordinary differential equation
(ODE). We also introduce one further parameter, representing a fixed cost
of recapitalisation (as in [Milne and Whalley(2003)], itself an extension of MR to analyse bank capital regulation). For high values of this fixed cost of recapitalisation firms do not recapitalise and the solution is similar to that of MR with liquidation on the lower net worth boundary (a boundary that can itself determined from a baseline version of the model without financing constraints, set out in Appendix A); but for lower recapitalisation costs firms choose to exercise their option to recapitalise on the lower boundary and so avoid liquidation. Section 3.1 states the model assumptions. Section 3.2 outlines the method of solution using the dynamic programming techniques developed in MR and explains how to obtain a solution for the ergodic distribution of firm capital across the range of possible values for the ratio of cash to capital. Section 3.4 presents some results for this initial model.

Section 4 extends the model to allow capital to be rented by firms to external debt providers (since transactions in capital are costless and there is no information asymmetry about the productivity of capital the purchase and sale of capital can be equally well modeled as a rental market). This makes the model almost identical to that of BS but with the incorporation of the additional parameter representing the fixed costs of recapitalisation. Section 4.1 presents the additional assumptions and the solution method. Firms, by renting capital to households, are able to reduce their risk exposure, but at the expense of a decline in their expected output. This decision parallels that of reinsurance in optimal dividend and reinsurance models of Taksar and co-authors, and also the output decision discussed in MR Section 4, and the lending decisions of banks in [Milne(2004)]. The level of risk transfer depends on an endogenous level of relative risk aversion to cash flow risks. Substituting in for optimal policy yields a different optimality equation, but this again this is a ordinary differential equation in a single state variable, which can be easily numerically solved. Section 4.2 presents the slightly different calculations now needed to obtain the ergodic distribution of firms. Section 5 provides some brief conclusions.

Three further appendices contain supporting technical detail. Appendix A discusses solution in the absence of financing constraints. Appendix B discusses the use of asymptotic expansions to determine boundary conditions on the ergodic distribution of cash to fixed capital. Appendix C describes the algorithms used for numerical solution.
2 Related literature

The modeling of both [Brunnermeier and Sannikov(2012)] (BS) and of [Milne and Robertson(1996)] (MR) is heuristic, without a full microeconomic analysis of the underlying contractual relationships. The presence of financing constraints can justified by the argument that they arise in several different fully micro-founded models, but the models available in the literature are all one period not fully dynamic (for example the widely used [Townsend(1979)] model of costly state verification). We are aware of only one paper [Gertler(1992)] that attempts to generalise costly state verification into a multiperiod setting. This shows that, in a dynamic setting, a similar relationship between net worth and investment can appear as that used in the standard single period models of the financial accelerator such as [Bernanke and Gertler(1989)]. A similar relationship also appears in the analysis pursued here but the microeconomic foundations are not quite the same. In particular we make the assumption that lenders, while risk-neutral, will not take any exposure to credit risk, rather they limit their exposure to the value that can be recovered after liquidation of the firm. This is convenient, because it is not then necessary to analyse the determination of credit spreads (all lending is at a risk-free rate of interest). The microeconomics of financing constraints in a fully dynamic setting remain relatively unexplored, but we anticipate (given the general 'folk theorem' of dynamic games) that many equilibrium outcomes will be possible and in some settings are likely to include equilibria in which lenders will not accept credit risk.

While there are few papers in the economics journals analysing the impact of financing constraints on firm behaviour in a dynamic setting, a much larger body of related work has appeared in the mathematical insurance literature, using continuous time methods to model optimal dividend, reserving and re-insurance policy. This work was initiated by [Jeanblanc-Picquè and Shiryaev(1995)], [Asmussen and Taksar(1997)] and (in a corporate finance setting almost identical to MR but with greater attention to mathematical foundations) by [Radner and Shepp(1996)]. Subsequent contributions explore dividend, reserving and reinsurance policy under a wide range of different assumptions, and in many different settings, a literature to which Taksar in particular has made very many contributions, listed on his MathSci website.1

This mathematical insurance literature focuses primarily on optimal reserving and reinsurance policies. This contrasts with MR, BS , and the standard models of the financial accelerator where the emphasis is on the

1http://ams.rice.edu/mathscinet/search/publications.html?pg1=IID&ts=170335
impact of financing constrains on investment, output and other real choices of the firm.

There is also the large mathematical finance literature using continuous time methods to explore optimal portfolio and consumption decisions. Firm investment choices can be interpreted as a portfolio decision (this is the approach adopted by BS). But the literature on portfolio choice subject to financing constraints assumes investor risk aversion, rather than risk-neutrality as in MR and BS. This literature extends the standard dynamic portfolio modeling of [Merton(1971)] to incorporate transaction costs, borrowing constraints and other frictions. This suggests that there is scope for extending the present analysis to incorporate investor risk aversion, using methods developed in the portfolio choice literature (BS present some results along these lines).

There are a number of papers using related continuous time methods to model intervention in exchange rates and in money markets, and how intervention rules affect market pricing in these markets, for example [Krugman(1991)] and [Mundaca and Ø ksendal(1998)]. [Korn(1999)] provides a useful survey article linking this work to that on both optimal portfolio allocation subject to transaction costs and the modeling of cash management problems faced by companies and insurance firms. Finally [Milne and Whalley(1999) ] , [Milne and Whalley(2003) ], [Milne(2004) ] and [Peura and Keppo() ] use a related framework to analyse bank capital regulation and bank behaviour.

3 The model

This section presents a first version of our model, combining elements from [Brunnermeier and Sannikov(2012)] and [Milne and Robertson(1996)]. Section 3.1 sets out the model assumptions. Section 3.2 discusses the solution method. Section 3.3 presents the solution.

3.1 Model assumptions

Firms manage two ‘state’ variables, net cash $c$ and capital $k$ (we later show that the model collapses to a single state variable $\eta = c/k$). These evolve
according to:
\[ dc = \left[ -\lambda + ak + rc - ik - \frac{1}{2} \theta (i - \delta)^2 k \right] dt + \sigma k dz \]
\[ dk = (i - \delta)k dt. \]

- \( r \) is both the rate of interest paid on borrowing \((c < 0)\) and the rate of interest received on cash deposits \((c > 0)\).
- Output \( ak \) is a linear function of the capital stock.
- The coefficient \( \theta \) captures costs of adjustment of the capital stock increasing with the rate of investment \( i \). Here the costs are quadratic, rather than the general convex function assumed in Brunnermeier and Sannikov (2012).

Note that in this specification the diffusion term \( \sigma k dz \) impacts on cash flow, as in [Milne and Robertson (1996)], rather than on the productivity of capital as in [Brunnermeier and Sannikov (2012)].

Firms choose rules for two control variables, the investment rate \( i \) and dividend payout \( \lambda \), subject to a non-negativity constraint on dividends \( \lambda \geq 0 \), in order to maximise the present value of expected future dividends:

\[ \Omega = \max_{\{i_t\},\{\lambda_t\}} \mathbb{E} \int_{t=0}^{\infty} e^{-\rho t} \lambda dt \quad (2) \]

The only other agents are outside investors (‘households’ in the terminology of [Brunnermeier and Sannikov (2012)]) who play two roles: first they lend to firms; second they can be a residual acquirer of a firm’s assets. Capital held directly by investors is less productive than when held by firms, generating an output of \( \bar{a}k < ak \). Investors are further assumed to avoid all credit risk on their lending. This means that the maximum amount they will lend to a firm is an amount \( \bar{\eta}k \) where \( \bar{\eta} \) is their own fundamental valuation of the underlying of a firm’s assets (see Appendix A for derivation of \( \bar{\eta} \)) and hence (since borrowing is the negative of cash holdings) \( c \geq -\bar{\eta}k \).

If \( c \) falls below (borrowing increases above) this bound then the firm is either liquidated with no further payment to shareholders or the firm is recapitalised at a fixed cost to shareholders of \( \chi \).

Like firms these investors are risk-neutral and seek to minimise the present discounted value of current and future consumption. Unlike firms there is no non-negativity constraint on their consumption. Since they are the marginal suppliers of finance, and there is no risk of credit losses, they lend to or
borrow from firms at a rate of interest $r$. We further assume that investors are more patient than firms i.e. $r < \rho$ (without this assumption firms will build up unlimited cash holdings instead of paying dividends).

### 3.2 Solution

#### 3.2.1 Characterisation of solution

Applying standard methods of stochastic dynamic programming, optimal policy by firms satisfies the Hamilton-Jacobi-Bellman equation

$$
\rho V = \max_{i, \lambda} \left\{ \lambda + \left[ ak + rc - \lambda - ik - \frac{1}{2} \theta (i - \delta)^2 k \right] V_c + (i - \delta)k V_k + \frac{1}{2} \sigma^2 k^2 V_{cc} \right\},
$$

with two first order conditions for maximisation. The first is:

$$
\begin{align*}
\lambda &\geq 0 \text{ of unbounded magnitude, } V_c = 1 \\
\lambda &= 0, \quad V_c > 1,
\end{align*}
$$

i.e. there is ‘bang-bang’ control with two distinct regions of dividend behaviour, one when whenever $c \geq c^* (k)$ with unlimited payout of dividends and $V_c = 1$, the other when $c < c^* (k)$ and with no payment of dividends and $V_c > 1$ (for further discussion and proof that there is a unique optimal choice of the boundary $c^* (k) > 0$ is the optimal policy see [Milne and Robertson (1996)]).

The other first-order condition is:

$$
(1 + \theta (i - \delta)) V_c = V_k
$$

yielding

$$
i = \delta + \theta^{-1} \left( \frac{V_k}{V_c} - 1 \right). \quad (3)
$$

Consider the HJB equation in the non-dividend paying region $c < c^* (k)$. Substituting for the optimal policy ($\lambda = 0$ and $i - \delta = \theta^{-1} (V_k / V_c - 1)$) yields:

$$
\rho V = \left[ (a - \delta) k + r c - \theta^{-1} \left( \frac{V_k}{V_c} - 1 \right) k - \frac{1}{2} \theta^{-1} \left( \frac{V_k}{V_c} - 1 \right)^2 k \right] V_c \\
+ \theta^{-1} \left( \frac{V_k}{V_c} - 1 \right) k V_k + \frac{1}{2} \sigma^2 k^2 V_{cc} \\
= \left[ a - \delta + \frac{r c}{k} + \frac{1}{2} \theta^{-1} \left( \frac{V_k}{V_c} - 1 \right)^2 \right] k V_c + \frac{1}{2} \sigma^2 k^2 V_{cc}.
$$
3.2.2 Restatement using a single state variable

Because of the linearity of production the value function is linearly homogeneous in $k$ and so we can define a new scale free value function $W$ with a new independent variable $\eta$:

$$V(c, k) = kW\left(\frac{c}{k}\right), \quad \frac{c}{k} = \eta,$$

or conversely $W(\eta) = k^{-1}V(\eta k, k) = V(\eta, 1)$. Optimal policy is given by the choice of dividend paying boundary $c^*(k) = \eta^*k$.

Substitution then yields a single state variable version of the HJB equation. From $V(c, k) = kW(c/k)$ we obtain the following relationships:

$$V_c = W', \quad V_k = W - \eta W',$$

$$V_{cc} = \frac{1}{k}W'', \quad V_{ck} = -\frac{\eta}{k}W'', \quad V_{kk} = \frac{\eta^2}{k}W''.$$

Using these substitutions results (after cancellation of terms in $k$) in:

$$\rho W = \left[a - \delta + r\eta + \frac{1}{2}\theta^{-1}\left(\frac{W}{W'} - \eta - 1\right)^2\right] W' + \frac{1}{2}\sigma^2 W''. \quad (5)$$

which can be solved subject to the optimality condition at the dividend paying boundary: $k^{-1}V_{cc} = W'' = 0$. On this boundary we have the additional scaling condition that $W' = 1$, so that here $\rho W = [a - \delta + r\eta + (W - \eta - 1)^2/2\theta]$.

Further insight can be gained by expressing the system using the function $q$ representing the ‘market value’ of the fixed assets of the firm per capital,

$$q = \frac{V_k}{V_c} = \frac{W}{W'} - \eta, \quad q' = -\frac{WW''}{W'W}, \quad (6)$$

The absolute market value of the fixed assets of the firm is given by $kq = kW/W' - c$ and so the mark-to-market accounting value of the equity of the firm is $kW/W'$. The actual value $V = kW$ is however always less than or equal to the accounting value because $W' \geq 1$.

The second order ODE for $W$, Eq. (5), can be recast as a first order ODE for $q$, by dividing by $W'$, and using the inverse of Eq. (6),

$$\frac{W'}{W} = \frac{1}{q + \eta}, \quad \frac{W''}{W'} = -\frac{q'}{q + \eta}, \quad (7)$$

to yield:

$$q' = \frac{2}{\sigma^2} \left[a - \delta - (\rho - r)\eta - \rho q + \frac{1}{2}\theta^{-1}(q - 1)^2\right](q + \eta). \quad (8)$$
In these variables, the first-order condition for $i$, Eq. (3), reads

$$ i = \delta + \theta^{-1}(q - 1). \quad (9) $$

Optimality condition at the dividend paying boundary $kV_c = W''(\eta^*) = 0$ implies that

$$ q'(\eta^*) = 0, $$

since $q' \propto W''$ by Eq. (7). This in turn means that the boundary is achieved when

$$ \eta^* = \frac{a - \delta - \rho q + \frac{1}{2} \theta^{-1} (q - 1)^2}{\rho - r}. $$

We can also invert this equation to solve for $q$ on the dividend paying boundary, yielding:

$$ q = 1 + \theta \left( \rho - \sqrt{\rho^2 - 2\theta^{-1} \{a - \delta - \rho - \eta^*(\rho - r)\}} \right). \quad (10) $$

The positive root is ruled out because this would result in both negative cash flows (expenditure of investment exceeding output $a$) and a growth rate higher than the discount rate, exactly as in the solution of the model with no financing constraints in Appendix A.

### 3.2.3 Lower boundary condition and the financing constraint

Solution also depends on the characterisation of the lower boundary, representing the maximum amount of debt that can be issued by the firm. We suppose that there is an externally imposed maximum level of debt, $\bar{\eta}$, defining the minimum value of $\eta$, together with a limiting value of $q$, $\bar{q}$. $\bar{\eta}$ is determined by assuming that the outside investors who lend to firms, will lend no more than their own fundamental valuation of the firm’s capital, were they to liquidate the firm and takeover the assets $k$, i.e. they will not accept any credit risk. In Appendix A we show that this fundamental valuation, i.e. the maximum amount of borrowing, is given by:

$$ \bar{\eta} = - \left[ 1 + \theta r - \theta \sqrt{r^2 - 2\theta^{-1} \{a - \delta - r\}} \right]. \quad (11) $$

When $\eta$ falls to $\bar{\eta}$ the company is either liquidated in order to satisfy the debt holders’ claims, in which case since $W = 0$ $q = \bar{q} = -\bar{\eta}$ or the firm is recapitalised. To model costs associated with raising equity, it is assumed that the shareholders incur fixed cost of $\chi k$ in order to recapitalize the firm.
\( \chi \) parameterises the strength of financing constraints experienced by the firm. Recapitalisation is chosen if the gains at least match the costs, in which case:

\[
W(\eta^*) - W(\bar{\eta}) = \eta^* - \bar{\eta} + \chi. \quad (12)
\]

Our numerical solution is presented in Appendix C. In outline, our method is as follows. We can numerically solve equation (8) for any chosen initial value \( \bar{q} \) on the lower boundary \( \bar{\eta} \), locating the upper boundary \( \eta^* \) using \( q'(\eta^*) = 0 \) and rescaling to enforce \( W'(\eta^*) = 1 \). \( W(\bar{\eta}), W(\eta^*) \) and \( \eta^* \) are thus all implicitly functions of \( \bar{q} \). If a value of \( \bar{q} > -\bar{\eta} \) exists for which the above matching condition (12) is satisfied, the firm has a possibility of choosing an investment policy that yields a higher \( W \) at \( \bar{\eta} \) and it will recapitalise rather than liquidate. This in turn implies that values of \( \bar{q} \) are bound from below by \( -\bar{\eta} \) and from above by the value of \( q \) on the dividend paying boundary (Eq. (10)) \( q = 1 + \theta \rho \). This can be summed up as follows:

\[
-\bar{\eta} \leq q(\eta^*) \leq q_{\text{max}} \equiv 1 + \theta \rho. \quad (13)
\]

Note that if \( \chi = 0 \), then \( \eta^* = \bar{\eta} \) satisfies Eq. (12). Firms can then always remain at the dividend paying boundary, recapitalising as necessary to avoid liquidation.

We must also be careful to ensure that the parameterisation is such that an optimal policy is well determined. In the case of the model without financing constraints explicit conditions on the parameters (stated in the proposition of Appendix A) ensure that firms are both able to generate positive cash flows (otherwise the value function representing the present discounted value of dividends would be negative) and cannot deliver unbounded payoffs through a growth rate of dividends that exceeds the shareholder rate of discount (in which case optimal policy is indeterminate). While we have no equivalent parameter restrictions for the case when there are financial constraints, we choose parameterisations that would be sensible in the case without financing constraints and in all the simulations of our model we obtain both a positive value function \( W(\eta) > 0, \) when \( \eta > \bar{\eta} \) and our solution method identifies an optimal policy.

### 3.3 Ergodic steady state

We can also solve for the ‘ergodic steady state’. This is mostly easily understood from a cross-sectional perspective (an ergodic distribution, if it exists, represents both the cross-sectional distribution of many firms subject to independent shocks to cash flow and the unconditional time distribution of a single firm across states).
Suppose that there is a large number of firms each behaving according to the assumptions of the model, but with independent diffusion. Let the distribution $f(t, \eta)$ represent the density function for the location of firms across the possible values of $\eta$ at the moment $t$, with the corresponding cumulative density being

$$F(t, \eta) = \int_{\eta^*}^{\eta} f(t, \eta') \, d\eta', \quad F(\eta^*) = 1.$$ 

The evolution of $f(t, \eta)$ is determined by the Kolmogorov forward, or Fokker-Planck, equation:

$$\frac{\partial f}{\partial t}(t, \eta) = -\frac{\partial}{\partial \eta} \left[ \mu^\eta(\eta) f(t, \eta) \right] + \frac{1}{2} \frac{\partial^2}{\partial \eta^2} \left[ \sigma^\eta(\eta)^2 f(t, \eta) \right], \quad (14)$$

where it is supposed that $\eta$ follows the equation of motion

$$d\eta = \mu^\eta \, dt + \sigma^\eta \, dz.$$

To gain an intuitive picture of the meaning of Eq. (14), consider the following. If $N$ is the total number of firms, the number of firms located in $\eta \in (\eta_1, \eta_2)$ at time $t$, $N_{\eta_1,\eta_2}(t) = N[F(t, \eta_2) - F(t, \eta_1)]$, satisfies

$$\frac{d}{dt} N_{\eta_1,\eta_2}(t) = N \left[ d(t, \eta_1) - d(t, \eta_2) \right],$$

where

$$d(t, \eta) = \mu^\eta(\eta) f(t, \eta) - \frac{1}{2} \frac{\partial}{\partial \eta} \left[ \sigma^\eta(\eta)^2 f(t, \eta) \right].$$

This is simply obtained by integrating Eq. (14) in $\eta$. The quantity $d(t, \eta)$ can be readily interpreted as the rate of flow of companies through the point $\eta$; $Nd(t, \eta_1)$ is the number of firms entering the interval $(\eta_1, \eta_2)$ at $\eta_1$, $-Nd(t, \eta_2)$ is the number leaving it at $\eta_2$, and the sum of these amounts to the total change in the number of firms in that interval, $dN_{\eta_1,\eta_2}/dt$.

An ergodic density is the stationary, $\partial f/\partial t = 0$, solution of Eq. (14), which we also denote by $f(\eta)$. Integration of the Kolmogorov forward equation in $\eta$ now gives

$$d = \mu^\eta(\eta) f(\eta) - \frac{1}{2} \frac{\partial}{\partial \eta} \left[ \sigma^\eta(\eta)^2 f(\eta) \right]. \quad (15)$$

In order to satisfy stationarity, $d$ is now a constant determined by the boundary conditions and representing the total rate of companies passing the non-dividend paying region.
The coefficients in the equation of motion for $\eta$ can be obtained simply by Itô differentiating $\eta = c/k$:

$$
\frac{d\eta}{k} = \left[ \mu^c - \eta \mu^k + \frac{\eta}{k} (\sigma^k)^2 \right] dt + \frac{\sigma^c}{k} dz_1 - \eta \frac{\sigma^k}{k} dz_2,
$$

where $\mu^{c,k}$ and $\sigma^{c,k}$ are respectively the drift and diffusion terms for $c$ and $k$: $dc = \mu^c dt + \sigma^c dz_1$, $dk = \mu^k dt + \sigma^k dz_2$. These can be directly read from Eq. (1):

$$
\mu^c = -\lambda + ak + rc - ik - \frac{1}{2} \theta (i - \delta)^2 k, \quad \mu^k = (i - \delta) k, \\
\sigma^c = \sigma, \quad \sigma^k = 0.
$$

Substituting these into the expression of $d\eta$, the drift and diffusion of $\eta$ can be found:

$$
\mu^\eta = a + r\eta - \frac{1}{2} \theta (\delta - \eta)(q - 1) - \frac{1}{2} \theta^{-1} (q - 1)^2, \\
\sigma^\eta = \sigma.
$$

Finally, the differential equation for the ergodic density comes out as

$$
\frac{1}{2} \sigma^2 f' - \left[ a + r\eta - \delta - \theta^{-1} (1 + \eta) (q - 1) - \frac{1}{2} \theta^{-1} (q - 1)^2 \right] f = -d.
$$

Boundary conditions must also be determined. Both in the case of liquidation and recapitalization, the minimum $\eta$ state is absorbing. Hence, any firm reaching $\eta = \bar{\eta}$ is removed from this point, giving

$$
f(\bar{\eta}) = 0.
$$

Note that $f'(\bar{\eta}) \geq 0$², and if the equality does not apply, by Eq. (15) there is a positive rate of liquidation (or recapitalisation), $-d > 0$, together with an equal rate of creation of new firms at the dividend paying curve (or transfer of recapitalised companies). The constant $d$ is determined endogenously by the condition (18) and normalisation $F(\eta^*) = 1$.

The numerical solution of $f$ for both boundary conditions is a straightforward task. Since Eq. (17) depends only on $\eta$ and $q$, the entire problem can be solved by solving Eq. (17) in parallel with Eq. (8) using standard ODE integration methods. For a transparent and efficient algorithm for enforcing the boundaries, and other details of the numerics, see App. C.

²If $f(\bar{\eta}) = 0$, $f'(\bar{\eta}) < 0$, there is $\varepsilon > 0$ such that $f(\bar{\eta} + \varepsilon) < 0$. 
3.4 Solution and interpretation

We have performed extensive simulations of the model equations, focussing on the issue of emphasised by [Brunnermeier and Sannikov(2012)], i.e. whether the ergodic distribution \( f(\eta) \) has two peaks and can therefore help explain a transition from a high output boom to a low out slump, or instead has a single peak. We find that for this first version of our model there is always a single peak and this usually is located at the maximum value \( \eta^* \).

Typical value functions \( W \) together with the corresponding ergodic densities \( f \) are presented in Fig. 1. Here, the chosen parameters are:

\[
\rho = 0.06, \quad r = 0.05, \quad \sigma = 0.3, \\
\theta = 100.0, \quad \chi = 0.5, \\
a = 0.1, \quad \bar{a} = 0.05, \quad \delta = 0.03. \tag{19}
\]

The shape of these plots are typical of what we find, with a monotonously rising value function \( W \), with single \( \eta^* \)–peaked ergodic densities \( f \). This feature appears to persistent. In a wide parameter space search, we have found only single peaked distributions of this kind.

Although a double peak were not found, we found that in some simulations the main peak normally at \( \eta^* \) can migrate into the central part of the \( \eta \) range. This occurs when choosing parameters for which cash-flows are non-positive \( (d\eta \leq 0) \) which is not economically sensible. We therefore do not report these simulations.

4 Extension to allow sale and repurchase of capital

This section extends the model, incorporating the [Brunnermeier and Sannikov(2012)] assumption that capital can be freely traded between firms and households. A simple way of solving this extended model is assume that capital can be rented by firms to outside investors (there is complete information about the productivity of capital so therefore this is equivalent to outside investors purchasing and holding capital for resale back to firms). This extension also nests [Milne and Robertson(1996)] Section 4, in which firms choose the level of output. By renting capital to households, instead of employing themselves, firms reduce both their own output and their exposure to risk [Milne and Robertson(1996)] Section 4 thus corresponds to the case where the rental income is zero, but where optimal policy may still be to ‘mothball’ capacity i.e. to take capital temporarily out of service.
Figure 1: Solutions of the model equations of Section 3 for baseline parameters $\rho = 0.06$, $r = 0.05$, $\sigma = 0.3$, $a = 0.1$, $\bar{a} = 0.05$, $\delta = 0.03$, $\theta = 100$. Subfigure (a): value function $W$, inset shows the function $q$ over the same $\eta$ range; (b) the ergodic density $f$ (solid curve) and the cumulative density function $F$ (dashed).
Figure 2: Solutions of the model of 4 with option to rent, using baseline parameters \( \rho = 0.06, r = 0.05, \sigma_1 = \sigma = 0.3, \sigma_2 = 0.0, a = 0.1, \bar{a} = 0.05, \delta = 0.03, \theta = 100 \). Contrast this to Fig. 1 where identical parameters were used, but without rental. Subfigure (a): value function \( W \), inset shows the functions \( q \) and \( \psi \) over the same \( \eta \) range; (b) the ergodic density \( f \) (solid curve) and the cumulative density function \( F \) (dashed). Notice the prominent peak in \( f \) towards the left-hand side boundary.
4.1 Additional assumptions and solution

In this extended setting, firms now continue to manage the same two ‘state’ variables, net cash $c$ and capital $k$, but these now evolve according to:

$$dc = \left\{-\lambda + (\psi a + (1 - \psi)\bar{a})k + rk - \frac{1}{2}\theta(i - \delta)^2k\right\} dt + \sigma\psi k dz_1,$$

$$dk = (i - \delta)k dt. \tag{20a-20b}$$

There is now an additional third control variable, the proportion of capital $\psi$ firms themselves manage (with remaining capital $1 - \psi$ rented to outside investors; because of competition amongst investors, this rented capital provides an income of $\bar{a}$, lower than the output $a$ of capital employed by the firm). The cash flow uncertainty $\sigma\psi k$ also depends on the proportion of capital firms manage for themselves. By renting out capital firms reduce both expected cash flow and the uncertainty of cash flow.

In order to fully nest the assumptions of [Brunnermeier and Sannikov(2012)] it would be necessary to include an additional term for a diffusion affecting the level of capital. This model could be solved using the same methods that are applied here, but we are able to reproduce results very similar to those reported by [Brunnermeier and Sannikov(2012)] without this additional complication (we presume that this additional parameter makes little difference to the model because the motion of the single state variable $\eta = c/k$ is little affected).

The HJB equation now becomes:

$$\rho V = \max_{i,\lambda,\psi} \left\{ \lambda + \left[-\lambda + (\bar{a} + (a - \bar{a})\psi)k + rk - \frac{1}{2}\theta(i - \delta)^2k\right] V_c 
+ (i - \delta)k V_k + \frac{1}{2}\sigma^2\psi^2k^2V_{cc} \right\},$$

with the additional first order condition for maximisation (subject to the constraint $0 \leq \psi \leq 1$):

$$(a - \bar{a})k V_c + \psi k^2 \sigma^2 V_{cc} = 0$$

This yields the further control rule:

$$\psi = (a - \bar{a}) \left[-k \frac{\sigma^2 V_{cc}}{V_c}\right]^{-1} = \frac{a - \bar{a}}{\sigma^2} \left[-\frac{W''}{W'}\right]^{-1} \tag{21}$$

where $-W''/W'$ measures the endogenous risk aversion created by the presence of financing constraints (see [Milne and Robertson(1996)] section 4 for
further discussion of this endogenous risk aversion and comparison with the endogenous risk loving behaviour that emerges in many standard discrete time models.) The greater this endogenous risk-aversion the lower the proportion of capital that is managed by firms instead of being rented out to households.

Restating the HJB equation in terms of the single-state value function $W$ and the independent variable $\eta$ the following is obtained:

$$
\rho W = \left\{ \bar{a} + (a - \bar{a})\psi - \delta + r\eta + \frac{1}{2}\theta^{-1} \left[ \frac{W}{W'} - 1 - \eta \right]^2 \right\} W' + \frac{1}{2}\sigma^2\psi^2W''.
$$

In the case that $0 < \psi < 1$, inserting the first order condition yields

$$
\rho W = \left\{ \bar{a} - \frac{1}{2}(a - \bar{a})^2 \left[ -\frac{W''}{W'} \right]^{-1} - \delta + r\eta + \frac{1}{2}\theta^{-1} \left[ \frac{W}{W'} - 1 - \eta \right]^2 \right\} W'.
$$

(22)

When the constraint $0 \leq \psi$ binds, i.e. when $\psi = 0$, then instead we have:

$$
\rho W = \left\{ \bar{a} - \delta + r\eta + \frac{1}{2}\theta^{-1} \left[ \frac{W}{W'} - 1 - \eta \right]^2 \right\} W'.
$$

$-W''/W'$ is a continuous and decreasing function of $\eta$ falling until, at the upper boundary $\eta^*$, $-W''/W' = 0$ and hence, according to equation (21), $\psi$ is a continuous and increasing function of $\eta$ which, without the constraint $\psi \leq 1$, would be unbounded over the region. This implies that there is some intermediate value of $\eta = \tilde{\eta} < \eta^*$ below which capital is rented and above which the firm retains and uses all its capital. This in turn means that this new version of the HJB equation (22) applies for $\tilde{\eta} \leq \eta \leq \tilde{\eta}$ and for $\tilde{\eta} < \eta < \eta^*$ the earlier version of the HJB equation (5) without rental applies, with appropriate matching conditions (continuity and smoothness) at the boundary point.

The HJB equation over the lower region $\eta < \tilde{\eta}$ can also be expressed as a differential equation in $q$ using Eq. (7):

$$
\rho(q + \eta) = \bar{a} + \frac{1}{2}(a - \bar{a})^2 \frac{q + \eta}{q'} - \delta + r\eta + \frac{1}{2}\theta^{-1}(q - 1)^2
$$
or

$$
q' = a - \bar{a} \frac{q + \eta}{\psi} = -\frac{1}{2}(a - \bar{a})^2 \frac{q + \eta}{\bar{a} - \delta + r\eta - \rho(q + \eta) + \frac{1}{2}\theta^{-1}(q - 1)^2}. \tag{23}
$$
We can also write down $\psi$ in terms of $\eta$ and $q$ when $0 < \psi < 1$:

$$\psi = -\frac{2}{a - \bar{a}} \left[ \bar{a} - \delta + r\eta - \rho(q + \eta) + \frac{1}{2} \theta^{-1}(q - 1)^2 \right]. \quad (24)$$

As in the model without the option to rent, the value function can be solved simply by integrating the equation for $q$, Eq. (23) beginning at the lower boundary $\bar{\eta}$. As before we allow for the possibility of recapitalisation. The boundary conditions are the same as in the model without the option to rent, but with the additional constraint set by the non-negativity of $\psi$ and the value of $\bar{\eta}$ is again given by Eq. (11). It turns out, somewhat to our surprise, that when $q(\bar{\eta}) = -\bar{\eta}$ then $\psi(\bar{\eta}) = 0$ and thus the non-negativity of $\psi$ does not impose any new restrictions on the value of $\bar{q}$.

### 4.2 Ergodic distribution

The equation for the ergodic distribution is obtained in exactly the same way as with the model without rental. The Itô differential of $\eta = \eta(c, k)$ given in Eq. (16) can again be used, but now with the drift and diffusion coefficients from Eq. (20). With the first order conditions for $\lambda$ and $i$ substituted in, these read

$$\mu^c = [\psi a + (1 - \psi)\bar{a} - \delta + r\eta - \theta^{-1}(q - 1) - \frac{1}{2} \theta^{-1}(q - 1)^2]k$$

$$\mu^k = \theta^{-1}(q - 1)k$$

$$\sigma^c = \sigma^k, \quad \sigma^k = 0$$

Substituting the expressions for $\mu^{c,k}$ and $\sigma^{c,k}$ into eq. (16) we have

$$d\eta = \left[ \bar{a} + (a - \bar{a})\psi - \delta + r\eta - (1 + \eta)\theta^{-1}(q - 1) - \frac{1}{2} \theta^{-1}(q - 1)^2 \right] dt + \sigma\psi \, dz. \quad (25)$$

There is however now a critical difference from the model without rental. If firms recapitalise on the lower boundary $\bar{\eta}$ then the boundary is once again absorbing and the boundary condition $f(\bar{\eta}) = 0$ applies exactly as in the case of the model without rental. But if firms do not recapitalise, the lower boundary is then reflective rather than absorbing. This is because when, on this lower boundary $\psi = 0$, firms have reduced their risk exposure to zero. They are then able to maintain this state without liquidation so we
no longer have $f(\bar{\eta}) = 0$. With a reflecting lower boundary the net flow $d$ through this point (and by Eq. (15) through any point) is zero, $0 = -d = -\mu\eta f + (1/2) [(\sigma\eta)^2 f]'$, where $d\eta = \mu\eta dt + \sigma\eta dz$, and $\mu\eta, \sigma\eta$ given by Eq. (25).

The equation for the ergodic density is now

$$\frac{1}{2} (\sigma\eta)^2 f' + \frac{d}{d\eta} \frac{(\sigma\eta)^2}{2} f - \left[ a + r\eta - \delta - \theta^{-1}(1 + \eta)(q - 1) - \frac{1}{2}\theta^{-1}(q - 1)^2 \right] f = -d$$

but this can be written down more conveniently in terms of $\phi$,

$$\phi = \frac{(\sigma\eta)^2}{2} f,$$

instead of $f$, resulting in:

$$\phi' = \begin{cases} 
\bar{a} + (a - \bar{a})\psi - \delta + r\eta - \theta^{-1}(q - 1)[\eta + \frac{1}{2}(q + 1)] \phi - d, & \text{when } \psi \in (0, 1), \\
\frac{1}{2} \sigma^2 \psi^2 & \text{when } \psi = 1.
\end{cases}$$

(26)

The numerical solution method of Eqs. (23) and (26) are described in Appendix C. The numerical solution in the case where there is no recapitalisation creates a technical challenge, because now we no longer have the condition $f(\bar{\eta}) = 0$. In appendix B, we show that in this case near $\bar{\eta}$, where $\bar{q} = -\bar{\eta}$ and $\psi = 0$, the form of the ergodic distribution can be expressed as the following asymptotic expansion:

$$f \propto (\eta - \bar{\eta})^{\alpha - 2},$$

(27)

where $\alpha$ is a constant that is dependent on the model parameters (see Eq. (37)). Typically, $\alpha < 2$ and the ergodic density diverges according to a power-law near the left-hand side boundary. Furthermore, $\alpha$ can be less than 1, indicating that the distribution can become degenerate.

This divergence and possible degeneracy corresponds very closely the findings of [Brunnermeier and Sannikov(2012)]where similarly divergent distributions were found (and can be described as ergodic instability although this term can be used also to refer to the slightly different situation of large probability densities arising in the lower end of $\eta$ rang). But our model is more general than theirs because we allow for the possibility of recapitalisation. In the case where the cost of recapitalisation $\chi$ parameter is sufficiently low, so that firms recapitalise on the lower boundary, then we once again have $f(\bar{\eta}) = 0$ and ergodic stability no longer arises. The simulations reported in the following section explore this parameter dependence in more detail.
4.3 Numerical solutions of the model with rental

As expected from the power-law shape of \( f \), Eq. (27), the option to rent can have a strong impact on the shape of the ergodic density. As an example of this, in Fig. 2 we have plotted the value function \( W \) together with \( q \), and the probability and cumulative densities using again our baseline parameters \( \rho = 0.06, r = 0.05, \sigma_1 = 0.35, \sigma_2 = 0.0, a = 0.1, \bar{a} = 0.05, \delta = 0.03, \theta = 100 \), cf. Fig. 1 where identical parameters were used. Whereas the value function \( W \) and \( q \) show little change when rental is introduced, the density function \( f \) changes dramatically. This time a second peak is clearly present near the left-hand side range of \( \eta \) values, in direct analogy to the ergodic instability of [Brunnermeier and Sannikov(2012)].

Where our results differ from those of [Brunnermeier and Sannikov(2012)] is in the parameter dependence of the ergodic instability. This parameter dependence emerges in two different ways: (i) the parameter dependence of the power-law exponent \( \alpha \), and (ii) the financing constraint \( \chi \). The ability to recapitalise or not has a major impact on the ergodic distribution. For any given parameters, there is a threshold \( \chi \), say \( \bar{\chi} \), above which recapitalisation is no longer worthwhile. If \( \chi \) is equal to or greater than this value, then \( \psi(\bar{\eta}) = 0 \), and the density diverges and the ergodic density follows the power-law \( f \propto (\eta - \bar{\eta})^{\alpha - 2} \) near \( \bar{\eta} \), which in turn can lead to infinite densities. Hence, the strength of the instability (i.e. the amount of probability mass near \( \bar{\eta} \)) is strongly controlled by the constraint \( \chi \). But when \( \chi \) is less than \( \bar{\chi} \) then \( f(\bar{\eta}) = 0 \).

This is illustrated in Fig. 3 where we show how the ergodic density changes as \( \chi \) is varied. For low values of \( \chi \), there is no significant left-hand side peak in the model with rental (Fig. 3a) and \( f \) largely resembles that of the model without the option to rent (Fig. 3b). As \( \chi \) approaches \( \bar{\chi} \) (indicated by the dotted lines on the floor of the two panels of this figure, where \( \bar{\chi} \simeq 0.55 \) with rental, \( \bar{\chi} \simeq 0.54 \) without), then in the model with rental a probability mass starts to appear near \( \bar{\eta} \). Crossing \( \bar{\chi} \), recapitalisation becomes no longer an option, and the density at \( \bar{\eta} \) diverges. Above \( \bar{\chi} \) there is no longer \( \chi \) dependence. Note that the distribution \( f \) changes quite sharply approaching \( \bar{\chi} \) is crossed, with a second peak of the distribution emerging close to \( \eta = \bar{\eta} \) something that in a variety of simulations with parameters not far from our baseline seems always to be the case.

In order to identify the parameters for which the ergodic instability is present, and to probe the extent to which it affects the distribution of firms, we have computed the median of \( f \) for varying parameters. Since the value of \( \bar{\eta} \) and \( \eta^* \), the range on which the distribution is defined, also varies with the parameters, it is convenient to scale the median on to the interval \([0, 1]\):
Figure 3: Comparison of ergodic densities $f$ between the option to rent (a) and no option to rent (b) as the financing constraint $\chi$ is varied. Other parameters are set to baseline. The lower boundary is recapitalising upto $\chi = \bar{\chi}$ ($\bar{\chi} \simeq 0.55$ in (a), 0.54 in (b)), indicated by the thick solid line on the graph and dashed line on the axis. In (a) a left-boundary peak emerges for $\chi$ just less than $\bar{\chi}$. Density is infinite at $\bar{\eta}$ for $\chi > \bar{\chi}$. Note the complete absence of the left-hand side peak in (b).
Let $m$ be the median, then the scaled median is defined as
\[ \tilde{m} = \frac{m - \bar{\eta}}{\eta^* - \bar{\eta}}, \quad F(m) = \frac{1}{2}. \] (28)

A value of $\tilde{m} \sim 0$ implies that most of the probability mass is concentrated near $\bar{\eta}$, while $\tilde{m} \sim 1$ suggests that firms are more probably found near $\eta^*$. This is a somewhat crude measure, the median cannot distinguish between e.g. distributions that are $\cup$- or $\cap$-shaped. Nonetheless, $\tilde{m} \lesssim 1/2$ is a strong indicator of an ergodic instability. Furthermore, in our simulations, the only reason for left-heavy distributions has been the appearance of the second peak near $\bar{\eta}$ so for these parts of the parameter space $\tilde{m}$ does serve well to summarise the shape of the distribution.

In Fig. 4 we present a countour plot $\tilde{m}$ as a function of the financing constraint $\chi$ and the volatility $\sigma$ (note that Fig. 3 represents a small slice of data presented in this figure). The solid heavy line represents the critical value $\bar{\chi}(\sigma)$, the firm choosing to recapitalise only when $\chi < \bar{\chi}(\sigma)$. Three roughly distinct regimes can be seen:

(i) the low volatility range $\sigma \lesssim 0.2$, in which the firm always prefers to recapitalise and where $\tilde{m} \gtrsim 0.8$ and so most of the probability is found near the dividend paying boundary.

(ii) a region of ergodic instability, where $\sigma \gtrsim 0.3$ and at the same time $\chi \gtrsim 0.5$, i.e. red region to the top right, where $\tilde{m} \sim 0$, and much of the probability mass is located near the boundary

(iii) an intermediate transition range where small changes in either $\sigma$ or in $\chi$, result in a very substantial change in $\tilde{m}$. This transition is especially abrupt for high values of $\sigma$.

We have examined the behaviour of $\tilde{m}$ as a functio of other model parameters, obtaining remarkably similar contour plots. For example as the relative impatience of shareholders $\rho - r$ is increased from relatively low to high values, there are also two distinct regions similar to those of Figure 4, with a relatively sharp transition in the balance of the probability distribution from near the upper boundary $\eta^*$ to the lower boundary $\bar{\eta}$.

We obtain one further finding which does not appear at all in [Brunnermeier and Sannikov(2012)] . As in [Milne and Robertson(1996)] we can measure endogenous aversion to cash flow risk by the ratio $-W''/W'$. We find that in the present model this too is strongly parameter dependent and in the model with rental when firms do not recapitalise they become extremely risk-averse close to the lower boundary $\bar{\eta}$.

This is revealed by an analysis of power-law behaviour of $W$ at the lower boundary $\bar{\eta}$ (see Appendix B), if $\bar{q} = -\bar{\eta}$ (i.e. if there is no recapitalisation)
Figure 4: The scaled median $\tilde{m}$ as a function of $\chi$ and $\sigma$. Other parameters are set to baseline values $\rho = 0.06$, $r = 0.05$, $\sigma = 0.3$, $a = 0.1$, $\bar{a} = 0.05$, $\delta = 0.03$, $\theta = 100$. Contours are plotted at level values of $\tilde{m}$ and are spaced at intervals of 0.1.
Figure 5: Endogeneous risk aversion $-W''/W'$ as a function of $\eta$. Parameters are set to baseline. Solid curve: model with option to rent; dashed curve: model without option to rent. Significantly, the risk aversion diverges strongly near $\bar{\eta}$ in the model without rent, in contrast to the model without rental.
then:

\[ W(\eta) \propto (\eta - \bar{\eta})^\beta, \]

where \( \beta = 1/(1 + q' (\bar{\eta})) \). Since \( q' (\bar{\eta}) > 0 \), \( \beta \in (0, 1) \), and so our measure of endogeneous risk aversion is always divergent at \( \bar{\eta} = -\bar{q} \).

\[ -\frac{W''}{W'} \simeq \frac{1 - \beta}{\eta - \bar{\eta}}. \]

In contrast, in the model without rental, \( W \) can be approximated by its Taylor series, and so the risk aversion tends to a finite value at \( \bar{\eta} \).

This finding is illustrated in Figure (5) which compares endogenous risk-aversion for the two version of the model, with and without the option to rent. The parameters here are the same as in Figures (1) and (2). For relatively large values of \( \eta \) close to \( \eta^* \) the option to rent provides protection against cash flow risk and \(-\frac{W''}{W'}\) is lower for the model with rental; but as \( \eta \) falls down towards \( \bar{\eta} \) then in the model with rental \(-\frac{W''}{W'}\) diverges upwards, rising increasingly rapidly, whereas it rises only modestly in the model without rental.

We offer the following analogies, hoping they provide some intuition for this result. It seems that, without an option to rent, the firm is rather like a boat in a stormy sea near a rocky shore, the probability of disaster is already very high, so it is worth taking some additional risk of shipwreck in order to escape the danger. But with the option to rent the situation is more walking on slippery slope near a cliff edge, by being very cautious and taking little risk eventually the danger can be escaped; so since rental provides a slow and steady route away from the danger risk must be radically reduced so as not to hamper the escape.

5 Conclusions

This paper presents a model nesting the recent analysis of the impact of the financial sector on the macroeconomy, of [Brunnermeier and Sannikov(2012)], and the earlier analysis of firm behaviour subject to financing constraints of [Milne and Robertson(1996)]. Both models use continuous time dynamic stochastic optimisation to investigate how financial frictions can affect firm behaviour over the course of the business cycle. This relatively technical modelling approach is worthwhile because it provides new insight into the standard model of the financial accelerator, due to [Bernanke and Gertler(1989)] and [Ben Bernanke Mark Gertler and Gilchrist(1999)].
The present paper makes both technical and analytical contributions. The technical contributions are to show that the solution methods of Milne and Robertson are relatively simple, more direct and more transparent, when compared to those employed by Brunnermeier and Sannikov. Because there is no feedback from financial market prices onto firm behavior, model solution can be conveniently characterized in terms of ratio of debt to capital. Solution, while still numerical, can then be obtained without the need for iteration. This makes it relatively easy to explore a wide range of parameter values and investigate how these affect firm behavior. The use of asymptotic approximations are also convenient to explore the behaviour of both the value function and of the ergodic distribution of firms near the lower boundary of maximum debt.

The analytical contribution is to show that the settings of Milne and Robertson and Brunnermeier and Sannikov, while more fully dynamic, can yield behavior that is very similar to that emerging in the standard static model of the financial accelerator. Because of constraints in the access to finance, various aspects of firm behavior (output, investment, and attitudes towards risk) are non-linear functions of the (accounting) the indebtedness of the firm. This is also a prediction of static one period models of the financial accelerator. But a more fully dynamic analysis clarifies when a linearisation - of the kind routinely employed in new Keynesian macroeconomic models - provided a reasonable approximation to the fully dynamic optimal behavior of the firm. This is appropriate as long as the great majority of the distribution of firms lies close to the maximum level of cash (or minimum level of debt), an outcome which will emerge for some parameter values and model specifications but not for others.

Our findings thus concur with those of Brunnermeier and Sannikov(2012) on the importance of allowing for non-linear behavior, either when firms are financially weak (generating comparatively low expected cash flows compared to cash flow uncertainty) or for modeling the impact of an unexpectedly large aggregate shock which drives many firms well below their desired ratio of debt to fixed capital. But our findings differ on the principal ‘ergodic instability’ result of Brunnermeier and Sannikov(2012), according to which there is possibility of a ‘net worth trap’ in which, following a large aggregate shock, limited access to external finance means that firms are unable to build up capital and escape from financing constraints. We find this result to be very parameter dependent, emerging only when firms have no opporutnity to re-capitalise as a alternative way of overcoming their lack of net worth, and instead turn to selling or equivalently renting out capital stock as a way of reducing their risk exposure.

That said, our results suggests that the general line of analysis pursued
here – previously explored by [Milne and Robertson(1996)], by a number of related contributions to the mathematical insurance literature, and most recently by [Brunnermeier and Sannikov(2012)] – using continuous time modelling to explore the dynamic implications of financing constraints, has a lot to offer in understanding firm behaviour, both at the microeconomic and macroeconomic level. Taken as a whole this literature suggests that the economic impact of financing constraints is very widespread. While there is unlikely to be a single canonical model which fully captures the impact of financing constraints on the macroeconomy, it is well worth exploring a range of such models of this kind to obtain insight into the interactions of the financial sector and the broader economy.

A Solution in the absence of financing constraints

This appendix considers the solution to the model of this paper in the situation where dividend payments can be negative (this case in which there are no financing constraints provides a useful benchmark for the solutions explored in the main text, and also determines the maximum amount of borrowing by the constrained firm for which dividends are non-negative).

First note that, given the assumed impatience of firm owners relative to market rates of interest ($\rho > r$), optimal policy in this case is to transfer funds between the firm and shareholders, so as to maintain the cash balances unchanged at the highest level of indebtedness allowed by lenders. There is therefore now only a single state variable $k$. The value function (the value of the objective function under optimal policy) is still linearly homogenous in $k$ and so can be written $V(k) = kW$ where $W$ is a constant that depends on the parameters representing preferences and the evolution of the state variable $k$. This in turn implies that $V_k = W$ and $V_{kk} = 0$. Note also that if the maximum level of indebtedness is given by $\bar{c} = \bar{\eta}k < 0$, then the growth of capital stock $k$ will allow additional dividends to be paid through the growth of $k$. The remaining policy decision is to choose a rate of investment $i$ and hence expected growth of the capital stock $g = i - \delta$ to maximise $\Omega$, Eq. (2)

The solution can then be summarised in the following proposition:

**Proposition 1** An optimal policy yielding positive payoffs for the owners of
the firm can be found provided that:

\[
2 \frac{a - \delta - \rho + (r - \rho)\bar{\eta}}{\rho^2} < \theta < \begin{cases} 
\infty & \text{if } a + r\bar{\eta} \geq \delta \\
\frac{1}{2} \frac{(1 + \bar{\eta})^2}{\delta - r\bar{\eta} - a} & \text{if } a + r\bar{\eta} < \delta 
\end{cases}
\]

in which case the growth rate is given by

\[
g = \rho - \sqrt{\rho^2 - 2\theta^{-1} [a - \delta - \rho + (r - \rho)\bar{\eta}]},
\]

and the value of the maximised objective is given by

\[
V = \frac{a - (1 + \bar{\eta})g - \delta + r\bar{\eta} - \frac{1}{2} \theta g^2}{\rho - g} k.
\]

**Proof.** In order to maintain a constant ratio \(\bar{\eta}\) of cash to capital, Dividends are paid out according to Eq. (20a) with \(dc = \bar{\eta}dk\) due to issuing new debt, implying:

\[
\lambda dt = \left[ a + r\bar{\eta} - (1 + \bar{\eta})g - \delta - \frac{1}{2} \theta g^2 \right] k dt + \sigma k dz
\]

The objective function can now be evaluated. Substituting in the expression for \(\lambda dt\) we get

\[
\Omega = \max_g \left\{ \mathbb{E} \int_0^\infty e^{-\rho t} \left[ a - (1 + \bar{\eta})g - \delta - \frac{1}{2} \theta g^2 \right] k dt + \int e^{-\rho t} \sigma k dz \right\}
\]

and performing the integrals, and noting that \(\mathbb{E}[k] = k(0) \exp(\rho t)\) and \(e^{-\rho t} \sigma k = 0\), the following is obtained:

\[
\Omega = k(0) \max_g \frac{a - (1 + \bar{\eta})g - \delta + r\bar{\eta} - \frac{1}{2} \theta g^2}{\rho - g}.
\]  

(29)

The growth rate \(g\) that maximises the right hand side of this expression is determined by the first order condition

\[
\frac{1}{2} g^2 - \rho g + \theta^{-1} [a - \delta - \rho + (r - \rho)\bar{\eta}] = 0,
\]

(30)
yielding two potential solutions

\[ g_\pm = \rho \pm \sqrt{\rho^2 - 2\theta^{-1}[a - \delta - \rho + (r - \rho)\bar{\eta}]}. \]

The first order condition \( (30) \) will characterise optimal policy, provided both that expected dividend payments are positive (otherwise the first order condition will identify a minimum of the value function not a maximum); and that the growth rate of expected dividends is greater less than the growth rate of expected capital \( \text{i.e. } \rho - g < 0 \). This rules out \( g = g_+ \) as a possible solution. It also implies the first inequality (the infimum) for \( \theta \). This is because if

\[ \theta < 2\frac{a - \delta - \rho + (r - \rho + \sigma_2^2)\bar{\eta}}{\rho^2}, \]

then the choice of \( g = \rho \) yields positive expected dividend payments and again an unbounded value function (mathematically this inequality \( \theta \) is reflected in the requirement that the optimal choice of \( g \) is real not imaginary; but we have the additional requirement that the two roots are not co-incident).

The second inequality condition on \( \theta \) is necessary to ensure that it is possible to achieve positive dividends per unit of capital (this is a weak condition since normally we would expect \( a > \delta \) in which case a policy of no planned growth in the stock of capital \( g = 0 \) and no indebtedness will always yield positive dividends; but if depreciation is larger than the productivity of capital, or if the firm can become highly indebted, then a further restriction on \( \theta \) is required). To see this note that expected dividends per unit of capital \( a - \delta + r\bar{\eta} - (1 + \bar{\eta})g - \frac{1}{2}\theta g^2 \) are maximised by choosing \( g = - (1 + \bar{\eta}) \theta^{-1} \) resulting in expected dividend payments per unit of capital of \( a - \delta + r\bar{\eta} + \frac{1}{2}\theta^{-1} (1 + \bar{\eta})^2 \). This is always greater than zero if \( a > \delta \), otherwise this requires that \( \theta < (1 + \bar{\eta})^2/2(\delta - a - r\bar{\eta}) \).

Finally note that the fundamental valuation of a firm’s capital by outside investors can be obtained by substituting \( r = \rho \), \( a = \bar{a} \) and \( \bar{\eta} = 0 \) into this solution. A finite positive valuation is obtained provided the parameters satisfy:

\[ \frac{2\bar{a} - \delta - r}{r^2} < \theta < \begin{cases} \infty & \text{if } \bar{a} \geq \delta \\ \frac{1}{2} \frac{1}{\delta - \bar{a}} & \text{if } \bar{a} < \delta \end{cases} \]

in which case the growth rate (when held by outside investors) is given by

\[ g = r - \sqrt{r^2 - 2\theta^{-1}[\bar{a} - \delta - r]}, \]
and the value of the maximised objective by

\[ V = \frac{\bar{a} - g - \delta - \frac{1}{2} \theta g^2}{r - g} k = \frac{\bar{a} - g - \delta - \frac{1}{2} \theta g^2}{\sqrt{r^2 - 2\theta^{-1}[\bar{a} - \delta - r]}} k = \left[ 1 + \theta r + \frac{2(\bar{a} - \delta - r) - \theta r^2}{\sqrt{r^2 - 2\theta^{-1}[\bar{a} - \delta - r]}} \right] k \]

This valuation of the firm’s assets by outside investors is also the maximum amount of debt that it can borrow from these investors, implying (after substitution for \( g \)) that the lower boundary for \( \eta \) is:

\[ \bar{\eta} = \frac{\bar{a} - g - \delta - \frac{1}{2} \theta g^2}{\sqrt{r^2 - 2\theta^{-1}[\bar{a} - \delta - r]}} = -\left[ 1 + \theta r - \theta \sqrt{r^2 - 2\theta^{-1}[\bar{a} - \delta - r]} \right] \]

B Behaviour of solutions near boundaries

The ordinary differential equations governing \( q, W, \) and \( f \) can in certain cases diverge as \( q \) or \( \eta \) is varied. Here we derive analytical formulas for the behaviour of these functions near singular points to provide insight into the dynamics of the model and to aid and to verify the consistency of the numerical solutions.

B.1 Model without option to rent

In the model without the option to rent, only the equation for \( W \) can have singularities. Given the behaviour of the function \( q \), the evolution of the value function \( W \) can be determined from \( W'/W = 1/(q + \eta) \), which tends to infinity as the initial point \( q(\bar{\eta}) = -\bar{\eta} \) is approached. Suppose now that \( q \) is of the form \( q(\eta) = -\bar{\eta} + q'(\bar{\eta})(\eta - \bar{\eta}) + \mathcal{O}((\eta - \bar{\eta})^2) \). Near the boundary, \( W \) follows

\[ W' = \frac{1}{1 + q'(\bar{\eta})} \frac{W}{\eta - \bar{\eta}} + \mathcal{O}(\eta - \bar{\eta}). \]

Denoting \( 1/(1 + q'(\bar{\eta})) = \beta \), the solution is

\[ W = C_W(\eta - \bar{\eta})^\beta (1 + \mathcal{O}(\eta - \bar{\eta})), \quad (31) \]

where \( C_W \) is a constant. If the option to rent is not available, it is clear from Eq. (8) that \( q'(\bar{\eta}) = 0 \) and so \( W \) is linear near \( \bar{\eta} \).
B.2 Model with option to rent

Turning now exclusively to the model with option to rent, there are three distinct cases where singularities can occur: (i) \( q = -\eta, \psi > 0 \), (ii) \( q > -\eta, \psi = 0 \), and (iii) \( q = -\eta, \psi = 0 \). The first and the last are the ones that occur in practice, but for completeness, we will also consider case (ii).

**Case (i)**  
This situation is identical the model without option to rent. Again, \( q \) is quadratic near \( \bar{\eta} \) and the finding that \( W \) is linear in \( \eta \), with \( W = 0 \) at \( q = -\eta \), again applies. No singularities are present in equations for \( q \) and \( f \).

**Case (ii)**  
Now \( q > -\eta \), but \( \psi = 0 \) and singularities appear in expressions for \( q' \) and \( f' \). The zero \( \psi \) condition can only apply at the lower boundary, and it is equivalent to

\[
\bar{a} - \delta + r\bar{\eta} - (\bar{q} + \bar{\eta})\rho + \frac{1}{2}\theta^{-1}(\bar{q} - 1)^2 = 0. \tag{32}
\]

Consider first the equation for \( q \) near \( \psi = 0 \). Assuming that the first derivative grows unbounded as the boundary is approached from above, we may suppose that \( q - \bar{q} \gg \eta - \bar{\eta} \), when \( \eta - \bar{\eta} \) is sufficiently small. Retaining only the dominant terms in the equation for \( q' \), and using Eq. (32), we have

\[
q' = -\frac{1}{2} \frac{(a - \bar{a})^2}{\sigma_1^2 + \bar{\eta}^2\sigma_2^2} \frac{\bar{q} + \bar{\eta}}{\theta^{-1}(\bar{q} - 1) - \rho(q - \bar{q})}. \tag{33}
\]

An analytic solution is readily obtained:

\[
q = \bar{q} + q_{1/2}(\eta - \bar{\eta})^{1/2}, \tag{34}
\]

where

\[
q_{1/2} = \left\{ \frac{(a - \bar{a})^2}{\sigma_1^2 + \bar{\eta}^2\sigma_2^2} \frac{\bar{q} + \bar{\eta}}{\rho - \theta^{-1}(\bar{q} - 1)} \right\}^{1/2}.
\]

Note that \( q_{1/2} \) is real when \( \bar{q} \geq -\bar{\eta} \) and if \( \bar{q} < 1 + \rho\theta \). Former condition is clearly always satisfied. Latter is also always true, since if \( \bar{q} > 1 + \rho\theta \), the dividend paying curve is necessarily reached above the maximum \( q \) value, \( q_{\text{max}} = 1 + \rho\theta \) (see Eq. (13)). Therefore \( q_{1/2} \) is real for relevant parameter values. The obtained form of the function \( q \) is consistent with the assumptions used in its derivation.
For the ergodic density (or for $\phi = (\sigma^2)f/2$, rather), we also need the limiting form of the function $\psi$. This is found starting from Eq. (24), substituting $q$ from Eq. (34), and using $\eta - \bar{\eta} \ll q - \bar{q}$:

$$\psi = 2 \left\{ \frac{\bar{q} + \bar{\eta}}{\sigma_1^2 + \bar{\eta}^2\sigma_2^2} \left[ \rho - \theta^{-1}(\bar{q} - 1) \right] \right\}^{1/2} (\eta - \bar{\eta})^{1/2}. $$

Finally, substituting this into Eq. (26) and using the limiting properties of $q$, we have (after numerous cancellations) a surprisingly simple expression:

$$\phi' = \frac{1}{2} \frac{1}{\eta - \bar{\eta}} \phi. $$

The solution is clearly of the form

$$\phi = C_\phi (\eta - \bar{\eta})^{1/2}, $$

where $C_\phi$ is a constant. The ergodic density $f$ will then diverge as $(\eta - \bar{\eta})^{-1/2}$.

**Case (iii)** The condition $\psi = 0$ and $q = -\eta$ applies only at a single point on $\eta, q$-plane, and hence this case should occur only rarely. However, $\bar{\eta}$ as given by Eq. (11) is exactly such that if $\bar{q} = -\bar{\eta}$, then $\psi = 0$, and so this case occurs always when we do not have recapitalisation.

Note now that in the equation for $q'$, Eq. (23), both the numerator and the denominator vanish. Applying the l'Hôpital’s rule, the derivative can be solved to be

$$q'(\bar{\eta}) = -\frac{(\rho - r - \gamma)}{\sigma_1^2 + \bar{\eta}^2\sigma_2^2} \pm \frac{\sqrt{\rho - r - \gamma)^2 + 4\gamma\theta^{-1}[1 + \theta\rho - \bar{q}]}{2\theta^{-1}[1 + \theta\rho - \bar{q}]}, \quad (35)$$

where $\gamma = (a - \bar{a})^2/2(\sigma_1^2 + \bar{\eta}^2\sigma_2^2)$. Above, only the plus sign applies. This can be seen by recalling that $\bar{q} < q_{\max} = 1 + \rho\theta$ must apply (see above for the reasoning), in which case only the plus sign gives a positive $q'$. Thus, the solution near $\bar{\eta}$ reads

$$q = \bar{q} + q'(\bar{\eta}) (\eta - \bar{\eta}). \quad (36)$$

The power-law form of $W$ given in Eq. (31) holds here as well. Since now $q'(\bar{\eta}) > 0$, the exponent $\beta = 1/(1 + q'(\bar{\eta}))$ is always greater than one, in contrast to the model without option to rent.

To find the behaviour of the ergodic density near $\bar{\eta}, \bar{q}$, we first need $\psi$. This time $\eta - \bar{\eta}$ is not negligible compared to $q - \bar{q}$, otherwise everything is analogous to case (ii). A straight-forward calculation gives:

$$\psi = \psi'(\bar{\eta})(\eta - \bar{\eta}), $$
where
\[ \psi'(\bar{\eta}) = \frac{2}{a - \bar{a}} \left\{ \rho - r + \theta^{-1} [1 + \theta \rho - \bar{q}] q'(\bar{\eta}) \right\}. \]

Next, the \( \eta \to \bar{\eta} \) limiting forms of \( q \) and \( \psi \) are substituted into the equation for \( \phi' \), and only terms up to \( O(\eta - \bar{\eta}) \) are kept. Notice that the numerator vanishes in the leading order, and hence \( \phi' \propto (\eta - \bar{\eta})^{-1} \) and not \( \propto (\eta - \bar{\eta})^{-2} \):
\[ \phi' = \alpha \frac{\phi}{\eta - \bar{\eta}}, \]
where
\[ \alpha = \frac{(a - \bar{a}) \psi'(\bar{\eta}) + r - \frac{1}{2} \theta^{-1} q'(\bar{\eta})(\bar{\eta} + 1) + \theta^{-1}(\bar{\eta} + 1)(1 + q'(\bar{\eta})/2)}{\frac{1}{2}(\sigma_1^2 + \bar{\eta}^2 \sigma_2^2) \psi'(\bar{\eta})^2}. \] (37)

This gives a power-law solution with \( \alpha \) as the exponent:
\[ \phi = C_\phi (\eta - \bar{\eta})^\alpha, \] (38)
where \( C_\phi \) is again a constant. The ergodic density is then \( f \propto (\eta - \bar{\eta})^{\alpha-2} \), and thus diverges if \( \alpha < 2 \) and becomes degenerate if \( \alpha \leq 1 \).

C  Numerical solution of the model equations

C.1  Model without option to rent

The ordinary differential equation governing \( q \) can be solved by forward integration using standard methods starting from the initial condition \( q(\bar{\eta}) = \bar{q} \). The right-hand side boundary at \( \eta = \eta^* \) is found by evaluating the function \( q'(\eta) \) during the integration. After a single integration step is found to bracket a root of \( q'(\eta) \), the critical value of \( \eta \) is pin-pointed using standard root finding methods, here the Brent’s method.

The value function \( W \) can be solved from \( W' = W/(\eta + q) \) parallel to integrating the equation for \( q \). The boundary condition \( W''(\eta^*) = 0 \) will be satisfied since the \( q \) variable integration is stopped at \( q' = 0 \), \( q' = WW''/W^2 \). In order to also satisfy the boundary condition \( W''(\eta^*) = 1 \), we solve \( W \) for an arbitrary initial value at \( \bar{\eta} \). Let the resulting solution be \( \tilde{W} \). Since the ODE for \( W \) is linear and homogeneous, we can simply multiply \( \tilde{W} \) ex post by \([W'(\eta^*)]^{-1}\) to get a solution for which \( W'(\eta^*) = 1 \).

In the case of liquidation, the lower boundary is \( \bar{\eta} = -\bar{q} \), and consequently, the derivative of \( W \), \( W' = W/(\eta + q) \) cannot be evaluated. In Appendix B we
have shown that $W \propto \eta - \bar{\eta}$, and so $W'(\bar{\eta})$ is finite. If $\bar{\eta}$ is indeed liquidating, we simply set $W'(\bar{\eta}) = 1$ and $W(\bar{\eta}) = 0$.

In order to satisfy the absorbing boundary condition for the ergodic density, $f(\bar{\eta}) = 0$, we use the following method. We solve two independent differential equations for two new densities $f_0$ and $f_1$:

$$f_0'(\eta) = \frac{2\mu(\eta)}{\sigma^2} f_0(\eta), \quad f_1'(\eta) = \frac{2\mu(\eta)}{\sigma^2} f_1(\eta) + 1. \quad (39)$$

These can be simply integrated starting from arbitrary initial conditions such that $f_{0,1}(\bar{\eta}) > 0$. Let $F_0$ and $F_1$ be the corresponding cumulative functions, $F_0' = f_0$, $F_1' = f_1$, $F_0(\bar{\eta}) = F_1(\bar{\eta}) = 0$ ($F_0(\eta^*) = F_1(\eta^*) = 1$ need not apply). We will next show that the probability density function $f$ can be written as

$$f(\eta) = a_0 f_0(\eta) + a_1 f_1(\eta), \quad (40)$$

where $a_0$ and $a_1$ are constants. By differentiating the above expression and using Eq. (39), we get

$$f'(\eta) = \frac{2\mu}{\sigma^2} f(\eta) + a_1,$$

and hence $a_1 = -2d/\sigma^2$ (cf. Eq. (15)). The coefficients $a_0$, $a_1$ are now determined by the conditions $f(\bar{\eta}) = 0$ and $F(\eta^*) = 1$. Upon substituting the trial solution (40), one obtains

$$a_0 f_0(\bar{\eta}) + a_1 f_1(\bar{\eta}) = 0, \quad a_0 F_0(\eta^*) + a_1 F_1(\eta^*) = 1.$$ 

The values of the functions $f_{0,1}$ and $F_{0,1}$ are given by the integration of Eqs. (39), and so solving for $a_0$ and $a_1$ is trivial. Note that if $f_0(\bar{\eta})/f_1(\bar{\eta}) = F_0(\eta^*)/F_1(\eta^*)$, the solutions $a_0$, $a_1$ fail to exist. In practice we have never encountered this situation.

The possibility for recapitalisation is tested by finding roots of

$$G(\tilde{q}, \bar{\eta}) = W[\tilde{q}, \eta^*(\tilde{q}, \bar{\eta})] - W(\tilde{q}, \bar{\eta}) - [\eta^*(\tilde{q}, \bar{\eta}) - \bar{\eta}] - \chi.$$ 

We have made the $\tilde{q}, \bar{\eta}$ dependence here explicit. Clearly $G = 0$ is equivalent to Eq. (12). Functions $\eta^*$, $q$ and $W$ are obtained using the method outlined above. First a coarse root bracketing is attempted by evaluating $G$ at $\tilde{q}_i = -\bar{\eta} + (q_1 + \bar{\eta})i/n_q$, where $i = 0 \ldots n_q$, $n_q$ an integer (we use $n_q = 10$), and $q_1$ is $q$ as given by Eq. (10) if that value is real, or $q_{\text{max}}$ if it is not. If sign of $G$ changes across a bracketing interval $(\tilde{q}_i, \tilde{q}_{i+1})$, the root is pin-pointed using standard root finding algorithms. If no roots are found, $\tilde{q} = -\bar{\eta}$ is taken as the initial value. As a final step in determining $\tilde{q}$, we evaluate $q^*$ and test if $q^* > q_{\text{max}}$. If the inequality holds, the input parameters are considered invalid.
C.2 Model with option to rent

The algorithm outline is same as in the model without the rental option. However, the solution near the lower boundary can become more involved as \( \psi = 0 \) if recapitalisation is not possible.

The differential equations for \( q \) can be solved by simple forward integration starting from \( q(\bar{\eta}) = \bar{q} \). If recapitalisation is available (\( \bar{q} > -\bar{\eta} \)), no singularities are present, and the equation for \( q \), Eq. (23), can be integrated directly to obtain \( q(\eta), \eta^* \), and now also \( \bar{\eta} \). The point \( \bar{\eta} \) is found in the same way as \( \eta^* \), i.e. by monitoring the function \( \psi - 1 \) as integration advances and polishing the root after a coarse approximation is found. Initial \( \bar{q} \) is found the same way as for the model without rental (but with \( q, W \) computed as described below).

If \( \bar{q} = -\bar{\eta} \), then \( \psi = 0 \) and singularities appear. As is shown in Appendix B, the derivative \( q'(\bar{\eta}) \) is finite. In order to evaluate it numerically, we use Eq. (33) since Eq. (23) is indeterminate at \( \bar{\eta} \) (in practice, numerical round-off would cause significant error in \( \bar{q} \)). Otherwise the solution of \( q \) proceeds the same way as with a recapitalising lower boundary.

Equations for \( f' \) and \( W' \) do not, contrary to \( q' \), tend to finite values at \( \bar{\eta} \), since \( \psi(\bar{\eta}) = 0 \) if \( q(\bar{\eta}) = -\bar{\eta}, q'(\bar{\eta}) > 0 \). Due to this divergence, the point \( \bar{\eta} \) cannot be reached by directly integrating the model equations, which in principle could be done backwards from, say, \( \bar{\eta} \) down to \( \bar{\eta} + \epsilon \), \( 0 < \epsilon \ll 1 \). Cutting the integration short in this way would lead to severe underestimation of the probability mass near \( \bar{\eta} \) if \( f \) diverges fast enough at this edge.

To resolve this issue, we use the analytically obtained power-law solutions, \( f_a \propto (\eta - \bar{\eta})^{\alpha - 2} \) (Eq. (27)), and \( W_a \propto (\eta - \bar{\eta})^3 \) (Eq. (31)), from \( \bar{\eta} \) up to a cross-over value \( \eta_x \). Numerical solutions are matched to the analytic ones so that the resulting functions are continuous. The cross-over point can determined by requiring that

\[
\left| \frac{f_a'(\eta_x)}{f_a(\eta_x)} \right| = \varepsilon^{-1},
\]

where \( 0 < \varepsilon \ll 1 \), implying that the divergent terms dominate the expression for the derivative of \( f \). However, since \( W' \) also tends to infinity, we write the same condition for \( W_a \) as well. This gives two different cross-over values, of which we will choose the smallest:

\[
\eta_x = \varepsilon \min(|\alpha|, \beta) + \bar{\eta},
\]

where \( \alpha \) is given by Eq. (37) and \( \beta = 1/(1 + q'(\bar{\eta})) \), with \( q'(\bar{\eta}) \) from Eq. (35). We typically use the value \( \varepsilon = 1.0 \times 10^{-3} \). Naturally, we use the analytic solution for \( f \) to obtain the cumulative density \( F \) below \( \eta_x \).
If the lower boundary is at $\bar{q} = -\bar{\eta}$, we can then directly integrate Eq. (26) with $d = 0$ from $\eta_x$ to $\eta^*$. The obtained solution can then be multiplied by a constant to make the cumulative distribution satisfy $F(\eta^*) = 1$. If $\bar{\eta}$ is absorbing (recapitalisation), we use the same trick as in the model without rent: we solve for $\phi_0$ and $\phi_1$ satisfying Eq. (26) with $d = 0$ and $d = 1$, respectively. The final $\phi$ is then constructed as a superposition of these two, $\phi = a_0\phi_0 + a_1\phi_1$. Coefficients $a_0$ and $a_1$ are determined from

$$\phi(\bar{\eta}) = 0, \quad \int_{\bar{\eta}}^{\eta^*} \frac{2}{(\sigma^2(\eta))} \phi(\eta) \, d\eta = 1.$$  

When needed, the same analytic solution, Eq. (27), can be used for both $\phi_0$ and $\phi_1$ ($\phi_{0,1} \propto (\eta - \bar{\eta})^\alpha/(\sigma^2)^2$), since $d$ term is negligible near $\bar{\eta}$.

Note that reverting to the analytic solution for $f$ is equivalent to using a truncated integration range with an additional correction term coming from the analytical solution near $\bar{\eta}$. Numerical simulations confirm that this approach is sound: (i) the analytical and numerical solutions are in very good agreement across a wide range of $\eta$, (ii) the obtained solutions are independent of $\epsilon$ provided it is small enough while keeping the numerical solution from reaching the singularity, and (iii) qualitative features of the solution do not change if the analytical correction is omitted.

We do not consider the possibility that $\bar{q} > -\bar{\eta}$ and $\psi(\bar{\eta}) = 0$, again, this does not occur when $\bar{\eta}$ follows Eq. (11). Extensions of the model where a more general $\bar{\eta}$ holds can be numerically solved by applying the analytical formulas for limit behaviour of $q$ and $\phi$ given in Appendix B.

References


