

# On the real roots of the Bernoulli polynomials and the Hurwitz zeta-function

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**Abstract.** The behaviour of the real roots of the Bernoulli polynomials  $B_m(a)$  for large  $m$  is investigated. It is shown that if  $N(m)$  is the number of these roots then

$$\lim_{m \rightarrow \infty} \frac{N(m)}{m} = \frac{2}{\pi e}$$

We also show that on the interval  $I_m : -[\frac{m-2}{2\pi e}] < a < 1 + [\frac{m-2}{2\pi e}]$ , the roots of  $B_m(a)$  are close to the half-integer lattices:  $a = 0, \pm 1/2, \pm 1, \pm 3/2, \pm 2, \dots$  if  $m$  is odd, and  $a = \pm 1/4, \pm 3/4, \pm 5/4, \pm 7/4, \dots$  if  $m$  is even. The proof is based on the relation between  $B_m(a)$  and the generalised Riemann zeta-function (Hurwitz zeta-function)  $\zeta(s, a)$ .

## 1 Introduction.

The classical *Bernoulli polynomials*  $B_k(a)$  are defined through the generating function:

$$\frac{ze^{za}}{e^z - 1} = \sum_{k=0}^{\infty} \frac{B_k(a)}{k!} z^k :$$

$$B_0(a) = 1 \quad B_1(a) = a - \frac{1}{2} \quad B_2(a) = a^2 - a + \frac{1}{6} \quad B_3(a) = a^3 - \frac{3a^2}{2} + \frac{a}{2}, \dots$$

The *Bernoulli numbers* are defined through the relation:  $B_k = B_k(0)$ . Bernoulli polynomials and Bernoulli numbers possess many interesting properties and arise in many areas of mathematics (see [1, 2]).

In this paper we will discuss the behaviour of the real roots of the Bernoulli polynomials  $B_m(a)$  for large values of the index  $m$ . We will prove that, for sufficiently large  $m$

- on the interval  $I_m : -\left[\frac{m-2}{2\pi e}\right] < a < 1 + \left[\frac{m-2}{2\pi e}\right]$ , where the square brackets refer to the integer part, the roots of  $B_m(a)$  are close to the half-integer lattices:  $a = 0, \pm 1/2, \pm 1, \pm 3/2, \pm 2, \dots$  if  $m$  is odd, and  $a = \pm 1/4, \pm 3/4, \pm 5/4, \pm 7/4, \dots$  if  $m$  is even;
- for any constant  $\alpha > 1$  the largest real root  $A(m)$  of  $B_m(a)$  satisfies the inequality

$$\frac{m}{2\pi e} - \frac{1}{2} < A(m) < \frac{m}{2\pi e} + \frac{\alpha}{4\pi e} \log m \quad (1)$$

- the number  $N(m)$  of the real roots of the Bernoulli polynomials  $B_m(a)$  satisfies a similar inequality

$$1 + 4 \left[ \frac{m-2}{2\pi e} \right] < N(m) < \frac{2m}{\pi e} + \alpha \log m$$

In particular,

$$\lim_{m \rightarrow \infty} \frac{N(m)}{m} = \frac{2}{\pi e}. \quad (2)$$

We should mention that in the recent paper [4] by S.C.Woon the regular behaviour of the real roots of Bernoulli polynomials has been predicted but on a different larger interval. One of the Woon's conjectures (see [4], page 10) states:

$$\lim_{m \rightarrow \infty} \frac{N(m)}{m} = \frac{1}{5} + \frac{4}{105} = \frac{5}{21},$$

which is actually larger than  $\frac{2}{\pi e}$ :  $\frac{5}{21} = 0.23809\dots > 0.234199\dots = \frac{2}{\pi e}$ . The results of [4] were based on a computer analysis of the real roots of the Bernoulli polynomials  $B_m(a)$  for  $m \leq 80$ .

Our approach is based on the relation between the Bernoulli polynomials and the *generalised Riemann zeta-function* also known as the *Hurwitz zeta-function* (see [1, 2, 3]):

$$\zeta(s, a) = \sum_{r=0}^{\infty} (a+r)^{-s} \quad a \neq 0, -1, -2, \dots$$

When  $a = 1$  it reduces to the classical Riemann zeta-function. It is also known (see e.g. [2], volume 1, page 27) that in the special cases when  $s$  is a negative integer this function (as a function of the parameter  $a$ ) reduces, up to a factor, to a Bernoulli polynomial: explicitly when  $s = -m, m = 0, 1, 2, 3, \dots$

$$\zeta(-m, a) = -\frac{B_{m+1}(a)}{m+1}.$$

Hurwitz [5] has found the following remarkable representation of the  $\zeta(s, a)$  in the domain  $0 < a \leq 1$  and  $\text{Re}(s) = \sigma < 0$ :

$$\zeta(s, a) = \frac{2\Gamma(1-s)}{(2\pi)^{1-s}} \sum_{r=1}^{\infty} \frac{\sin(2\pi r a + \frac{1}{2}\pi s)}{r^{1-s}},$$

in which  $\Gamma$  is the classical Euler gamma-function (see e.g [2], volume 1, page 26).

Our main observation is that for a large negative  $\sigma$  this formula gives a good approximation for  $\zeta(\sigma, a)$  on a much wider interval:  $0 < a < -\frac{\sigma}{2\pi e}$ . As a result we prove that

$$\frac{\zeta(\sigma, a)}{Q(\sigma)} = \sin(2\pi a + \frac{1}{2}\pi\sigma) + o(1) \quad \text{as } \sigma \rightarrow -\infty \quad \text{provided } 0 < a < \frac{-\sigma}{2\pi e}$$

where  $Q(s) = \frac{2\Gamma(1-s)}{(2\pi)^{1-s}}$ .

In particular for the integer  $\sigma = -m + 1$  we can prove that the Bernoulli polynomials  $B_m(a)$  behave sinusoidally on the interval  $I_m : -[\frac{m-1}{2\pi e}] < a < 1 + [\frac{m-1}{2\pi e}]$ :

$$\frac{B_m(a)}{Q_m} = \cos(2\pi a - \frac{1}{2}\pi m) + o(1) \quad \text{as } m \rightarrow \infty,$$

where  $Q_m = -\frac{2(m!)}{(2\pi)^m}$ . This is in good agreement with the computer-generated graphs of the Bernoulli polynomials (see fig.1). We should mention that because of the rapid growth of the term  $|Q_m|$  (for example  $|Q_{30}| \approx 6 \times 10^8$ ), the sinusoidal behaviour in the Bernoulli polynomials is easy to overlook.

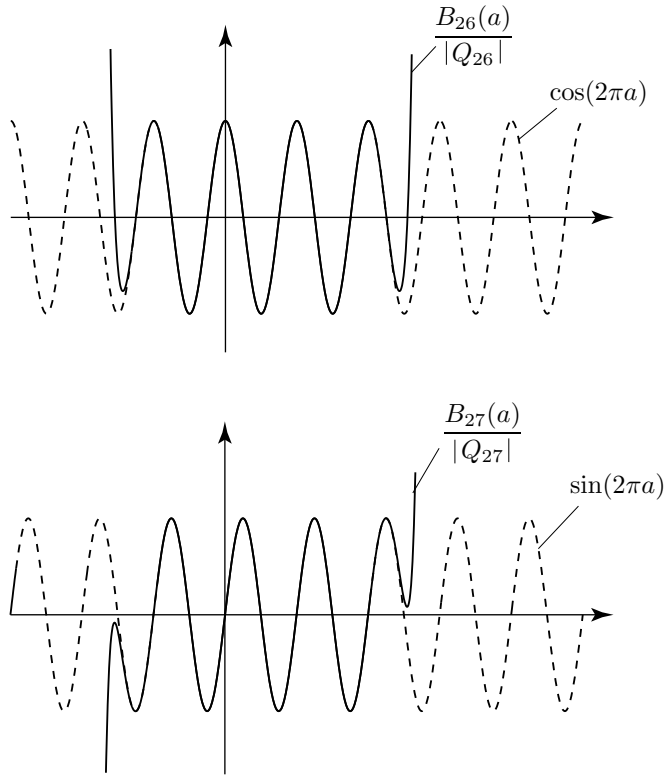


Figure 1

We show also that for any  $c > \frac{1}{4\pi e}$  in the region  $a > \frac{-\sigma}{2\pi e} + c \log(-\sigma)$  the Hurwitz zeta-function has no real roots for large negative  $\sigma$ . This implies the asymptotic formulas (1), (2) for the roots of the Bernoulli polynomials.

## 2 Asymptotic behaviour of the Hurwitz zeta-function $\zeta(\sigma, a)$ for large negative $\sigma$ .

The Hurwitz zeta-function (or generalised Riemann zeta-function) is defined as a series

$$\zeta(s, a) = \sum_{r=0}^{\infty} (a+r)^{-s} \quad a \neq 0, -1, -2, \dots$$

in the complex domain  $\text{Re}(s) > 1$  and can be analytically continued to a meromorphic function in the whole complex plane with the only pole at  $s = 1$  (see [1],[2],[3]). When  $a = 1$  it reduces to the classical Riemann zeta-function

$$\zeta(s) = \sum_{k=1}^{\infty} k^{-s}.$$

The Hurwitz zeta-function can be extended to the whole of the complex  $s$ -plane through the formula

$$\zeta(s, a) = -\frac{\Gamma(1-s)}{2\pi i} \int_{\infty}^{(0+)} \frac{(-z)^{s-1} e^{-az}}{1-e^{-z}} dz \quad a > 0$$

in which the integral is taken over a curve starting at ‘infinity’ on the real axis, encircles the origin in a positive direction and returns to the starting point (see [2]). By using an alternative integral formulation for  $\zeta(s, a)$  it can be shown that  $\zeta(s, a)$  is analytic everywhere except for the simple pole at  $s = 1$ .

The Hurwitz zeta-function obviously satisfies the functional relation:

$$\zeta(s, a) = \zeta(s, n+a) + \sum_{r=0}^{n-1} (r+a)^{-s} \quad n = 1, 2, \dots \quad (3)$$

Since each term in this relation is analytic we can assume this relation is true for the whole of the complex  $s$  plane, except for  $s = 1$ .

In this paper we restrict ourselves to the case when  $s$  is real:  $s = \sigma \in \mathbb{R}$ . When  $\sigma$  is negative Hurwitz has found the following Fourier representation for  $\zeta(\sigma, a)$  on the interval  $0 < a \leq 1$ :

$$\zeta(\sigma, a) = \frac{2\Gamma(1-\sigma)}{(2\pi)^{1-\sigma}} \sum_{r=1}^{\infty} \frac{\sin(2\pi r a + \frac{1}{2}\pi\sigma)}{r^{1-\sigma}} \quad (4)$$

From this formula we see that

$$\frac{\zeta(\sigma, a)}{Q(\sigma)} = \sin(2\pi a + \frac{1}{2}\pi\sigma) + o(1) \quad \text{when } \sigma \rightarrow -\infty,$$

where  $Q(\sigma) = \frac{2\Gamma(1-\sigma)}{(2\pi)^{1-\sigma}}$ . Our first theorem proves that this is actually true on a much larger interval.

As part of the theorem proofs we will use the following inequality for the function  $S(p, n) = 1^p + 2^p + \dots + n^p$ :

$$S(p, n) < n^p \left( \frac{1 - e^{-p}}{1 - e^{-p/n}} \right)$$

Indeed,

$$S(p, n) = n^p \left( 1 + \left(1 - \frac{1}{n}\right)^p + \left(1 - \frac{2}{n}\right)^p + \dots + \left(1 - \frac{n-1}{n}\right)^p \right)$$

But since  $1 - x < e^{-x}$  for  $x < 1$  we have

$$1 - \frac{1}{n} < e^{-1/n} \quad 1 - \frac{2}{n} < e^{-2/n} \quad \dots \quad 1 - \frac{n-1}{n} < e^{-(n-1)/n}$$

and therefore

$$S(p, n) < n^p \left( 1 + e^{-p/n} + e^{-2p/n} + \dots + e^{-(n-1)p/n} \right) = n^p \left( \frac{1 - e^{-p}}{1 - e^{-p/n}} \right).$$

We shall also need an estimate for the sum of the series:  $\sum_{r=2}^{\infty} \frac{1}{r^{1+p}}$ . Now

$$\frac{1}{2^{1+p}} + \frac{1}{3^{1+p}} + \frac{1}{4^{1+p}} + \dots = \frac{1}{2^p} \left( \frac{1}{2} + \frac{1}{3} \left(\frac{2}{3}\right)^p + \frac{1}{4} \left(\frac{2}{4}\right)^p + \dots \right)$$

But  $\left(\frac{2}{n}\right)^p < \frac{1}{n}$  if  $p > 4$  and  $n \geq 3$  so we easily deduce

$$\sum_{r=2}^{\infty} \frac{1}{r^{1+p}} < (\zeta(2) - \frac{3}{4}) 2^{-p} \quad p > 4$$

Finally, we will also make particular use of Stirling's inequality for the gamma function:

$$(2\pi p)^{\frac{1}{2}} p^p e^{-p} < \Gamma(1+p) < (2\pi p)^{\frac{1}{2}} p^p e^{-p} e^{\frac{1}{12p}} \quad p \gg 1$$

**Theorem 1.** Let  $\sigma = -p, p \geq 0$  and  $0 < a < \alpha p$  where  $\alpha < \frac{1}{2\pi e}$  then the Hurwitz zeta-function satisfies the inequality

$$\left| \frac{\zeta(-p, a)}{Q(-p)} - \sin(2\pi a - \frac{1}{2}\pi p) \right| < C_1 p^{-1/2} (2\pi e \alpha)^p + C_2 2^{-p}, \quad (5)$$

where  $C_1, C_2$  are constants, which do not depend on  $p$ . In particular, on the interval  $0 < a < \frac{1}{2\pi e} p$  we have the asymptotic behaviour

$$\frac{\zeta(-p, a)}{Q(-p)} = \sin(2\pi a - \frac{1}{2}\pi p) + o(1) \quad \text{when } p \rightarrow \infty.$$

### Proof

Using the functional relation (3) we have for any integer  $n$

$$\left| \frac{\zeta(\sigma, n+a)}{Q(\sigma)} - \frac{\zeta(\sigma, a)}{Q(\sigma)} \right| \leq \frac{1}{|Q(\sigma)|} \sum_{r=0}^{n-1} (r+a)^{-\sigma}$$

Now assuming that  $0 < a \leq 1$  we obviously have

$$\sum_{r=0}^{n-1} (r+a)^{-\sigma} \leq S(p, n) \quad 0 < p = -\sigma$$

and, as we obtained above,

$$S(p, n) < n^p \left( \frac{1 - e^{-p}}{1 - e^{-p/n}} \right).$$

Also, from the Stirling formula for the  $\Gamma$ -function we have the following asymptotically exact inequality  $\Gamma(1 + p) > (2\pi p)^{1/2} p^p e^{-p}$  and therefore

$$\frac{1}{Q(-p)} = \frac{(2\pi)^{1+p}}{2\Gamma(1+p)} < \left\{ \frac{2\pi e}{p} \right\}^p \frac{\pi}{\sqrt{2\pi p}}$$

Thus for a large  $p$

$$\begin{aligned} \left| \frac{\zeta(-p, n+a)}{Q(-p)} - \frac{\zeta(-p, a)}{Q(-p)} \right| &< \left\{ \frac{2\pi en}{p} \right\}^p \frac{\pi}{\sqrt{2\pi p}} \left( \frac{1 - e^{-p}}{1 - e^{-p/n}} \right) \\ &< \left\{ \frac{2\pi en}{p} \right\}^p \frac{\pi}{\sqrt{2\pi p}} \frac{1}{(1 - e^{-\frac{1}{\alpha}})} \quad \text{if } \frac{n}{p} < \alpha \end{aligned} \quad (6)$$

However, from Hurwitz' formula (4) it follows that

$$\frac{\zeta(-p, a)}{Q(-p)} = \sum_{r=1}^{\infty} \frac{\sin(2\pi r a - \frac{1}{2}\pi p)}{r^{1+p}} = \sin(2\pi a - \frac{1}{2}\pi p) + \frac{\sin(4\pi a - \frac{1}{2}\pi p)}{2^{1+p}} + \frac{\sin(6\pi a - \frac{1}{2}\pi p)}{3^{1+p}} + \dots$$

Therefore

$$\left| \frac{\zeta(-p, a)}{Q(-p)} - \sin(2\pi a - \frac{1}{2}\pi p) \right| < \sum_{r=2}^{\infty} r^{-p-1} < (\zeta(2) - \frac{3}{4}) 2^{-p} \quad (7)$$

if  $p > 4$ . The estimates (6) and (7) imply the theorem.

By a slight modification of the previous arguments we can prove the following:

**Theorem 2** For any constant  $c > \frac{1}{4\pi e}$  there exists  $p_0(c)$  such that  $\zeta(-p, a)$  is negative for any  $p > p_0$  and  $a > \frac{p}{2\pi e} + c \log p$ .

**Proof** From the same functional relation (3)

$$\frac{\zeta(-p, a+n)}{Q(-p)} < \frac{\zeta(-p, a)}{Q(-p)} - \frac{S(p, n-1)}{Q(-p)} < \frac{\zeta(-p, a)}{Q(-p)} - \frac{(n-1)^p}{Q(-p)}$$

Now we can use Stirling's inequality  $\Gamma(1+p) < \sqrt{2\pi p} \left(\frac{p}{e}\right)^p e^{\frac{1}{12p}}$ , so

$$Q(-p) = \frac{2\Gamma(1+p)}{(2\pi)^{1+p}} < \frac{\sqrt{2\pi p}}{\pi} \left(\frac{p}{2\pi e}\right)^p e^{\frac{1}{12p}}$$

However, we know from the Hurwitz formula, that when  $p \rightarrow \infty$

$$\frac{\zeta(-p, a)}{Q(-p)} = \sin(2\pi a - \frac{1}{2}\pi p) + o(1)$$

Therefore if  $\frac{(n-1)^p}{Q(-p)} > 1$  and  $p$  large enough then  $\zeta(-p, a+n) < 0$ . But as we have shown

$$\frac{(n-1)^p}{Q(-p)} > \left( \frac{2\pi e(n-1)}{p} \right)^p \sqrt{\frac{\pi}{2p}} e^{-\frac{1}{12p}}$$

which is greater than 1 if

$$n - 1 > \frac{p}{2\pi e} \left( \frac{2p}{\pi} \right)^{\frac{1}{2p}} e^{\frac{1}{12p^2}} = \frac{p}{2\pi e} e^{\left\{ \frac{(\log 2p - \log \pi)}{2p} + \frac{1}{12p^2} \right\}}$$

Now, using the inequality  $e^x < 1 + \alpha x$  if  $\alpha > 1$  and  $x$  is sufficiently small we find that  $\frac{(n-1)^p}{Q(-p)} > 1$  if

$$n - 1 > \frac{p}{2\pi e} + \frac{\alpha}{4\pi e} \log p$$

where  $\alpha > 1$  and  $p$  is sufficiently large. This implies the theorem.

### 3 The Real Roots of the Bernoulli Polynomials.

As we have already mentioned, when  $p = -\sigma = m$ ,  $m \in \mathbb{Z}^+$  the Hurwitz-zeta function reduces to certain polynomials related in a simple way to the Bernoulli polynomials:

$$\zeta(-m, a) = -\frac{B_{m+1}(a)}{m+1} \quad (8)$$

Because of the well-known symmetry properties of the Bernoulli polynomials

$$B_m(1-a) = (-1)^m B_m(a) \quad (9)$$

we can extend all of the results we have for positive roots of  $B_m(a)$  to the general case. In particular we can apply the results of the previous section to the problem of real roots of Bernoulli polynomials.

Let  $N(m)$  be the number of real roots of  $B_m(a)$ , and  $A(m)$  be the largest of these roots.

**Theorem 3** *Let  $\alpha$  be any constant such that  $\alpha > 1$ . For  $m$  sufficiently large*

$$1 + 4 \left[ \frac{m-2}{2\pi e} \right] < N(m) < \frac{2m}{\pi e} + \alpha \log m \quad (10)$$

$$\frac{m}{2\pi e} - \frac{1}{2} < A(m) < \frac{m}{2\pi e} + \frac{\alpha}{4\pi e} \log m \quad (11)$$

*The roots of  $B_m(a)$  on the interval  $-\left[\frac{m-2}{2\pi e}\right] < a < 1 + \left[\frac{m-2}{2\pi e}\right]$  are simple and close to the half-integer lattices:*

$$a = 0, \pm 1/2, \pm 1, \pm 3/2, \pm 2, \dots \text{ if } m \text{ is odd, and}$$

$$a = \pm 1/4, \pm 3/4, \pm 5/4, \pm 7/4, \dots \text{ if } m \text{ is even;}$$

**Proof** From theorem 1 and (8) it follows that

$$\frac{B_m(a)}{Q_m} = \cos(2\pi a - \frac{1}{2}\pi m) + o(1)$$

on the interval  $I_m : -\left[\frac{m-1}{2\pi e}\right] < a < 1 + \left[\frac{m-1}{2\pi e}\right]$  when  $m \rightarrow \infty$ . Actually this is true also for the  $k^{\text{th}}$  derivative of  $\frac{B_m(a)}{Q_m}$  but on a smaller interval:  $I_{m-k-1}$ . Indeed,

$$B_m^{(k)}(a) = m(m-1)\dots(m-k+1)B_{m-k}(a)$$

and

$$Q_m = -\frac{2m!}{(2\pi)^m} = \frac{m(m-1)\dots(m-k+1)}{(2\pi)^k} Q_{m-k}$$

Now, according to theorem 1 and to (8) when  $m \rightarrow \infty$

$$\begin{aligned} \frac{B_m^{(k)}(a)}{Q_m} &= \frac{m(m-1)\dots(m-k+1)B_{m-k}(a)}{m(m-1)\dots(m-k+1)Q_{m-k}}(2\pi)^k \\ &= (2\pi)^k \cos\left(2\pi a - \frac{1}{2}\pi(m-k)\right) + o(1) = \cos^{(k)}\left(2\pi a - \frac{1}{2}\pi m\right) + o(1) \\ &\text{if } -\left[\frac{m-k-1}{2\pi e}\right] < a < 1 + \left[\frac{m-k-1}{2\pi e}\right]. \end{aligned}$$

In particular, on the interval  $I_{m-2}$  the function  $\frac{B_m(a)}{Q_m}$  (and its derivative) tend to  $\cos(2\pi a - \frac{1}{2}\pi m)$  (and its derivative) when  $m \rightarrow \infty$  which ensures that for large  $m$  all the roots of  $B_m(a)$  on this interval are simple and located near the points  $a = \frac{m+1}{4} + \frac{k}{2}$   $k \in \mathbb{Z}^+$ . This implies the last statement of the theorem and the lower estimates of (10) and (11).

The upper estimates for  $A(m)$  follows directly from theorem 2. To prove the upper estimates for  $N(m)$  we need the following simple lemma.

**Lemma** If a function  $f(x)$  (with a continuous  $n^{\text{th}}$  derivative) on some interval  $(a, b)$  has the property that the sign of the  $n^{\text{th}}$  derivative is constant throughout the interval then  $f$  has no more than  $n$  roots on this interval.

Now we apply this lemma to the function  $f(a) \equiv B_{m+1}(a)$  on the interval  $J_m : (\frac{m}{2\pi e}, \frac{m}{2\pi e} + \frac{\alpha}{4\pi e} \log m)$  to estimate the number of roots there. The idea of this calculation is clear from figure 2 (in which  $\kappa \equiv \frac{1}{2\pi e}$ ).

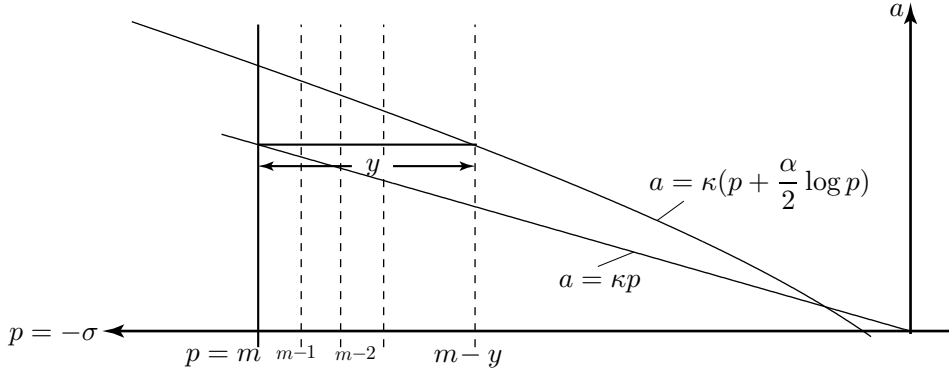


Figure 2

Using the well-known fact that

$$B'_{m+1}(a) = (m+1)B_m(a)$$

(see e.g. [2]) we differentiate  $B_{m+1}$  many times until we have a positive function and then apply the lemma. As one can see from Fig. 2 if  $n > [y]$  then  $B_{m+1}^{(n)}(a)$  will be positive in the interval  $J_m$  and, as such, cannot have more than  $n$  simple roots in this interval. Now  $y$  is the solution to the equation

$$\kappa \left( (m-y) + \frac{\alpha}{2} \log(m-y) \right) = \kappa m$$

or

$$-y + \frac{\alpha}{2} \log(m - y) = 0$$

We claim that the solution to this equation satisfies the inequality

$$y < \frac{\alpha}{2} \log m$$

Indeed the function  $F(y) = -y + \frac{\alpha}{2} \log(m - y)$  is monotonically decreasing and

$$F\left(\frac{\alpha}{2} \log m\right) = -\frac{\alpha}{2} \log m + \frac{\alpha}{2} \log\left(m - \frac{\alpha}{2} \log m\right) < 0.$$

Thus according to the lemma  $B_{m+1}(a)$  has no more than  $\frac{\alpha}{2} \log m$  roots on the interval  $(\frac{m}{2\pi e}, \frac{m}{2\pi e} + \frac{\alpha}{4\pi e} \log m)$ . Using the symmetry relations (9) of  $B_{m+1}(a)$  and the regular behaviour of the roots on the middle interval we obtain the inequality (10). This proves theorem 3.

## 4 Concluding Remarks.

In this paper we have proved the regular lattice behaviour of almost all of the real roots of the Bernoulli polynomials  $B_m(t)$  and found asymptotic formulae for the number of these roots, for large values of the index  $m$ . It is surprising that these results seem to have not been noted before, particularly as the proofs are elementary. The only relevant paper we have found is [4] by S.C.Woon, who observed the regular behaviour of the real roots of  $B_m(t)$  using a numerical investigation. We should mention that our interest also started from a graphical analysis of the Bernoulli polynomials using computer-generated graphs.

Computer experiments carried out by S.Woon also demonstrate a remarkably regular structure of the complex roots of  $B_m(t)$ . It would be very interesting to find a mathematical explanation for this and to find explicitly the curves along which the complex roots of  $B_m(t)$  lie.

As we have seen, this problem is closely related to the problem of the behaviour of the zeroes of the Hurwitz-zeta function  $\zeta(s, a)$ , and therefore belongs to the circle of problems connected to the famous Riemann hypothesis (see [3]). However, we believe that our problem is not as difficult and we hope to make progress in this direction in the future.

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