Egyptian fraction representations of 1 with odd denominators

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Abstract

The two optimisation problems associated with the representation of 1 by Egyptian fractions with odd denominators are solved. The unique solution with denominators up to 105 is given by 3, 5, 7, 9, 11, 33, 35, 45, 55, 77, 105. There are five solutions when only nine denominators are used.

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Abstract

The two optimisation problems associated with the representation of 1 by Egyptian fractions with odd denominators are solved. The unique solution with denominators up to 105 is given by 3, 5, 7, 9, 11, 33, 35, 45, 55, 77, 105. There are five solutions when only nine denominators are used.

1. *Statement of results*

By an Egyptian fraction with length ℓ we mean an expression of the form

$$\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_\ell},$$

where a_i are distinct positive integers. We consider the representation of 1 by such fractions with denominators greater than 1, the simplest being

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{6}=1.$$

The problem becomes much more complicated if we impose the condition that the denominators used are not divisible by the first few primes. In particular, the problem of finding a representation with odd denominators is already interesting. Thus we are concerned with solutions to the equation

$$\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_\ell}=1, \qquad 3\le a_1<a_2<\cdots<a_\ell, \qquad (a_i,2)=1; \tag{1.1}$$

note that ℓ has to be odd. John Leech (see [**1**], Problem D11) found the solution

$$\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\frac{1}{15}+\frac{1}{21}+\frac{1}{27}+\frac{1}{35}+\frac{1}{63}+\frac{1}{105}+\frac{1}{135}=1, \tag{1.2}$$

which has $\ell=11$ and $a_\ell=135$; he also remarked that any solution must have $a_\ell\ge 105$ and $\ell\ge 9$.

We solve the two optimisation problems posed by Leech, showing in particular that his bounds are sharp. Thus, we now have the following solution

$$\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\frac{1}{11}+\frac{1}{33}+\frac{1}{35}+\frac{1}{45}+\frac{1}{55}+\frac{1}{77}+\frac{1}{105}=1, \tag{1.3}$$

which has $a_\ell=105$, and also the following five solutions which have $\ell=9$:

$$\left.\begin{aligned}
\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\frac{1}{11}+\frac{1}{15}+\frac{1}{21}+\frac{1}{135}+\frac{1}{10395}&=1,\\
\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\frac{1}{11}+\frac{1}{15}+\frac{1}{21}+\frac{1}{165}+\frac{1}{693}&=1,\\
\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\frac{1}{11}+\frac{1}{15}+\frac{1}{21}+\frac{1}{231}+\frac{1}{315}&=1,\\
\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\frac{1}{11}+\frac{1}{15}+\frac{1}{33}+\frac{1}{45}+\frac{1}{385}&=1,\\
\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\frac{1}{11}+\frac{1}{15}+\frac{1}{35}+\frac{1}{45}+\frac{1}{231}&=1.
\end{aligned}\right\} \tag{1.4}$$

We prove the following two theorems:

Theorem 1. *Any solution to* (1.1) *must have* $a_\ell \geq 105$*, and* (1.3) *is the unique solution with* $a_\ell = 105$.

Theorem 2. *Any solution to* (1.1) *must have* $\ell \geq 9$*, and the solutions* (1.4) *are the only ones with* $\ell = 9$.

It is clear that, for an Egyptian fraction representation of an *integer*, a prime power which divides some denominator must also divide another denominator. This is the crucial Diophantine requirement, which we use to develop into a constructive method whereby all the solutions with a_ℓ below a given a bound can be found. When trying to show that the solution (1.2) given by Leech was optimal we discovered that there are 29 solutions with $a_\ell < 135$, the optimal one being that in (1.3). A motivated account of this is given in the proof of Theorem 1, instead of just establishing that there is no solution with $a_\ell < 105$.

The nature of the two problems necessarily involves some tedious computations, while the mathematics used is entirely elementary. Indeed the proof of Theorem 2 may even be considered crude, but we do not wish to lengthen the argument merely to reduce an already trivial amount of calculations done on a machine.

2. Proof of Theorem 1

We let $A \subset \{3, 5, 7, \ldots, 133\}$, write $f(A)$ for the Egyptian fraction with denominators $a \in A$, and proceed to find all such sets A with $f(A) = 1$. For the bound 133, the largest admissible odd multiples of the primes p are given by $b = 1$ for $47 \leq p \leq 133$; $b = 3$ for $29 \leq p \leq 43$; $b = 5$ for $p = 23$; $b = 7$ for $p = 17, 19$; and $b = 9$ for $p = 13$.

For each of the 31 non-zero vector $(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5)$, where $\epsilon_i = 1$ or 0, we let

$$\epsilon_1 + \frac{\epsilon_2}{3} + \frac{\epsilon_3}{5} + \frac{\epsilon_4}{7} + \frac{\epsilon_5}{9} = \frac{L}{M}, \qquad (L, M) = 1. \tag{2.1}$$

If a prime $p \geq 13$ does not divide any of the 31 values L in (2.1) then, by the Diophantine requirement, no member of A can be a multiple of p. The argument applies also to a prime power, with an obvious modification. Of the 31 equations in (2.1) only the following four are relevant to our investigations

$$1 + \frac{1}{3} + \frac{1}{5} = \frac{23}{15}, \quad 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} = \frac{11\times16}{105}, \quad 1 + \frac{1}{3} + \frac{1}{9} = \frac{13}{9}, \quad \frac{1}{5} + \frac{1}{7} + \frac{1}{9} = \frac{11\times13}{315}. \tag{2.2}$$

Thus members of A are not divisible by primes $p \geq 17$, except possibly $p = 23$, and we also find that $25, 27, 49, 75, 81, 125 \notin A$. We now write

$$A_1 = 23\{1, 3, 5\}, \quad A_2 = 11\{1, 3, 5, 7\}, \quad A_3 = 13\{1, 3, 9\}, \quad A_4 = 11\{5, 7, 9\}, \quad A_5 = 13\{5, 7, 9\}, \tag{2.3}$$

where $p\{a, b, c\}$ means $\{pa, pb, pc\}$. Let

$$A_0 = \{d : 1 < d < h,\ d|h\}, \qquad h = 315, \tag{2.4}$$

the set of 10 proper divisors of $h = 3^2 5\times7$, so that $A_0 = \{3, 5, 7, 9, 15, 21, 35, 45, 63, 105\}$, and note that $a \in A_0$ if and only if $h/a \in A_0$. According to our argument A must be the union of a subset of A_0 and the whole of one or more of the sets A_i in (2.3). We note that $A_i \cap A_j$ is empty, except for $\{i, j\} = \{2, 4\}, \{3, 5\}$. Therefore the five sets A_i in (2.3) form 17 disjoint unions, to each of which we further attach the set A_0.

The six Egyptian fractions $f(A_i)$ have the following values

$$f(A_i) = \frac{44}{45}, \ \frac{1}{15}, \ \frac{16}{105}, \ \frac{1}{9}, \ \frac{13}{315}, \ \frac{11}{315}, \qquad i = 0, 1, 2, 3, 4, 5, \tag{2.5}$$

and the sum of these six numbers is $1 + 121/h$. The Egyptian fractions corresponding to the 17 disjoint unions have values $1 + r/h$, where $1 < r < 121$. We call r the *residue* associated with either the union or its corresponding Egyptian fraction, and we define the *valency* of r to be the number of ways that r can be decomposed into a sum of distinct members of A_0 in (2.4). All the solutions with $a_\ell \le 133$ can now be found, and the number of solutions is the sum of the 17 valencies. We proceed to search for one with $a_\ell \le 105$.

By (2.3) any solution obtained from a union of sets involving A_1, A_3, A_5 must contain a denominator having the value $23{\times}5 = 115$ or $13{\times}9 = 117$. Consequently, if there is a solution with $a_\ell \le 105$ it can only come from a union of A_0 together with either A_2 or A_4. By (2.5) the union $A_0 \cup A_2$ has the residue 41, with a valency 1. In fact, we have

$$f(A_0 \cup A_2) = \frac{44}{45} + \frac{16}{105} = 1 + \frac{41}{315} = 1 + \frac{5+15+21}{315} = 1 + \frac{1}{63} + \frac{1}{21} + \frac{1}{15}.$$

Thus 41 has the unique decomposition as the sum of $a = 5, 15, 21 \in A_0$, with their conjugate divisors $h/a = 63, 21, 15 \in A_0$. We therefore have a solution to (1.1), namely

$$f(A_0 \cup A_2 \setminus \{15, 21, 63\}) = \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{35} + \frac{1}{45} + \frac{1}{105} + \frac{1}{11}\Big(1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7}\Big) = 1,$$

which is (1.3). Moreover, since the residue of $A_0 \cup A_4$ is 6, which has a valency 0, the solution here is the only one with $a_\ell \le 105$. The theorem is proved.

The union $A_0 \cup A_5$ has the residue 4, which also has valency 0. The residues r corresponding to the remaining 14 unions satisfying $14 \le r \le 97$, all with valency 1, 2 or 3. The sum of all 17 valencies is 29, the number of solutions to (1.1) with $a_\ell \le 133$. For an example of a union with valency 2, we take $A_0 \cup A_1 \cup A_4$, which has the residue 27, with the decompositions $27 = 5 + 7 + 15 = 3 + 9 + 15$. The two corresponding solutions are

$$\frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{15} + \frac{1}{35} + \frac{1}{105} + \frac{1}{23}\Big(1 + \frac{1}{3} + \frac{1}{5}\Big) + \frac{1}{11}\Big(\frac{1}{5} + \frac{1}{7} + \frac{1}{9}\Big) = 1,$$

and

$$\frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{15} + \frac{1}{45} + \frac{1}{63} + \frac{1}{23}\Big(1 + \frac{1}{3} + \frac{1}{5}\Big) + \frac{1}{11}\Big(\frac{1}{5} + \frac{1}{7} + \frac{1}{9}\Big) = 1.$$

3. Proof of Theorem 2

It is almost trivial that (1.1) has no solution if $\ell = 7$. For if $a_7 \ge 23$ then

$$\frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} + \frac{1}{23} < 1,$$

and it is obvious that there is no solution if $a_7 = 15, 17, 19, 21$.

We now fix $\ell = 9$ in (1.1), so that $a_9 \ge 105$ by Theorem 1; in fact $a_9 \ge 135$ if we consider the proof of Theorem 1, but the smaller lower bound is more than enough.

First, from

$$\frac{1}{3} + \frac{1}{5} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} + \frac{1}{15} + \frac{1}{17} + \frac{1}{19} + \frac{1}{105} < 1,$$

we deduce at once that $a_1 = 3$, $a_2 = 5$, $a_3 = 7$. Similarly we find that $a_4 = 9$ or 11, and from

$$\frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} + \frac{1}{67} + \frac{1}{69} + \frac{1}{105} < 1,$$

we also deduce that $a_7 \le 65$.

Next, for each of the $\binom{28}{3} + \binom{27}{3} = 6201$ sets of odd numbers (a_4, a_5, a_6, a_7) satisfying

$$a_4 = 9, 11, \qquad a_4 < a_5 < a_6 < a_7 \le 65,$$

we write

$$\frac{2m}{n} = 1 - \Big(\frac{1}{3} + \frac{1}{5} + \frac{1}{7}\Big) - \Big(\frac{1}{a_4} + \frac{1}{a_5} + \frac{1}{a_6} + \frac{1}{a_7}\Big). \tag{3.1}$$

Then a necessary condition for (1.1) to be soluble is

$$0 < \frac{2m}{n} \le \frac{1}{a_7 + 2} + \frac{1}{105},$$

which is satisfied by only 78 values $2m/n$ given by (3.1). Moreover, on rewriting (1.1) as

$$\frac{1}{a_8} + \frac{1}{a_9} = \frac{2m}{n}, \tag{3.2}$$

the condition $a_8 < a_9$ now implies $a_8 < n/m$. The search for all the solutions to (1.1) is no longer onerous, because the least positive value in (3.1) is 1438/1036035, corresponding to $(a_4, a_5, a_6, a_7) = (9, 11, 13, 23)$, so that $a_8 < 1441$ for the worst of the possible 78 cases.

It turns out that there are only three values in (3.1) for which (3.2) is soluble. More specifically, we need to have $a_4 = 9$, $a_5 = 11$, $a_6 = 15$ and $a_7 = 21, 33, 35$, giving the only value $n = 3465 = 3^2 5\times7\times11$ and the three corresponding values $m = 13, 43, 46$. The solutions to (3.2) are given in the following.

When $a_7 = 21$, $m = 13$, we have $(a_8, a_9) = (135, 10395), (165, 693), (231, 315)$.

When $a_7 = 33$, $m = 43$, we have $(a_8, a_9) = (45, 385)$.

When $a_7 = 35$, $m = 46$, we have $(a_8, a_9) = (45, 231)$.

The corresponding five solutions to (1.1) are already given in (1.4). The theorem is proved.

References

[**1**] Richard K. Guy, *Unsolved Problems in Number Theory*, Springer-Verlag, Second edition, 1994.

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