

**Loughborough University**  
**Department of Mathematical Sciences**  
**MATHEMATICAL CHALLENGE**  
**CHRISTMAS - 2022**

**Problem 1.** Two sequences  $(a_n)_{n=1}^{\infty}$  and  $(b_n)_{n=1}^{\infty}$  are geometric progressions. Define  $c_n = a_n + b_n$  for all  $n \in \mathbb{N}$ . If the first 4 terms of  $(c_n)$  are:

$$c_1 = 2, c_2 = 12, c_3 = 58, c_4 = 264,$$

prove that all other terms in this sequence are also integers and find  $c_{2023}$ . How many decimal digits are in  $c_{2023}$  and what is the first digit?

**Problem 2.** Find all monotonically increasing functions  $f : \mathbb{R} \rightarrow \mathbb{R}$ , satisfying the functional relation

$$f(f(x)) = 2022x - 2f(x).$$

**Problem 3.** Marley and Scrooge play the following game. On a blackboard they write all natural numbers from 1 to 65. Marley erases 32 numbers of his choice, then Scrooge erases 16 numbers of his choice from the remaining numbers. Then Marley erases 8 numbers and Scrooge 4 numbers. Finally Marley erases 2 numbers and Scrooge one number. There are only two remaining numbers on the board. The difference between these numbers is the amount of shillings Scrooge must pay Marley.

What are the best strategies for each of them? How much Scrooge will have to pay Marley if each of them choose the optimal strategy? Justify your answer.

*Remarks.*

1. There will be a first prize of £50 to the person handing in what will be considered to be the best effort to these problems. There may also be special prizes for the most original solutions.
2. Any student registered on one of the undergraduate programmes in the Department of Mathematical Sciences may submit solutions to any or all of these problems.
3. Solutions should be scanned and e-mailed on or before January 30, 2023 to either Prof. A.P. Veselov ([a.p.veselov@lboro.ac.uk](mailto:a.p.veselov@lboro.ac.uk)) or Dr. B. Winn ([b.winn@lboro.ac.uk](mailto:b.winn@lboro.ac.uk)), who will be the judges for the Challenge.