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Loughborough University

Department of Mathematical Sciences

MATHEMATICAL CHALLENGE

CHRISTMAS - 2020

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Problem 1. Consider the solution $y(x)$ of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{1 + y^{2020}}$$

determined by the initial data $y(0) = 0$.

Prove that all the derivatives of this function at zero are integer and find the first non-zero value of the derivative $y^{(n)}(0)$ with $n > 1$.

Problem 2. Let A_n be the $n \times n$ matrix with entries $A_{ij} = (i, j)$, $1 \le i, j \le n$, where (i, j) is the greatest common divisor of i and j.
Compute the determinants of the matrices A_n for all $n \le 10$.
What is the determinant of A_n for general n? Justify your answer.

Problem 3. Alice split all natural numbers into two infinite groups such that the sum of any 3 numbers from one group belongs to the same group. Bob claims that he knows exactly how she did this. Is that possible? Justify your answer.
Will the same be true if we replace the number 3 by 5 in the question?

Remarks.

1. There will be a first prize of £50 to the person handing in what will be considered to be the best effort to these problems. There may also be special prizes for the most original solutions.
2. Any student registered on one of the undergraduate programmes in the Department of Mathematical Sciences may submit solutions to any or all of these problems.
3. Solutions should be scanned and e-mailed on or before January 31, 2021 to either Prof. A.P. Veselov (`a.p.veselov@lboro.ac.uk`) or Dr. B. Winn (`b.winn@lboro.ac.uk`), who will be the judges for the Challenge.